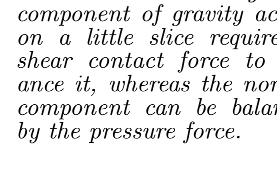
7

Hydrostatic shapes

It is primarily the interplay between gravity and contact forces that shapes the macroscopic world around us. The seas, the air, planets and stars all owe their shape to gravity, and even our own bodies bear witness to the strength of gravity at the surface of our massive planet. What physics principles determine the shape of the surface of the sea? The sea is obviously horizontal at short distances, but bends below the horizon at larger distances following the planet's curvature. The Earth as a whole is spherical and so is the sea, but that is only the first approximation. The Moon's gravity tugs at the water in the seas and raises tides, and even the massive Earth itself is flattened by the centrifugal forces of its own rotation.

Disregarding surface tension, the simple answer is that in hydrostatic equilibrium with gravity, an interface between two fluids of different densities, for example the sea and the atmosphere, must coincide with a surface of constant potential, an equipotential surface. Otherwise, if an interface crosses an equipotential surface, there will arise a tangential component of gravity which can only be balanced by shear contact forces that a fluid at rest is unable to supply. An iceberg rising out of the sea does not obey this principle because it is solid, not fluid. But if you try to build a little local "waterberg", it quickly subsides back into the sea again, conforming to an equipotential surface.



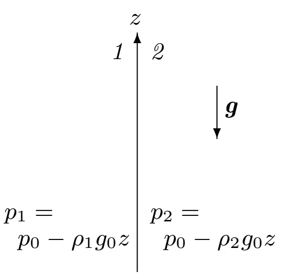
A triangular "waterberg" in the sea. The tangential component of gravity acting on a little slice requires a shear contact force to balance it, whereas the normal component can be balanced by the pressure force.

Hydrostatic balance in a gravitational field also implies that surfaces of constant pressure, isobars, must coincide with the equipotential surfaces. When surface tension plays a role, as it does for a drop of water hanging at the tip of an icicle, this is still true, but the shape bears little relation to equipotential surfaces, because surface tension creates a finite jump in pressure at the interface. Likewise for fluids in motion. Waves in the sea are "waterbergs" that normally move along the surface, but under special circumstances they are able stay in one place, as for example in a river flowing past a big stone.

7.1 Fluid interfaces in hydrostatic equilibrium

The intuitive argument about the impossibility of hydrostatic "waterbergs" must in fact follow from the equations of hydrostatic equilibrium. We shall now show that hydrostatic equilibrium implies that the interface between two fluids with different densities ρ_1 and ρ_2 must be an equipotential surface.

Since the gravitational field is the same on both sides of the interface, hydrostatic balance $\boldsymbol{\nabla} p = \rho \boldsymbol{g}$ implies that there is a jump in the pressure gradient across the interface, because on one side $(\boldsymbol{\nabla} p)_1 = \rho_1 \boldsymbol{g}$ and on the other $(\boldsymbol{\nabla} p)_2 = \rho_2 \boldsymbol{g}$. If the field of gravity has a component tangential to the interface, there will consequently be a jump in the tangential pressure gradient. The difference in tangential gradients on the two sides of the interface implies that even if the pressures are equal in one point, they must be different a little distance away along the surface. Newton's third law, however, requires pressure to be continuous everywhere, also across an interface (as long as there is no surface tension), so this problem can only be avoided if the tangential component of gravity vanishes everywhere at the interface, implying that it is an equipotential surface.

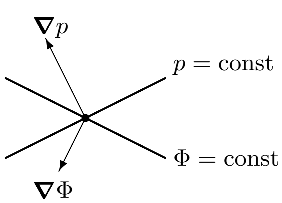


An impossible vertical interface between two fluids at rest with different densities. Even if the hydrostatic pressures on the two sides are the same for $z = 0$ they will be different everywhere else.

If, on the other hand, the fluid densities are the exactly same on both sides of the interface but the fluids themselves are different, the interface is not forced to follow an equipotential surface. This is, however, an unusual and highly unstable situation. The smallest fluctuation in density will call gravity in to make the interface horizontal. Stable vertical interfaces between fluids are simply not seen.

