

Part II

Solids at rest

Solids at rest

In the following chapters we only consider matter at permanent rest without any macroscopic local or global motion. Microscopically, matter is never truly at rest because of random molecular motion (heat), and internal atomic motion.

The essential chapters for a minimal curriculum are 6, 7, and 8. Readers that want to focus on fluid mechanics alone need only chapter 6.

List of chapters

6. **Stress:** Friction is discussed as an example of shear stress. The stress tensor field is formally introduced. The equations of mechanical equilibrium in all matter are formulated. The symmetry of the stress tensor is discussed.
7. **Strain:** The displacement field is introduced and analyzed in the limit of infinitesimal strain gradients. The strain tensor is introduced and its geometrical meaning exposed. Work and energy are analyzed, and large deformation briefly discussed.
8. **Elasticity:** Hooke's law is discussed and generalized to linear homogeneous materials. Young's modulus and Poisson's ratio are introduced. Fundamental examples of static uniform deformation are discussed. Expressions for the elastic energy are obtained.
9. **Basic elastostatics:** The most important elementary applications of linear elasticity are analyzed (gravitational settling, bending and twisting beams, spheres and tubes under pressure).
10. **Slender rods:** Slender elastic rods are analyzed and presented with characteristic examples, such as cantilevers, bridges, and coiled springs.
11. **Computational elastostatics:** The basic theory behind numeric computation of linear elastostatic solutions is presented, and a typical example is analyzed and computed.

6

Stress

In fluids at rest pressure is the only contact force. For solids at rest or in motion, and for viscous fluids in motion, this simple picture is no longer valid. Besides pressure-like forces acting along the normal to a contact surface, there may also be shearing forces acting in any tangential direction to the surface. The relevant local quantity analogous to pressure is the shear force per unit of area, called the *shear stress*. Friction, so well-known from everyday life, is always a result of shear stress acting in the contact areas between material bodies. The integrity of a solid body is largely secured by internal shear stresses.

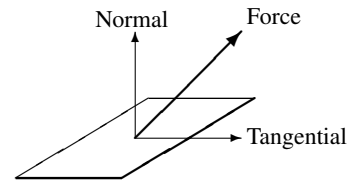
The two major classes of materials, fluids and solids, react differently to external stresses. Whereas fluids respond by *flowing*, solids respond by *deforming*. Elastic deformation usually grows linearly with stress, although every material in the end becomes plastic and deforms permanently, and may eventually rupture. We shall as far as possible maintain a general view of the physics of continuous systems, applicable to all types of materials, also those that do not fall into the major classes. The field of exotic materials is rapidly growing because of the needs for special material properties in technology.

In this chapter the emphasis is on the theoretical formalism for contact forces, independent of whether they occur in solids, fluids, or intermediate forms. The vector notation used up to this point is not adequate to the task, because contact forces not only depend on the spatial position but also on the orientation of the surface on which they act. A collection of nine stress components, called the *stress tensor*, is used to describe the full range of contact forces that may come into play. The concept of stress and its properties is the centerpiece of the continuum mechanics, and therefore of this book.

6.1 Friction

Friction between solid bodies is a shear force known to us all. We hardly think of friction forces, even though all day long we are served by them and do service to them. Friction is the reason that the objects we hold are not slippery as a piece of soap in the bathtub, but instead allow us to grab, drag, rub and scrub. Most of the work we do is done against friction, from stirring the coffee to making fire by rubbing two pieces of wood against each other.

The law of static friction goes back to Amontons (1699) and the law of dynamic friction to Coulomb (1779). The full story of friction between solid bodies is complicated, and in spite of our everyday familiarity with friction, there is still no universally accepted microscopic explanation of the phenomenon [GM01, Kes01].



The force on a small piece of a surface can be resolved by a normal pressure force and a tangential shear force.

Guillaume Amontons (1663–1705). French physicist. Made improvements to the barometer, hygrometer and thermometer.

Charles-Augustin de Coulomb (1736–1806). French physicist, best known from the law of electrostatics and the unit of electric charge that carries his name.

	μ_0	μ
Glass/glass	0.9	0.4
Rubber/asphalt	0.9	0.7
Steel/steel	0.7	0.6
Metal/metal	0.6	0.4
Wood/wood	0.4	0.3
Steel/ice	0.1	0.05
Steel/teflon	0.05	0.05

Typical friction coefficients for various combinations of materials.

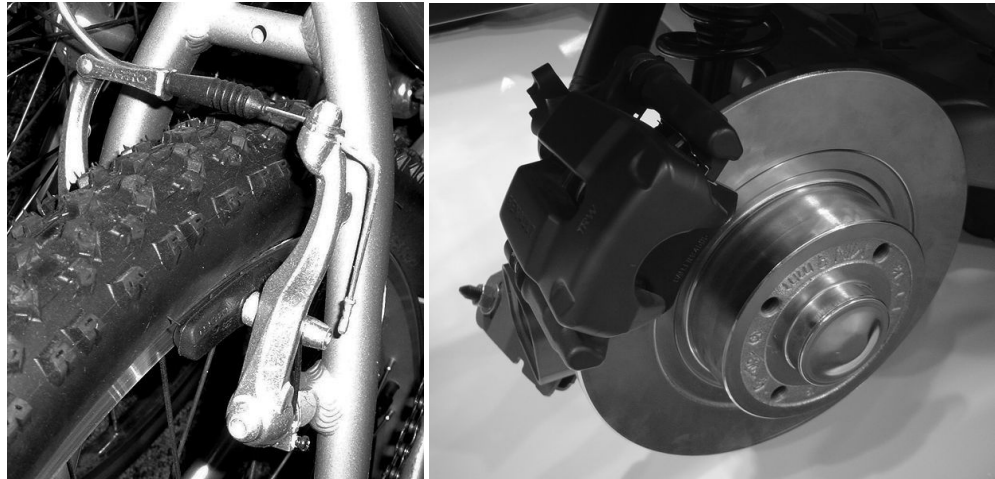
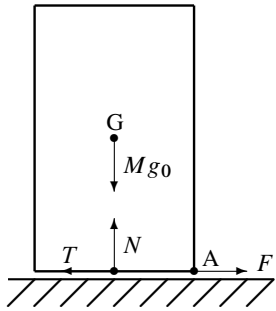
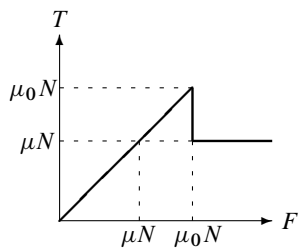


