10346: Continuum Physics, spring 2002

Homework assignment 2: *Collapsing and oscillating bubbles.*

During the first world war Lord Rayleigh was approached by the English Navy. They wanted him to solve the problem of cavitation, which is one of the major causes of erosion of ship propellers. If a fluid locally experiences large (negative) pressures or velocities, i.e. when it passes a sharp edge, it can "break" and form small voids, *cavities* which will then very rapidly collapse. The collapse generates noise and large pressures which can destroy solid bodies. This lead Lord Rayleigh to investigate the process of collapse of a spherical bubble. Recently bubble dynamics has become a very active field, because a gas-filled bubble can emit light when subjected to an intense sound field (*sonoluminescence*). In fact, some people claim that it is possible to obtain fusion in this way, and this has created a lot of publicity and controversy.

Imagine that a spherical cavity of radius R_0 is formed at time t_0 in an infinite, static fluid. Assume first that the fluid is inviscid and incompressible and neglect surface tension. The pressure of the fluid is p_0 far away from the bubble and the desity is ρ. We further assume that the bubble retains the spherical shape through the collapse and that the velocity field is purely radial (with origin in the center of the bubble).

1. Use dimensional analysis to find an expression for the collapse time T, assuming that it only depends on R_0, p_0 and ρ.

2. Let the bubble radius be $R(t)$. Show that the incompressibility of the flow together with the fact that it is purely radial, allows us to express it as

$$v = v_r = \frac{R^2 \dot{R}}{r^2} \tag{1}$$

3. Use the Euler equation together with the condition that the pressure on the surface of the cavity vanishes (since it is empty) to derive the equation of motion

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = -\frac{p_0}{\rho} \tag{2}$$

for R. *Hint*: integrate the Euler equation from $r = R$ to $r = \infty$.

4. Solve the equation of motion (2) numerically (say for $p_0 = 10^5$ Pa and $\rho = 10^3$ kg/m^3). Plot $R(t)$.

5. Solve (2) analytically. Use this to calculate the collaps time T and compare with the result of question 1. *Hint*: You can try the following trick: regard the velocity $U = \dot{R}$ as a function of R. More precisely, transform (2) by using the independent variable $y = \ln(R/R_0)$ and the dependent variable $f = U^2$.

6. At the collaps $U(t) \sim t^s$. Determine s numerically and analytically.

We now include surface tension. Across an interface with principal radii of curvature R_1 and R_2 there will be a pressure jump $\Delta P = \alpha(1/R_1 + 1/R_2)$ with increased pressure inside the convex region. Here α is the coefficient of surface tension. For water at room temprature $\alpha = 7.3 \times 10^{-2}$ N/m.

7. Show that surface tension changes the equation of motion to

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = -\frac{p_0}{\rho} - \frac{2\alpha}{\rho R} \qquad (3)$$

Repeat questions 1-6 for this case.

Including viscosity, has two effects. It changes the Euler equations to the Navier-Stokes equations and it changes the boundary conditions at the bubble.

8. Show that the first of these effects disappears for the case studied above, i.e. the viscous term in the Navier-Stokes equation vanishes for the flow around a spherical bubble.

We shall now imagine that the bubble is filled with an ideal gas of pressure p_g. The gas is assumed to be polytropic (see Lautrup, section 4.5) with index γ, and the variations in the gas are assumed to be adiabatic. The gas will prevent the bubble from collapsing completely, but forcing it with a time dependent pressure field can make it oscillate. We assume that the acoustic wave field which traps the bubble causes periodic pressure variations i.e. that the pressure far away from the bubble has the form

$$p_\infty = p_0 - p_D \sin(\omega_D t) \qquad (4)$$

9. Show that the equation of motion now becomes

$$\begin{aligned} R\ddot{R} + \frac{3}{2}(\dot{R})^2 &= \frac{p_g}{\rho} - \frac{2\alpha}{\rho R} - \frac{4\mu\dot{R}}{\rho R} - \frac{p_\infty}{\rho} \\ &= \frac{1}{\rho}((p_0 + \frac{2\alpha}{R_0})\left(\frac{R_0}{R}\right)^{3\gamma} - p_0 - \frac{2\alpha}{R} - \frac{4\mu\dot{R}}{R} + p_D\sin(\omega_D t)) \qquad (5) \end{aligned}$$

Note that the only non-vanishing component of the stress tensor is $\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r}$.

10. Even without the exciting sound field (i.e. with $p_D = 0$) the bubble can perform damped oscillations. Find the frequency ω_0 and damping rate of small oscillations with bubble radius around R_0. Compute the numerical values for $R_0 = 5 \times 10^{-6}$m.

11. Solve (5) numerically. Assume that the fluid is water and take $\gamma = 1.33$ for the gas in the bubble. Try $R_0 = 10^{-3}$m (although experimentally they are typically alot smaller). Vary the driving frequency ω_D and the driving amplitude p_D. Try frequencies both above and below ω_0. Find periodic solutions $R(t)$. Are they always in phase with the driving pressure? Do they always have the same period as the driving pressure? Can you find solutions that are not periodic. For p_D around 1 Atm, plot the temperature and pressure of the gas inside the bubble. Use dimensionless variables and write the coefficients in terms of dimensionless numbers. Make a complete list of such numbers, indicating their meaning and size. Motivate your choices.

Due *Tuesday, April 23* — Have fun!