Isobars and equipotential surfaces

Surfaces of constant pressure, satisfying $p(\boldsymbol{x}) = p_0$, are called *isobars*. Through every point of space runs one and only one isobar, namely the one corresponding to the pressure in that point. The gradient of the pressure is everywhere normal to the local isobar. Gravity, $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, is likewise everywhere normal to the local equipotential surface, defined by $\Phi(\boldsymbol{x}) = \Phi_0$. Local hydrostatic equilibrium, $\boldsymbol{\nabla} p = \rho \boldsymbol{g} = -\rho \boldsymbol{\nabla}\Phi$, tells us that the normal to the isobar is everywhere parallel with the normal to the equipotential surface. This can only be the case if *isobars coincide with equipotential surfaces in hydrostatic equilibrium.* For if an isobar crossed an equipotential surface anywhere at a finite angle the two normals could not be parallel.



If isobars and equipotential surfaces cross, hydrostatic balance $\boldsymbol{\nabla} p + \rho\boldsymbol{\nabla}\Phi = \boldsymbol{0}$ becomes impossible.

Since the "curl" of a gradient trivially vanishes, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$, it follows from hydrostatic equilibrium that

$$\boldsymbol{0} = \boldsymbol{\nabla} \times (\rho \boldsymbol{g}) = \boldsymbol{\nabla}\rho \times \boldsymbol{g} + \rho \boldsymbol{\nabla} \times \boldsymbol{g} = -\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}\Phi \ . \tag{7-1}$$

This implies that $\boldsymbol{\nabla}\rho \sim \boldsymbol{\nabla}\Phi$ so that the surfaces of constant density must also coincide with the equipotential surfaces in hydrostatic equilibrium.

The pressure function

The relation between pressure and gravitational potential in hydrostatic equilibrium may be often be expressed in an even simpler way. For an incompressible fluid with constant density, $\rho(\boldsymbol{x}) = \rho_0$, it follows that the quantity

$$\boxed{H = \frac{p}{\rho_0} + \Phi} , \tag{7-2}$$

takes the same constant value everywhere in the fluid. Calculating the gradient of both sides, we find $\boldsymbol{\nabla} H = \boldsymbol{\nabla} p/\rho_0 - \boldsymbol{g} = \boldsymbol{0}$, which vanishes because of hydrostatic equilibrium $\boldsymbol{\nabla} p = \rho_0 \boldsymbol{g}$. This shows that p and Φ are two sides of the same coin.

Even for compressible fluids we may integrate the equation of hydrostatic equilibrium, provided the material obeys a barotropic equation of state of the form $\rho = \rho(p)$. Defining the so-called *pressure function*,

$$w(p) = \int \frac{dp}{\rho(p)} , \tag{7-3}$$

its gradient becomes

$$\boldsymbol{\nabla} w(p) = \boldsymbol{\nabla} p \, \frac{dw(p)}{dp} = \frac{\boldsymbol{\nabla} p}{\rho(p)} .$$

Since $\boldsymbol{\nabla} p + \rho \boldsymbol{\nabla} \Phi = \boldsymbol{0}$ in hydrostatic equilibrium, it follows that

$$\boxed{H = w(p) + \Phi} \tag{7-4}$$

satisfies $\boldsymbol{\nabla} H = 0$ and is thus a constant everywhere in the fluid. Again we conclude fromt he constancy of H that isobaric and equipotential surfaces must coincide.

For an isothermal ideal gas with constant absolute temperature T_0 we obtain from the ideal gas law (4-23),

$$w = \int \frac{RT_0}{M_{\mathsf{mol}}} \frac{dp}{p} = \frac{RT_0}{M_{\mathsf{mol}}} \log p + \text{const} . \tag{7-5}$$

Similarly we find for a ideal gas under isentropic conditions with $p = A\rho^\gamma$,

$$w = \int \frac{A\gamma\rho^{\gamma-1}\, d\rho}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} . \tag{7-6}$$

Using the ideal gas law this becomes,

$$w = \frac{\gamma}{\gamma - 1} \frac{RT}{M_{\mathsf{mol}}} = c_p T \tag{7-7}$$

where c_p is the specific heat at constant pressure (4-51). This shows that in a homentropic gas, the isotherms will also coincide with the equipotential surfaces.

* The Münchausen effect

In the discussion of Archimedes principle in section 5.1 we assumed that a submerged or floating body did not change the physical conditions in the fluid around it. The question of what happens to buoyancy when the gravitational field of a body itself is taken into account, can now be answered. One might, for example, think of an Earth-sized planet floating in the atmosphere of Jupiter, and surrounding itself with a shroud of compressed atmosphere due to its own gravity.