Figure 6.1. The construction of a friction brake depends on how much energy it has to dissipate. In bicycles (left) the pads are pressed against the metallic rim of the wheel by suitably levered finger forces. In cars (right) disc brakes require hydraulics to press the pads against the metallic discs. Both images from Wikipedia Commons.



Balance of forces on a crate at rest on a horizontal floor. The point of attack A is here chosen at floor level to avoid creating a moment of force which could turn over the crate.



Sketch of tangential reaction (traction) T as a function of applied force F . Up to $F = \mu_0 N$, the traction adjusts itself to the applied force, $T = F$. At $F = \mu_0 N$, the tangential reaction drops abruptly to a lower value, $T = \mu N$, and stays there regardless of the applied force.

Static friction

Consider a heavy crate at rest, standing on a horizontal floor. The bottom of the crate acts on the floor with a force equal to its weight Mg_0 , and the floor acts back on the bottom of the crate with an equal and opposite normal reaction force of magnitude $N = Mg_0$. If you try to drag the crate along the floor by applying a horizontal force, F , you may discover that the crate is so heavy that you are not able to budge it, implying that the force you apply must be fully balanced by an opposite tangential or shearing friction force (also called traction), $T = F$, between the floor and the crate.

Empirically, such *static friction* can take any magnitude up to a certain maximum, which is proportional to the normal load,

$$T < \mu_0 N. \quad (6.1)$$

The dimensionless constant of proportionality μ_0 is called the *coefficient of static friction*. In our daily doings it may take a quite sizable value, say 0.5 or greater, making dragging comparable to lifting. Its value depends on what materials are in contact and on the roughness of the contact surfaces (see the margin table).

Dynamic friction

If you are able to pull with a sufficient strength, the crate suddenly starts to move, but friction will still be present and you will have to do real work to move the crate any distance. Empirically, this *dynamic* (or sliding) friction is proportional to the normal load,

$$T = \mu N, \quad (6.2)$$

with a coefficient of dynamic friction, μ , that is always smaller than the corresponding coefficient of static friction, $\mu < \mu_0$. This is why you have to heave strongly to get the crate set into motion, whereas afterwards a smaller force, $F = \mu N$, suffices to keep it going at constant speed. Notice that whereas static friction requires no work to be done, the rate of work, $P = FU$, performed against dynamic friction grows linearly with velocity U , and this work is irretrievably lost to heat.

Static and dynamic friction are both independent of the size of the contact area, so that a crate on legs is as hard to drag as one without. The best way to diminish the force necessary to drag the crate is to place it on wheels.

What do cars have wheels?: The contact between a rolling wheel and the road surface is always static, although the actual contact area moves as the wheel rolls. Static friction performs no work, so friction losses are only found in the bearing of the wheels and other parts of the transmission (and some in the deformation of the tires). Being lubricated these bearings are much less lossy than dynamic friction would be (see chapter 17).

When a moving car brakes, energy is dissipated by sliding friction in the brake assembly (see figure 6.1) where the normal load is controlled by the force applied to the brake pedals. A larger weight generates proportionally larger static friction between wheel and road, so a car's minimal braking distance will in the first approximation be independent of how heavily it is loaded, and that is lucky for the flow of traffic. Braking a car, it is best to avoid blocking the wheels so that the car skids because dynamic (sliding) friction is smaller than static (rolling) friction (see problem 6.3). More importantly, a skidding car has no steering control, so modern cars are equipped with anti-skid brake systems that automatically adjust brake pressure to avoid skidding.

6.2 Stress fields

The above arguments indicate that it is necessary to introduce one or more fields describing the distribution of normal and tangential forces acting on a surface. In analogy with pressure, such *stress fields* represent the local force per unit of area, whether it is normal or tangential, and are measured in the same units as pressure, namely the pascal ($\text{Pa} = \text{N m}^{-2}$). A stress that acts along the normal to a surface is called a *tension stress* whereas a stress that acts against the normal is called a *pressure stress*. A stress that acts along a tangential direction to a surface is called a *shear stress*.

External and internal stresses

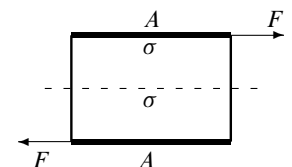
The stresses acting in the interface between a body and its environment are *external*. As for pressure, we shall also speak about *internal stresses* acting across any imagined surface in the body. Internal stresses abound in the macroscopic world around us. Whenever we come into contact with the environment (and when do we not?) internal stresses are set up in the materials we touch, as well as in our own bodies. The precise distribution of stress in a body depends not only on the external forces applied to the body, but also on the types of material the body is made from and on other macroscopic quantities such as temperature.

There is no guarantee that the normal stress is the same for any choice of surface through a given point. The proof of Pascal's law (page 26)—that the pressure is the same in all directions of the normal—assumed that the material was a fluid at rest. For fluids in motion and for solids in general, this proof does not hold, but there is a generalized version of the theorem.