Before the body is immersed in the barotropic fluid we assume for simplicity that the fluid is in hydrostatic equilibrium with a constant external gravitational field of strength g_0 and a pressure $p = p(z)$ satisfying

$$w(p_0) = w(p) + g_0 z \ , \tag{7-8}$$

where p_0 is the pressure at $z = 0$. The presence of the body with its own gravitational potential $\Delta\Phi(\boldsymbol{x})$ will generate a pressure change, $\Delta p = \Delta p(\boldsymbol{x})$, satisfying the equation,

$$w(p_0) = w(p + \Delta p) + g_0 z + \Delta\Phi(\boldsymbol{x}) \ . \tag{7-9}$$

The left hand side is the same as before the body arrived because far away from the body where $\Delta\Phi = 0$, the hydrostatic equilibrium is not disturbed, so that also $\Delta p = 0$. Provided the pressure change Δp is small relative to p and using that $dw/dp = 1/\rho$, we may expand to first order in Δp,

$$w(p_0) = w(p) + \frac{\Delta p}{\rho} + g_0 z + \Delta\Phi(\boldsymbol{x}) \ . \tag{7-10}$$

Freiherr Karl Friedrich Hieronymus von Münchhausen (1720-1797). *German (Hanoveran) soldier, hunter, nobleman, and delightful story-teller. The stories of his travels to Russia were retold and further embroidered by others and published as "The Adventures of Baron Munchausen" in 1793. In one of these, he lifts himself (and his horse) out of deep snow by his bootstraps. Incidentally, this story is also the origin of the expression "bootstrapping", or more recently just "booting", a computer.*

Making use of (7-8) we find,

$$\boxed{\Delta p(\boldsymbol{x}) = -\rho(z)\Delta\Phi(\boldsymbol{x}) \ ,} \tag{7-11}$$

where $\rho(z)$ is the density of the fluid in the absence of the body. Since $\Delta\Phi < 0$, the pressure change is positive everywhere in the vicinity of the body. This is also expected for physical reasons, because the body's gravity pulls the surrounding fluid in and compresses it. The net change in buoyancy may, however, be of both signs, depending on the body's internal mass distribution (which determines $\Delta\Phi$).

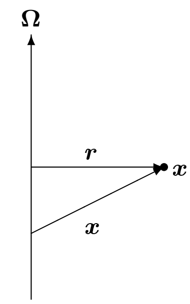
If the body is a uniform sphere with radius a, the potential is constant at the surface. Its value is $\Delta\Phi = -g_1 a$, where g_1 is the surface gravity. But then $\Delta p = \rho(z) g_1 a$, and since $\rho(z)$ is larger for negative z than for positive because the fluid is compressed by the ambient gravity, g_0, the pressure increase is larger below the sphere than above it. The net effect is therefore an increase in buoyancy. The body in a sense "lifts itself by its bootstraps" by means of its own gravity, and it is quite appropriate to call this phenomenon the Münchausen effect.

7.2 Shape of rotating fluids

Newton's second law of motion is only valid in *inertial* coordinate systems, where free particles move on straight lines with constant velocity. In rotating, or otherwise accelerated, non-inertial coordinate systems, one may formally write the equation of motion in their usual form, but the price to be paid is the inclusion of certain force-like terms that do not have any obvious connection with material bodies, but derive from the overall motion of the coordinate system (see chapter 20 for a more detailed analysis). Such terms are called *fictitious forces*, although they are by no means pure fiction, as one becomes painfully aware when standing up in a bus that suddenly stops. A more reasonable name might be *inertial* forces, since they arise as a consequence of the inertia of material bodies.

Antigravity of rotation

A material particle at rest in a coordinate system rotating with constant angular velocity Ω in relation to an inertial system will experience only one fictitious force, the *centrifugal force*. We all know it from carroussels. It is directed perpendicularly outwards from the axis of rotation and of magnitude $dm\, r\, \Omega^2$, where r is the shortest distance to the axis.



The geometry of a rotating system is characterized by a rotation vector $\boldsymbol{\Omega}$ directed along the axis of rotation with magnitude equal to the angular velocity. The vector $\boldsymbol{r}$ is directed orthogonally out from the axis to a point $\boldsymbol{x}$.