In the absence of external forces there is usually no internal stress in a material, although fast cooling may freeze stresses permanently into certain materials, for example glass, and provoke an almost explosive release of stored energy when triggered by a sudden impact.

Estimating stress

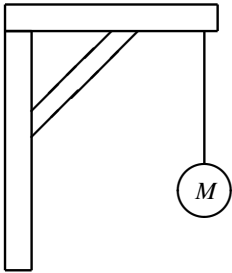
In many situations it is quite straightforward to estimate average stresses in a body from the external forces that act on the body. Consider, for example, a slab of homogeneous solid material bounded by two stiff flat clamps of area A , firmly glued to it. A tangential force F applied to one clamp with the other held fixed will deform the slab a bit in the direction of the applied force, so that there is an average shear stress $\sigma = F/A$ acting on the upper surface of the slab in the direction of the force.



Clamped slab of homogeneous material. The shear force F at the upper clamp is balanced by an oppositely directed fixation force F on the lower clamp. The shear stress $\sigma = F/A$ is the same on all inner planes parallel with the clamps (dashed).

The fixed clamp will by Newton's Third Law act back on the slab with a force of the same magnitude but in the opposite direction. If we make an imaginary cut through the slab parallel with the clamps, then the upper part of the slab must likewise act like a clamp on the lower with the shear force F , so that the average internal shear stress acting in the cut must again be $\sigma = F/A$. Had a normal external force also been applied to the clamps, we would have gone through the same type of argument to convince ourselves that the average normal stress would be the same everywhere in the cut.

For bodies with a more complicated geometry and non-uniform external load, internal stresses are not so easily calculated, although their average magnitudes may often be estimated, especially for symmetric bodies. In analogy with friction one may assume that variations in shear and normal stresses are roughly of the same order of magnitude, provided the material, the external load, and the body geometry are unexceptional.



The classic gallows construction.

Example 6.1 [Classic gallows]: The classic gallows is constructed from a vertical pole, a horizontal beam and a diagonal strut. A body of mass $M = 70$ kg hangs at the extreme end of the horizontal beam of cross section $A = 100$ cm². The body's weight must be balanced by a shear stress in the beam of average magnitude $\sigma \approx Mg_0/A \approx 70\,000$ Pa ≈ 0.7 bar. The actual distribution of shear stress will generally vary over the cross section of the beam and the position of the chosen cross section, but its average magnitude should be of the estimated value. There will also be a pressure force that has to vary over the cross section to balance the moment of force that the body exerts on the beam (see section 9.3).

Example 6.2 [Water pipe]: The half-inch water mains in your house have an inner pipe radius $a \approx 0.6$ cm. Tapping water at a high rate, internal friction in the water (viscosity) creates shear stresses opposing the flow, and the pressure drops perhaps by $\Delta p \approx 0.1$ bar = 10^4 Pa over a length of $L \approx 10$ m of the pipe. In this case, we may calculate the shear stress on the water at the inner surface of the pipe, because the pressure difference between the ends of the pipe is the only other force acting on the water. Setting the force due to the pressure difference equal to the total shear force on the inner surface, we get,

$$\pi a^2 \Delta p = 2\pi a L \sigma, \quad (6.3)$$

from which it follows that $\sigma = \Delta p a/2L \approx 3$ Pa. This stress is of the same magnitude as the pressure drop $\Delta p \cdot 2a/L$ over a stretch of pipe of length equal to the diameter.

	Yield [MPa]	Tensile [MPa]
Tungsten		3450
Titanium	275	420
Steel	250	415
Nickel	58	317
Copper	70	220
Cast iron	NA	172
Zinc	124	172
Magnesium	90	172
Silver	55	160
Gold		130
Aluminium	35	90
Tin	12	22
Lead		17

Tensile strength

When external forces grow large, a solid body will deform strongly, become plastic, or even fracture and break apart. The maximal tension, i.e. negative pressure or pull, a material can sustain without failure is called the *tensile strength* of the material. Similarly, the *yield stress* is defined as the stress beyond which otherwise elastic solids begin to undergo plastic deformation from which the material does not automatically recover when released. Such materials are also called *ductile*. Copper is an example of a ductile metal which can be drawn or stretched without breaking. Brittle materials, such as concrete or cast iron, do not have a yield point but fail suddenly when then tension reaches the tensile strength.

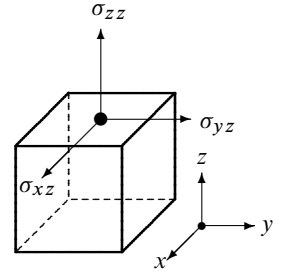
For metals the tensile strength lies typically in the region of several hundred megapascals (i.e. several kilobars) as shown in the margin table. Modern composite carbon fibers can have tensile strengths up to several gigapascals, whereas "ropes" made from single-wall carbon nanotubes are reported to have tensile strengths of up to 50 gigapascals or even larger [YFAR00].

Example 6.3 [Carbon steel rod]: Plain carbon steel has a tensile strength of 415 MPa. A quick estimate shows that a steel rod with a diameter of 2 cm breaks if loaded with more than 13 000 kg. Adopting a safety factor of 10, one should not load it with more than 1300 kg, which also brings it well below the yield point for carbon steel.

Typical yield stress and tensile strength for common metals. The values may vary widely for different specimens, depending on purity, heat treatment and other factors.