In a rotating coordinate system placed with its origin on the rotation axis, and z-axis coincident with it, the shortest vector to a point $\boldsymbol{x} = (x, y, z)$ is $\boldsymbol{r} = (x, y, 0)$. The centrifugal force is proportional to the mass of the particle and thus mimics a gravitational field

$$\boldsymbol{g}_{\text{centrifugal}}(\boldsymbol{r}) = \boldsymbol{r}\Omega^2 = (x, y, 0)\Omega^2 \ . \tag{7-12}$$

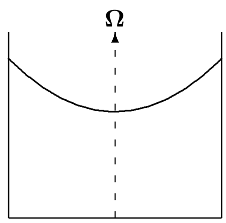
This fictitious gravitational field may be derived from a (fictitious) potential

$$\Phi_{\text{centrifugal}}(\boldsymbol{r}) = -\frac{1}{2}r^2\Omega^2 = -\frac{1}{2}\Omega^2(x^2 + y^2) \ . \tag{7-13}$$

Since the centrifugal field is directed away from the axis of rotation the centrifugal field is a kind of *antigravity* field, which will try to split things apart and lift objects off a rotating planet. The antigravity field of rotation is, however, cylindrical in shape rather than spherical and has consequently the greatest influence at the equator of Earth. If our planet rotated once in a little less than $1\frac{1}{2}$ hours, people at the equator could (and would) actually levitate!

Newton's bucket

A bucket of water on a rotating plate is an example going right back to Newton himself. Internal friction (viscosity) in the water will after some time bring it to rest relative to the bucket and plate, and the whole thing will end up rotating as a solid body. In a rotating coordinate system with z-axis along the axis of rotation, the total gravitational field becomes $\boldsymbol{g} = (\Omega^2 x, \Omega^2 y, -g_0)$, including



The water surface in rotating bucket as a parabolic shape because of centrifugal forces.

both "real" gravity and the "fictitious" centrifugal force. Correspondingly, the total gravitational potential becomes

$$\Phi = -\boldsymbol{g} \cdot \boldsymbol{x} = g_0 z - \frac{1}{2}\Omega^2 (x^2 + y^2) \ . \tag{7-14}$$

The pressure becomes

$$p = p_0 - \rho_0 \Phi = p_0 - \rho_0 g_0 z + \frac{1}{2}\rho_0 \Omega^2 \left(x^2 + y^2\right) \tag{7-15}$$

where p_0 is the pressure at the origin of the coordinate system. It grows towards the rim, reflecting everywhere the change in height of the water column.

The isobars and equipotential surfaces are in this case rotation paraboloids

$$z = z_0 + \frac{\Omega^2}{2g_0}(x^2 + y^2) \ , \tag{7-16}$$

where z_0 is a constant. In a bucket of diameter 20 cm rotating once per second the water stands 2 cm higher at the rim than in the center.

Example 7.2.1: An ultracentrifuge of radius 10 cm contains water and rotates at $\Omega = 60,000$ rpm $\approx 6300\ \mathrm{s}^{-1}$. The centrifugal acceleration becomes $400,000\ g_0$ and the maximal pressure 2 kBar, which is the double of the pressure at the bottom of the deepest abyss in the sea. At such pressures, the change in water density is about 10%.

∗ Stability of rotating bodies

Including the centrifugal field (7-12) in the fundamental field equation (6-5) we may calculate the divergence of the total acceleration field $\boldsymbol{g} = \boldsymbol{g}_{\text{gravity}} + \boldsymbol{g}_{\text{centrifugal}}$,

$$\nabla \cdot \boldsymbol{g} = -4\pi G \rho + 2\Omega^2 \ . \tag{7-17}$$

Effectively, centrifugal forces create a negative mass density $-\Omega^2/2\pi G$. This is, of course, a purely formal result, but it nevertheless confirms the "antigravity" aspect of centrifugal forces, which makes gravity effectively repulsive wherever $\Omega^2/2\pi G\rho > 1$.

For a spherical planet stability against levitation at the equator requires the centrifugal force at the equator to be smaller than surface gravity, which leads to the stronger condition,

$$q = \frac{\Omega^2 a}{g_0} = \frac{3}{2}\frac{\Omega^2}{2\pi G \rho_0} < 1 \ . \tag{7-18}$$

Inserting the parameters of the Earth we find $q \approx 1/291$. At the end of section 7.4 the influence of the deformation caused by rotation is also taken into account, leading to an even stricter stability condition.

7.3 The Earth, the Moon and the tides

Kepler thought that the Moon would influence the waters of Earth and raise tides, but Galilei found this notion of Kepler's completely crazy and compared it to common superstition. After Newton we know that the Moon's gravity acts on everything on Earth, also on the water in the sea, and attempts to pull it out of shape, thereby creating the tides. But since high tides occur roughly at the same time at antipodal points of the Earth, and twice a day, the explanation is not simply that the Moon lifts the sea towards itself but a little more sophisticated.

Johannes Kepler (1580–1635). *German mathematician and astronomer. Discovered that planets move in elliptical orbits and that their motion obeys mathematical laws.*

> Galileo wrote about Kepler: "But among all the great men who have philosophized about this remarkable effect, I am more astonished at Kepler than at any other. Despite his open and acute mind, and though he has at his fingertips the motions attributed to the earth, he has nevertheless lent his ear and his assent to the moon's dominion over the waters, and to occult properties, and to such puerilities." (see [25, p. 145]).

The best natural scientists and mathematicians of the eighteenth and nineteenth centuries worked on the dynamics of the tides, but here we shall only consider the simplest possible case of a quasistatic Moon. For a full discussion, including the dynamics of tidal waves, see for example Sir Horace Lamb's classical book [9] or ref. [24] for a modern account.

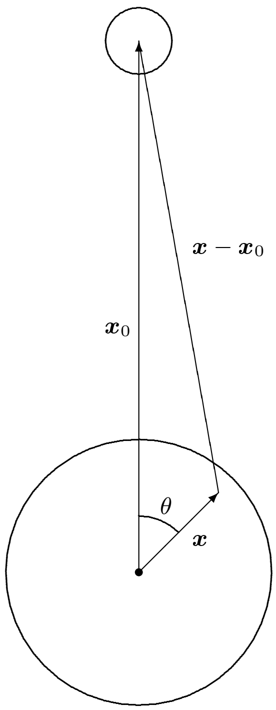
The Earth

We shall limit ourselves to study the Moon's influence on a liquid surface layer of the Earth. The solid parts of the Earth will of course also react to the Moon's field, but the effects are somewhat smaller and are due to elastic deformation rather than flow. This deformation has been indirectly measured to a precision of a few percent in the daily 0.1 ppm variations in the strength of gravity (see fig. 7.1 on page 127). There are also tidal effects in the atmosphere, but they are dominated by other atmospheric motions.

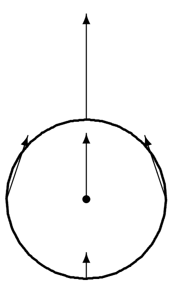
We shall furthermore disregard the changes to the Earth's own gravitational potential due to the shifting waters of the tides themselves, as well as the centrifugal antigravity of Earth's rotation causing it to deviate from a perfect sphere (which increases the tidal range by slightly more than 10 %, see section 7.4). Under all these assumptions the gravitational potential at a height h over the surface of the Earth is to first order in h given by

$$\boxed{\Phi_{\text{Earth}} = g_0 h \ ,} \tag{7-19}$$

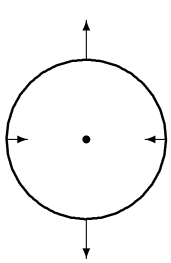
where g_0 is the magnitude of the surface gravity.



Geometry of the Earth and the Moon (not to scale)



How the Moon's gravity varies over the Earth (exaggerated).



The Moon's gravity with common acceleration cancelled. This also explains why the variations in tidal height have a semi-diurnal period.

The Moon

The Moon is not quite spherical, but nevertheless so small and far away that we may approximate its potential across the Earth with that of a point particle $-Gm/|\boldsymbol{x}-\boldsymbol{x}_0|$ situated at the Moon's position $\boldsymbol{x}_0$ with the Moon's mass m. Choosing a coordinate system with the origin at the center of the Earth and the z-axis in the direction of the Moon, we have $\boldsymbol{x}_0 = (0, 0, D)$ where $D = |\boldsymbol{x}_0|$ is the Moon's distance. Since the Moon is approximately 60 Earth radii (a) away, *i.e.* $D \approx 60a$, the Moon's potential across the Earth (for $r = |\boldsymbol{x}| \leq a$) may conveniently be expanded in powers of $\boldsymbol{x}/D$, and we find to second order

$$\begin{aligned}
\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|} &= \frac{1}{\sqrt{x^2+y^2+(z-D)^2}} = \frac{1}{\sqrt{D^2-2zD+r^2}} \\
&= \frac{1}{D}\frac{1}{\sqrt{1-\dfrac{2z}{D}+\dfrac{r^2}{D^2}}} \\
&\approx \frac{1}{D}\left(1-\frac{1}{2}\left(-\frac{2z}{D}+\frac{r^2}{D^2}\right)+\frac{3}{8}\left(-\frac{2z}{D}\right)^2\right) \\
&= \frac{1}{D}\left(1+\frac{z}{D}+\frac{3z^2-r^2}{2D^2}\right) .
\end{aligned}$$