6.3 The nine components of stress

Shear forces are more complicated than normal forces, because there is an infinity of tangential directions on a surface. In a coordinate system where a force $d\mathcal{F}_x$ is applied along the x -direction to a material surface element dS_z with its normal in the z -direction, the shear stress will be denoted $\sigma_{xz} = d\mathcal{F}_x/dS_z$, instead of just σ . Similarly, if the shear force is applied in the y -direction, the stress would be denoted $\sigma_{yz} = d\mathcal{F}_y/dS_z$, and if a normal force is applied along the z -direction, the normal stress is consistently denoted $\sigma_{zz} = d\mathcal{F}_z/dS_z$. The sign is chosen such that a positive value of σ_{zz} corresponds to a pull or *tension*.



Components of stress acting on a surface element dS_z in the xy -plane.

Cauchy's stress hypothesis

Continuing this argument for surface elements with normal along x or y , it appears to be necessary to use at least nine numbers to indicate the state of stress in a given point of a material in a Cartesian coordinate system. *Cauchy's stress hypothesis* (to be proved below) asserts that these nine numbers are in fact all what is needed to determine the force $d\mathcal{F} = (d\mathcal{F}_x, d\mathcal{F}_y, d\mathcal{F}_z)$ on an arbitrary surface element, $d\mathcal{S} = (dS_x, dS_y, dS_z)$, according to the rule

$$\begin{aligned} d\mathcal{F}_x &= \sigma_{xx}dS_x + \sigma_{xy}dS_y + \sigma_{xz}dS_z, \\ d\mathcal{F}_y &= \sigma_{yx}dS_x + \sigma_{yy}dS_y + \sigma_{yz}dS_z, \\ d\mathcal{F}_z &= \sigma_{zx}dS_x + \sigma_{zy}dS_y + \sigma_{zz}dS_z. \end{aligned} \quad (6.4)$$

This may also be written more compactly with index notation,

$$d\mathcal{F}_i = \sum_j \sigma_{ij}dS_j, \quad (6.5)$$

where the indices range over the coordinate labels, here x , y , and z . Each coefficient $\sigma_{ij} = \sigma_{ij}(\mathbf{x}, t)$ depends on the position (and time), and is thus a field in the normal sense of the word.

*** Proof of Cauchy's stress hypothesis:** As in the proof of Pascal's law (page 26) we take again a surface element in the shape of a tiny triangle with area vector $d\mathcal{S} = (dS_x, dS_y, dS_z)$. This triangle and its projections on the coordinate planes form together a little body in the shape of a right tetrahedron. Let the external (vector) forces acting on the three triangular faces of the tetrahedron be denoted respectively $d\mathcal{F}_x$, $d\mathcal{F}_y$ and $d\mathcal{F}_z$. Adding the external force $d\mathcal{F}$ acting on the skew face and a possible volume force $f dV$, Newton's Second Law for the small tetrahedron becomes

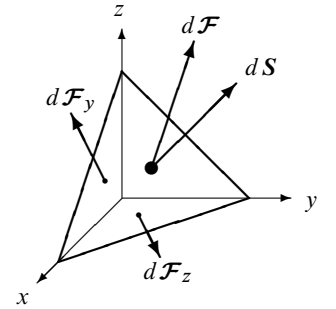
$$\mathbf{w} dM = f dV + d\mathcal{F}_x + d\mathcal{F}_y + d\mathcal{F}_z + d\mathcal{F}, \quad (6.6)$$

where \mathbf{w} is the center-of-mass acceleration of the tetrahedron and $dM = \rho dV$ its mass.

The volume of the tetrahedron scale like the third power of its linear size, whereas the surface areas only scale like the second power. Making the tetrahedron progressively smaller, the body force term and the acceleration term will vanish faster than the surface terms. In the limit of a truly infinitesimal tetrahedron, only the surface terms survive, so that we must have $d\mathcal{F}_x + d\mathcal{F}_y + d\mathcal{F}_z + d\mathcal{F} = \mathbf{0}$. Taking into account that the area projections dS_x , dS_y and dS_z point *into* the tetrahedron, we define the stress vectors along the coordinate directions as $\boldsymbol{\sigma}_x = -d\mathcal{F}_x/dS_x$, $\boldsymbol{\sigma}_y = -d\mathcal{F}_y/dS_y$ and $\boldsymbol{\sigma}_z = -d\mathcal{F}_z/dS_z$. Consequently,

$$d\mathcal{F} = \boldsymbol{\sigma}_x dS_x + \boldsymbol{\sigma}_y dS_y + \boldsymbol{\sigma}_z dS_z. \quad (6.7)$$

This shows that the force on an arbitrary surface element may be written as a linear combination of three basic stress vectors along the coordinate axes. Introducing the nine coordinates of the three stress vectors, $\boldsymbol{\sigma}_x = (\sigma_{xx}, \sigma_{yx}, \sigma_{zx})$, $\boldsymbol{\sigma}_y = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy})$ and $\boldsymbol{\sigma}_z = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$, we arrive at Cauchy's hypothesis (6.4).



The tiny triangle and its projections form a tetrahedron. The (vector) force acting on the triangle in the xy -plane is called $d\mathcal{F}_z$, and the force acting on the triangle in the xz -plane $d\mathcal{F}_y$. The force $d\mathcal{F}_x$ acting on the triangle in the yz -plane is hidden from view. The force $d\mathcal{F}$ acts on the skew triangle.



Augustin-Louis, Baron Cauchy (1789–1857). French mathematician who produced an astounding 789 papers. Contributed to the foundations of elasticity, hydrodynamics, partial differential equations, number theory and complex functions.

The stress tensor

Together the nine fields, $\{\sigma_{ij}\}$, make up a single geometric object, called the *stress tensor field*¹, first introduced by Cauchy in 1822. Vector and tensor calculus is developed in some detail in appendix B. It must be emphasized that this collection of nine fields cannot be viewed geometrically as consisting of either nine scalar or three vector fields, but must be considered together as one geometric object, a *tensor field* $\{\sigma_{ij}(\mathbf{x}, t)\}$. There is unfortunately no simple, intuitive way of visualizing the stress tensor field.