The first term in this expression leads to a constant potential $-Gm/D$, which may of course be ignored. The second term corresponds to a constant gravitational field in the direction towards the moon $g_z = Gm/D^2 \approx 30\ \mu\text{m/s}^2$, which is precisely cancelled by the centrifugal force due to the Earth's motion around the common center-of-mass of the Earth-Moon system (an effect we shall return to below). Spaceship Earth is therefore completely unaware of the two leading terms in the Moon's potential, and these terms cannot raise the tides. Galilei was right to leading non-trivial order, and that's actually not so bad.

Tidal effects come from the *variation* in the gravitational field across the Earth, to leading order given by the third term in the expansion of the potential. Introducing the angle θ between the direction to the Moon and the observation point on Earth, we have $z = r\cos\theta$, and the Moon's potential becomes (after dropping the two first terms)

$$\boxed{\Phi_{\text{Moon}} = -\frac{1}{2}(3\cos^2\theta - 1)\left(\frac{r}{D}\right)^2\frac{Gm}{D}} . \tag{7-20}$$

This expansion may of course be continued indefinitely to higher powers of r/D. The coefficients $P_n(\cos\theta)$ are called Legendre polynomials (here $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$).

The gravitational field of the Moon is found from the gradient of the potential. It is simplest to convert to Cartesian coordinates, writing $(3\cos^2\theta - 1)r^2 =$

$2z^2 - x^2 - y^2$, before calculating the gradient. In the xz-plane we get at the surface $r = a$

$$\boldsymbol{g}_{\text{Moon}} = (-\sin\theta, 0, 2\cos\theta)\frac{aGm}{D^3} \ .$$

Projecting on the local normal $\boldsymbol{e}_r = (\sin\theta, 0, \cos\theta)$ and tangent $\boldsymbol{e}_\theta = (\cos\theta, 0, -\sin\theta)$ to the Earth's surface, we finally obtain the vertical and horizontal components of the gravitational field of the Moon at any point of the Earth's surface

$$g_{\text{Moon}}^{\text{vert}} = \boldsymbol{g}_{\text{Moon}} \cdot \boldsymbol{e}_r = (3\cos^2\theta - 1)\,\frac{aGm}{D^3} \ , \tag{7-21}$$

$$g_{\text{Moon}}^{\text{horiz}} = \boldsymbol{g}_{\text{Moon}} \cdot \boldsymbol{e}_\theta = -\sin 2\theta\,\frac{3aGm}{2D^3} \ . \tag{7-22}$$

The magnitude of the horizontal component is maximal for $\theta = 45^o$ (and of course also 135^o because of symmetry).

Concluding, we repeat that tide-generating forces arise from variations in the Moon's gravity across the Earth. As we have just seen, the force is generally not vertical, but has a horizontal component of the same magnitude. From the sign and shape of the potential as a function of angle, we see that effectively the Moon lowers the gravitational potential just below its position, and at the antipodal point on the opposite side of the Earth, exactly as if there were shallow "valleys" at these places. Sometimes these places are called the Moon and anti-Moon positions.

And the tides

If the Earth did not rotate and the Moon stood still above a particular spot, water would rush in to fill up these "valleys", and the sea would come to equilibrium with its open surface at constant total gravitational potential. The total potential near the surface of the Earth is

$$\Phi = \Phi_{\text{Earth}} + \Phi_{\text{Moon}} = g_0 h - \frac{1}{2}(3\cos^2\theta - 1)\left(\frac{a}{D}\right)^2 \frac{Gm}{D} \ , \tag{7-23}$$

Requiring this potential to be constant we find the tidal height

$$\boxed{h = h_0 + \frac{1}{2}(3\cos^2\theta - 1)\left(\frac{a}{D}\right)^2 \frac{Gm}{g_0 D}} \ , \tag{7-24}$$

where h_0 is a constant. Since the average over the sphere of the second term becomes,

$$\frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} \sin\theta\, d\phi\,(3\cos^2\theta - 1) = \frac{1}{2}\int_{-1}^{+1}(3z^2 - 1)dz = 0 \ ,$$

we conclude that h_0 is the average water depth.