It is often convenient to write the stress tensor a matrix,

$$\boldsymbol{\sigma} = \{\sigma_{ij}\} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}, \quad (6.8)$$

so that the force may be written compactly as a matrix equation,

$$d\mathcal{F} = \boldsymbol{\sigma} \cdot d\mathbf{S}. \quad (6.9)$$

Notice how a dot is used to denote the product of a matrix and a vector. This is consistent with the rule for dot-products of vectors.

Writing $d\mathbf{S} = \hat{\mathbf{n}} dS$ where $\hat{\mathbf{n}}$ is the normal to the surface, the force per unit of area becomes $d\mathcal{F}/dS = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, often called the *stress vector*. Although it is a vector, it is *not a vector field* in the strict sense of the word, because it also depends the normal to the surface on which it acts. The following example demonstrates this point.

Example 6.4 [A tensor field]: A tensor field of the form,

$$\{\sigma_{ij}\} = \{x_i x_j\} = \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix}, \quad (6.10)$$

is a tensor product (see the definition (B.8) on page 599) and thus by construction a true tensor. The “stress vector” acting on a surface with normal in the direction of the x -axis becomes

$$\boldsymbol{\sigma}_x = \boldsymbol{\sigma} \cdot \hat{\mathbf{e}}_x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} x. \quad (6.11)$$

It is not a true vector because of the factor x on the right-hand side.

Total force

Including a true body force, $d\mathcal{F} = \mathbf{f} dV$, for example gravity $\mathbf{f} = \rho\mathbf{g}$, the total force on a body of volume V with surface S becomes,

$$\mathcal{F} = \int_V \mathbf{f} dV + \oint_S \boldsymbol{\sigma} \cdot d\mathbf{S}, \quad (6.12)$$

which in index notation takes the form

$$\mathcal{F}_i = \int_V f_i dV + \oint_S \sum_j \sigma_{ij} dS_j. \quad (6.13)$$

The circle through the surface integral is only there to remind us that the surface is closed.

¹Some texts use the symbols $\{\tau_{ij}\}$ or $\{T_{ij}\}$ for the stress tensor field.

The surface integral can be converted into a volume integral by means of Gauss' theorem, eq. (2.13) on page 25, applied to each component of the stress tensor,

$$\mathcal{F}_i = \int_V f_i dV + \int_V \sum_j \nabla_j \sigma_{ij} dV \equiv \int_V f_i^* dV. \quad (6.14)$$

In the last step we have—in complete analogy with the hydrostatic definition of the effective force density (2.21) on page 28—introduced the *effective force density*,

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}. \quad (6.15)$$

In matrix notation the second term may be written compactly as a tensor divergence,

$$\mathbf{f}^* = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}^\top. \quad (6.16)$$

where $\boldsymbol{\sigma}^\top$ is the *transposed* matrix, defined by $\sigma_{ji}^\top = \sigma_{ij}$.

The effective force density is not just a formal quantity because the total force on a material particle is indeed $d\mathcal{F} = \mathbf{f}^* dV$ (see the margin figure). That we can write the total force as a sum over contributions of this kind confirms that matter may be viewed as a collection of material particles. It must as before be emphasized that the effective force density \mathbf{f}^* is not a long-range volume force, but a local expression which for a tiny material particle equals the sum of the true long-range force (here gravity) and all the short-range contact forces acting on its surface.

Mechanical pressure

In hydrostatic equilibrium, where the only contact force is pressure and the force element is $d\mathcal{F} = -pd\mathbf{S}$, comparison with eq.(6.9) shows that the stress tensor must be

$$\boldsymbol{\sigma} = -p \mathbf{1} \quad \text{or equivalently} \quad \sigma_{ij} = -p \delta_{ij}, \quad (6.17)$$

where $\mathbf{1}$ is the (3×3) unit matrix. In index notation the unit matrix is represented by the Kronecker delta tensor, δ_{ij} , which equals 1 for $i = j$ and 0 otherwise (see appendix B).

Generally, however, the stress tensor will also have off-diagonal non-vanishing components. A diagonal component behaves like a (negative) pressure, and often the negative diagonal components are called the “pressures” along the coordinate directions,

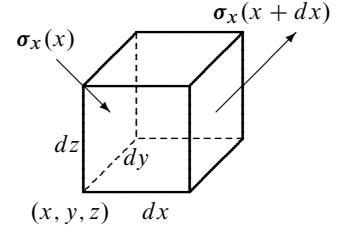
$$p_x = -\sigma_{xx}, \quad p_y = -\sigma_{yy}, \quad p_z = -\sigma_{zz}. \quad (6.18)$$

Here it should be remembered that the triplet (p_x, p_y, p_z) of diagonal elements of the stress tensor does not behave like a Cartesian vector (see section B.6 on page 606). It is in fact not a well-defined geometric object at all.

There is no unique way of defining *the* pressure in general materials, so wherever pressure is used, it must be accompanied by a suitable definition. Sometimes the pressure will be identified with the *mechanical pressure*, defined as the average of the three pressures along the axes,

$$p = \frac{1}{3}(p_x + p_y + p_z) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \quad (6.19)$$

This makes sense because the sum over the diagonal elements of a matrix, the trace $\text{Tr } \boldsymbol{\sigma} \equiv \sum_i \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$, is invariant under Cartesian coordinate transformations (see appendix B). Defining pressure in this way ensures that it is a scalar field, taking the same value in all coordinate systems. There is in fact no other scalar linear combination of the nine stress components with that property.



The total contact force on a small box-shaped material particle is calculated from the differences of the stress vectors acting on the sides. Suppressing the dependence on y and z , the resultant contact force on dS_x becomes $d\mathcal{F} = (\sigma_x(x + dx) - \sigma_x(x))dS_x \approx \nabla_x \sigma_x dV$. Adding the similar contributions from dS_y and dS_z , one arrives at the total surface force, $d\mathcal{F} = (\nabla_x \sigma_x + \nabla_y \sigma_y + \nabla_z \sigma_z) dV = \nabla \cdot \boldsymbol{\sigma}^\top dV$.

Example 6.5: For the stress tensor given in example 6.4 the pressures along the coordinate axes become $p_x = -x^2$, $p_y = -y^2$ and $p_z = -z^2$. Evidently, they do not form a vector, but the mechanical pressure,

$$p = -\frac{1}{3}(x^2 + y^2 + z^2), \quad (6.20)$$

is clearly a scalar, invariant under rotations of the Cartesian coordinate system.

6.4 Mechanical equilibrium

In mechanical equilibrium, the total force on any body must vanish, for if it does not the body must begin to move. Accordingly, the general condition for mechanical equilibrium is that $\mathcal{F} = \mathbf{0}$ for all volumes V . Applied to material particles, it implies that the force on each and every material particle must vanish, $\mathbf{f}^* = \mathbf{0}$. We thus arrive at *Cauchy's equilibrium equation*,

$$\mathbf{f} + \nabla \cdot \boldsymbol{\sigma}^T = \mathbf{0}. \quad (6.21)$$

which in tensor notation becomes,

$$\boxed{f_i + \sum_j \nabla_j \sigma_{ij} = 0}, \quad (6.22)$$

In spite of their simplicity *these partial differential equations govern mechanical equilibrium in all kinds of continuous matter*, be it solid, fluid, or whatever. For a fluid at rest with pressure being the only stress component, we have $\boldsymbol{\sigma} = -p\mathbf{1}$, and recover with $\mathbf{f} = \rho\mathbf{g}$ the equation of hydrostatic equilibrium, $\rho\mathbf{g} - \nabla p = \mathbf{0}$.

It is instructive to write out the three coupled partial differential equations in full,

$$\begin{aligned} f_x + \nabla_x \sigma_{xx} + \nabla_y \sigma_{xy} + \nabla_z \sigma_{xz} &= 0, \\ f_y + \nabla_x \sigma_{yx} + \nabla_y \sigma_{yy} + \nabla_z \sigma_{yz} &= 0, \\ f_z + \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{zz} &= 0. \end{aligned} \quad (6.23)$$

These three equations are in themselves not sufficient to determine the stress distribution in continuous matter, but must be supplemented by *constitutive* equations connecting the stress tensor with the variables describing the state of the matter. For fluids at rest, the equation of state serves this purpose by relating pressure to mass density (and temperature), as we saw in chapter 2. In solids at rest the constitutive equations are more complicated and relate stress to strain, a tensor that describes the state of deformation (chapter 7).

Symmetry

There is one additional condition (also going back to Cauchy) which is normally imposed on the stress tensor, namely that it must be *symmetric*,

$$\boxed{\sigma_{ij} = \sigma_{ji}, \quad \text{or} \quad \boldsymbol{\sigma}^T = \boldsymbol{\sigma}.} \quad (6.24)$$

Symmetry only affects the shear stress components,

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{yz} = \sigma_{zy}, \quad \sigma_{zx} = \sigma_{xz}, \quad (6.25)$$

and thereby reduces the number of independent stress components from nine to six. Here we shall now present a simple, though not entirely correct, argument for symmetry. In the following section we analyze the properties of asymmetric stress tensors and the deeper reasons for imposing symmetry.

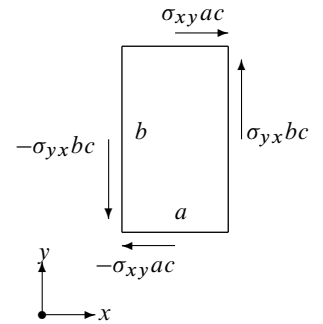
Consider a material particle in the shape of a tiny rectangular box with sides a , b and c . The shear force acting in the y -direction on a face in the x -plane is $\sigma_{yx}bc$ whereas the shear force acting in the x -direction on a face in the y -plane is $\sigma_{xy}ac$. On opposite faces the contact forces have opposite sign in mechanical equilibrium (their difference is as we have seen of order abc). Since the total force vanishes, the total moment of force on the box may be calculated around any point we wish. Using the lower left corner, we get (since the moments of the diagonal elements of the stress tensor cancel automatically)

$$\mathcal{M}_z = a \sigma_{yx}bc - b \sigma_{xy}ac = (\sigma_{yx} - \sigma_{xy})abc.$$

This shows that if the stress tensor is asymmetric, $\sigma_{xy} \neq \sigma_{yx}$, there will be a resultant moment on the box. In mechanical equilibrium this cannot be allowed, since such a moment would begin to rotate the box, and consequently the stress tensor must be symmetric. Conversely, when the stress tensor is symmetric, mechanical equilibrium of the forces alone guarantees that all local moments of force will also vanish.

Being a symmetric matrix, the stress tensor may be diagonalized. The eigenvectors of a symmetric stress tensor define the principal directions of stress and the eigenvalues the principal tensions or stresses. In the principal basis, there are no off-diagonal elements, i.e. no shear stresses, only pressure-like stresses. The principal basis is generally different from point to point in space.