Tidal range

The maximal difference between high and low tides, called the tidal range, occurs between the extreme positions at $\theta = 0$ and $\theta = 90°$,

$$\boxed{H_0 = \frac{3}{2}\left(\frac{a}{D}\right)^2 \frac{Gm}{g_0 D} = \frac{3}{2} a \frac{m}{M}\left(\frac{a}{D}\right)^3}\,, \tag{7-25}$$

where the last equation is obtained using $g_0 = GM/a^2$ with M being the Earth's mass. Inserting the values for the Moon we get $H_0 \approx 54$ cm. Interestingly, the range of the tides due to the Sun turns out to be half as large, about 25 cm. This makes spring tides when the Sun and the Moon cooperate almost three times higher than neap tides when they don't.

For the tides to reach full height, water must move in from huge areas of the Earth as is evident from the shallow shape of the potential. Where this is not possible, for example in lakes and enclosed seas, the tidal range becomes much smaller than in the open oceans. Local geography may also influence tides. In bays and river mouths funnelling can cause tides to build up to huge values. Spring tides in the range of 15 meters have been measured in the Bay of Fundy in Canada.

∗ Quasistatic tidal cycles

The rotation of the Earth cannot be neglected. If the Earth did not rotate, or if the Moon were in a geostationary orbit, it would be much harder to observe the tides, although they would of course be there (problem 7.8). It is, after all, the cyclic variation in the water level observed at the coasts of seas and large lakes, which makes the tides observable. Since the axis of rotation of the Earth is neither aligned with the direction to the Moon nor orthogonal to it, the tidal forces acquire a diurnal cycle superimposed on the 'natural' semidiurnal one (see Fig. 7.1).

For a fixed position on the surface of the Earth, the dominant variation in the lunar zenith angle θ is due to Earth's diurnal rotation with angular rate $\Omega = 2\pi/24$ hours $\approx 7 \times 10^{-5}$ radians per second. On top of that, there are many other sources of periodic variations in the lunar angle [24], which we shall ignore here.

> The dominant such source is the lunar orbital period of a little less than a month. Furthermore, the orbital plane of the Moon inclines about 5° with respect to the ecliptic (the orbital plane of the Earth around the Sun), and precesses with this inclination around the ecliptic in a little less than 19 years. The Earth's equator is itself inclined about 23° to the ecliptic, and precesses around it in about 25,000 years. Due to lunar orbit precession, the angle between the equatorial plane of the Earth and the plane of the lunar orbit will range over $23 \pm 5°$, *i.e.* between 18° and 28°, in about 9 years.

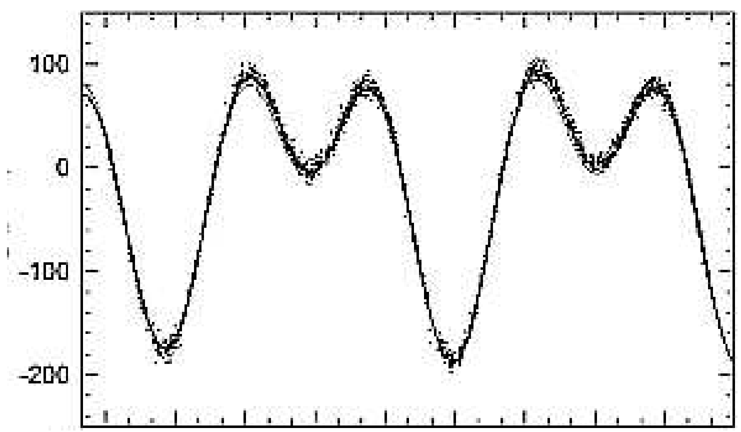
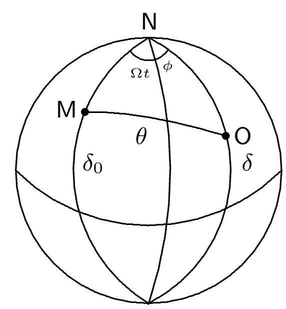


Figure 7.1: *Variation in the gravitational acceleration over a period of 56 hours in units of* $10^{-9} g_0$ *(measured in Stanford, California, on Dec. 8–9, 1996 [19], reproduced here with the permission of the authors (to be obtained)). The semidiurnal as well as diurnal tidal variations are prominently visible as dips in the curves. Modelling the Earth as a solid elastic object and taking into account the effects of ocean loading, the measured data is reproduced to within a few times* $10^{-9} g_0$.