Example 6.6: The stress tensor in example 6.4 has one radial eigenvector $\hat{e}_r = x/r$ with eigenvalue $r^2 = x^2 + y^2 + z^2$ and two degenerate eigenvectors with eigenvalue 0 orthogonal to \hat{e}_r , for example the spherical unit vectors \hat{e}_θ and \hat{e}_ϕ .



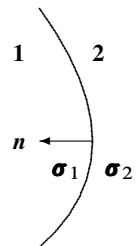
An asymmetric stress tensor will produce a non-vanishing moment of force on a small rectangular box (the z -direction not shown).

Boundary conditions

Cauchy's equation of mechanical equilibrium (6.22) constitutes a set of coupled partial differential equations valid inside a body's volume, and such equations require boundary conditions to be imposed on the surface of the body. The stress tensor is a local physical quantity, or rather collection of quantities, and may, like pressure in hydrostatics, be assumed to be continuous in regions where material properties change continuously. Across real boundaries, surfaces or interfaces, where material properties may change abruptly, Newton's third law demands (in the absence of surface tension) that the two sides act on each other with equal and opposite forces. Since the surface elements of the two sides are opposite to each other, it follows that the stress vector, $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \{\sum_j \sigma_{ij} n_j\}$, must be continuous across a surface with normal $\hat{\mathbf{n}}$ (in the absence of surface tension). This is best expressed by the vanishing of the surface discontinuity,

$$\Delta \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \mathbf{0} \tag{6.26}$$

This does *not* mean that all the components of the stress tensor should be continuous. Since it is a vector condition, it imposes continuity on three linear combinations of stress components, but leaves three other combinations free to jump discontinuously (for the symmetric stress tensor). Surprisingly, it does not follow that the mechanical pressure (6.19) is continuous. In full-fledged continuum mechanics, the mechanical pressure loses the appealing intuitive meaning it acquired in hydrostatics.



Contact surface separating body 1 from body 2. Newton's third law requires continuity of the stress vector $\boldsymbol{\sigma} \cdot \mathbf{n}$ across the boundary, i.e. $\boldsymbol{\sigma}_1 \cdot \mathbf{n} = \boldsymbol{\sigma}_2 \cdot \mathbf{n}$, or $\Delta \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$.

Example 6.7 [What may jump and what may not]: Consider a plane interface in the yz -plane with normal along x . The stress components σ_{xx} , σ_{yx} and σ_{zx} must then be continuous, because they specify the stress vector on such a surface. Symmetry implies that σ_{xy} and σ_{xz} are likewise continuous. The remaining three independent components σ_{yy} , σ_{zz} and $\sigma_{yz} = \sigma_{zy}$ are allowed to jump at the interface. In particular the mechanical pressure, $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$, may (and will usually) be discontinuous.

* 6.5 Asymmetric stress tensors

The stress tensor was introduced at the beginning of this chapter as a quantity which furnished a complete description of the contact forces that may act on any surface element. There is nothing in the definition of the stress tensor that requires it to be symmetric, so we cannot exclude the possibility that it could be asymmetric. In normal elastic materials where stress only depends on deformation it will automatically be symmetric, as we shall see in chapter 8.

Total moment of force

The basic principle of continuum physics is that matter may be viewed as a collection of material particles obeying Newton's laws. We have seen that the total force (6.14) is the integral over individual forces $\mathcal{F} = \mathbf{f}^* dV$ acting on each and every material particle, so in keeping with this principle, the total moment of force on a body with volume V must be

$$\mathcal{M} = \int_V \mathbf{x} \times d\mathcal{F} = \int_V \mathbf{x} \times \mathbf{f}^* dV. \quad (6.27)$$

In chapter 20 we shall see that this moment is indeed the time derivative of the total angular momentum of the body, as it ought to be. Since mechanical equilibrium requires $\mathbf{f}^* = \mathbf{0}$ everywhere, it follows that the total moment of force vanishes automatically in mechanical equilibrium, whether the stress tensor is symmetric or not.

How can it then be that we (on the preceding page) were able to demonstrate the symmetry of the stress tensor by demanding that the moment of force on a material particle should vanish? To find the answer we first write the total moment in index notation,

$$\mathcal{M} = \sum_i \hat{e}_i \mathcal{M}_i = \int_V \sum_{ijk} \epsilon_{ijk} \hat{e}_i x_j \left(f_k + \sum_l \nabla_l \sigma_{kl} \right) dV, \quad (6.28)$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor, defined in eq. (B.25) on page 602. Rewriting the stress term by means of the trivial identity,

$$x_j \sum_l \nabla_l \sigma_{kl} = \sum_l \nabla_l (x_j \sigma_{kl}) - \sigma_{kj}, \quad (6.29)$$

and applying Gauss theorem to the divergence, the total moment can be rewritten as the sum of the moment of external body and surface forces, plus an extra term,

$$\mathcal{M} = \int_V \mathbf{x} \times \mathbf{f} dV + \oint_S \mathbf{x} \times \boldsymbol{\sigma} \cdot d\mathbf{S} + \int_V \sum_{ijk} \hat{e}_i \epsilon_{ijk} \sigma_{jk} dV. \quad (6.30)$$

The extra contribution to the total moment may be viewed as a kind of internal moment associated with the material of the body. It vanishes for all volumes if and only the stress tensor is symmetric.

In our discussion of symmetry above, we only calculated the moment of the external surface stresses acting on a material particle, i.e. the second term in eq. (6.30). But the external moment does not have to vanish in equilibrium if the stress tensor is asymmetric, only the sum of the external and internal moments must vanish. So the argument fails in general, and is vacuous if the stress tensor is already symmetric. We are, however, left with an unanswered question about the nature of the extra internal moment distribution due to the asymmetry.

Non-classical continuum theories

The stress tensor allows us to calculate the force on any open surface in a body. But open surfaces do not define physical bodies, only closed surfaces do. This gives rise to an ambiguity in the stress tensor which makes it possible to convert any asymmetric stress tensor into a symmetric tensor without changing the forces on physical bodies (see [MPP72, Appendix A] and [Landau and Lifshitz 1986, p. 7]).

Even if it is thus formally possible to construct an equivalent symmetric stress tensor from an asymmetric one, the relation between them is, however, non-local. This is an embarrassment because such non-locality is in disagreement with the basic principle in continuum physics that forces and moments should be calculable from the local state of matter, i.e. from the properties of the material particles.

Manifestly asymmetric stress tensors have been used in various generalizations of classical continuum theory, generally called *Cosserat theory*. In such theories, a material particle is endowed with geometric properties in addition to its position, for example angles describing its spatial orientation. These theories have turned out to be useful for the continuum treatment of modern complex materials, which the classical theory is incapable of handling. We shall not go further into these extensions of classical continuum physics; see for example [Narasimhan 1993, p. 493] for a short review.

Eugène Maurice Pierre Cosserat (1866–1931). French mathematician. Worked in mathematics and astronomy, and created in 1909 together with his brother, an engineer, the now famous extension to the classical continuum theory of elasticity.

Problems

6.1 A crate is dragged over a horizontal floor with sliding friction coefficient μ . Determine the angle α with the vertical of the total reaction force.

6.2 Explain what forces and moments that act on an accelerating car. Explain why rear-wheel drive does not spin the wheels as easily as front-wheel drive.

6.3 A car with mass m moves with a speed U . Estimate the minimal braking distance without skidding and the corresponding braking time. Do the same if it skids from the beginning to the end. For numerics use $m = 1000$ kg and $v = 100$ km h⁻¹. The static coefficient of friction between rubber and the surface of a road may be taken to be $\mu_0 = 0.9$, whereas the sliding friction is $\mu = 0.7$.

6.4 A strong man pulls a jumbo airplane slowly but steadily exerting a force of $\mathcal{F} = 2000$ N on a rope. The plane has $N = 32$ wheels, each touching the ground in a square area $A = 40 \times 40$ cm². **(a)** Estimate the shear stress between the rubber and the tarmac. **(b)** Estimate the shear stress between the tarmac and his feet, each with area $A = 5 \times 25$ cm².

6.5 Estimate the maximal height h of a mountain made from rock with a density of $\rho = 3,000$ kg m⁻³ when the maximal stress the material can tolerate before it deforms permanently is $\sigma = 300$ MPa. How high could it be on Mars where the surface gravity is 3.7 m s⁻²?

6.6 A stress tensor has all components equal, i.e. $\sigma_{ij} = \tau$ for all i, j . Find its eigenvalues and eigenvectors.

6.7 Show that if the stress tensor is diagonal in all coordinate systems, then it can only contain pressure.

6.8 A stress tensor and a rotation matrix are given by,

$$\boldsymbol{\sigma} = \begin{pmatrix} 15 & -10 & 0 \\ -10 & 5 & 0 \\ 0 & 0 & 20 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{pmatrix}.$$

Calculate the stress tensor in the rotated coordinate system $\mathbf{x}' = \mathbf{A} \cdot \mathbf{x}$.

6.9 (a) Show that the average of a unit vector \mathbf{n} over all directions obeys

$$\langle n_i n_j \rangle = \frac{1}{3} \delta_{ij}. \quad (6.31)$$

(b) Use this to show that the average of the normal stress acting on an arbitrary surface element in a fluid equals (minus) the mechanical pressure (6.19).

* **6.10** A body of mass m stands still on a horizontal floor. The coefficients of static and kinetic friction between body and floor are μ_0 and μ . An elastic string with string constant k is attached to the body in a point close to the floor. The string can only exert a force on the body when it is stretched beyond its relaxed length. When the free end of the string is pulled horizontally with constant velocity v , intuition tells us that the body will have a tendency to move in fits and starts.

- (a) Calculate the amount s that the string is stretched, just before the body begins to move.
- (b) Write down the equation of motion for the body when it is just set into motion, for example in terms of the distance x that the point of attachment of the string has moved and the time t elapsed since the motion began.
- (c) Show that the solution to this equation is

$$x = \frac{v}{\omega} (\omega t - \sin \omega t) + (1 - r)s(1 - \cos \omega t)$$

where $\omega = \sqrt{k/m}$, $r = \mu/\mu_0$.

- (d) Assuming that the string stays stretched, calculate at what time $t = t_0$ the body stops again?
- (e) Find the condition for the string to be stretched during the whole motion.
- (f) How long time will the body stay in rest, before moving again?

* **6.11** One may define three *invariants*, i.e. scalar functions, of the stress tensor in any point. The first is the trace $I_1 = \sum_i \sigma_{ii}$, the second $I_2 = \frac{1}{2} \sum_{ij} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji})$ which has no special name, and the determinant $I_3 = \det \boldsymbol{\sigma}$. Show that the characteristic equation for the matrix $\boldsymbol{\sigma}$ can be expressed in terms of the invariants. Can you find an invariant for an asymmetric stress tensor which vanishes if symmetry is imposed?

6.12 A space elevator (fictionalized by Arthur C. Clarke in *Fountains of Paradise* (1978)) can be created if it becomes technically feasible to lower a line down to Earth from a geostationary satellite at height h . Assume that the line is unstretchable and has constant cross section A and constant density ρ . Calculate the maximal tension $\sigma = -\mathcal{F}/A$ (force per unit of area) in the line. Determine the numerical value of the ratio of tension to density, σ/ρ , and compare with the tensile strength (breaking tension) of a known material.