Spherical triangle formed by Moon (M), observer (O) and north pole (N).

Let the fixed observer position at the surface of the Earth have (easterly) longitude ϕ and (northerly) latitude δ. The lunar angle θ is then calculated from the spherical triangle NMO formed by the north pole, the lunar position and the observer's position,

$$\cos\theta = \sin\delta \sin\delta_0 + \cos\delta \cos\delta_0 \cos(\Omega t + \phi) \ . \tag{7-26}$$

Here δ_0 is the latitude of the lunar position and the origin of time has been chosen such that the Moon at $t = 0$ is directly above the meridian $\phi = 0$.

Inserting this into the static expression for the tidal height (7-24), we obtain the quasistatic height variation with time at the observer's place, which becomes the sum of a diurnal and a semidiurnal cycle

$$\boxed{h = \langle h\rangle + h_1 \cos(\Omega t + \phi) + h_2 \cos 2(\Omega t + \phi)} \ . \tag{7-27}$$

Here $\langle h\rangle$ is the time-averaged height, and $h_1 = \frac{1}{2} H_0 \sin 2\delta \sin 2\delta_0$ and $h_2 = \frac{1}{2} H_0 \cos^2\delta \cos^2\delta_0$ are the diurnal and semidiurnal tidal amplitudes. The full tidal range is not quite $2h_1 + 2h_2$, because the two cosines cannot simultaneously take the value -1 (see problem 7.9).

To go beyond the quasistatic approximation, the full theory of fluid dynamics on a rotating planet becomes necessary. The tides will then be controlled not only by the tide-generating forces, but also by the interplay between the inertia of the moving water and friction forces opposing the motion. High tides will no more be tied to the Moon's instantaneous position, but may both be delayed and advanced relative to it.

∗ Influence of the Earth-Moon orbital motion

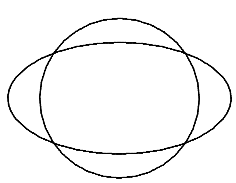
A question is sometimes raised concerning the role of centrifugal forces from the Earth's motion around the center-of-mass of the Earth-Moon system. This point lies a distance $d = Dm/(m + M)$ from the center of the Earth, which is actually about 1700 km below the surface, and during a lunar cycle the center of the Earth and the center of the Moon move in circular orbits around it. Were the Earth (like the Moon) in bound rotation so that it always turned the same side towards the Moon, one would in the corotating coordinate system, where the Moon and the Earth have fixed positions, have to add a centrifugal potential to the previously calculated potential (7-23), and the tidal range would become up to 27 times larger (problem 7.10), *i.e.* about 15 m!

Luckily, this is not the case. The Earth's own rotation is fixed with respect to the inertial system of the fixed stars (disregarding the precession of its rotation axis). A truly non-rotating Earth would, in the corotating system, rotate backwards in synchrony with the lunar cycle, cancelling the centrifugal potential. Seen from the inertial system, the circular orbital motion imparts the same centripetal acceleration $\Omega^2 d$ (along the Earth-Moon line) to all parts of the Earth. This centripetal acceleration must equal the constant gravitational attraction, Gm/D^2, coming from the linear term in the Moon's potential, and equating the two, one obtains $\Omega^2 = Gm/D^2 d = GM/D^3$, which is the usual (Kepler) equation relating the Moon's period of revolution to its mass and distance.

The Moon always turns the same side towards Earth and the bound rotation adds in fact a centrifugal component on top of the tidal field from Earth. Over time these effects have together deformed the Moon into its present egg-like shape.

∗ 7.4 Shape of a rotating fluid planet

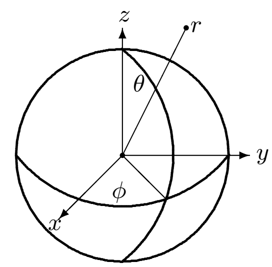
On a rotating planet, centrifugal forces will add a component of "antigravity" to the gravitational acceleration field, making the road from the pole to equator slightly downhill. At Earth's equator the centrifugal acceleration amounts to only $q \approx 1/291$ of the surface gravity, so a first guess would be that there is a centrifugal "valley" at the equator with a depth of 1/291 of the Earths radius, which is about 22 km. If such a difference suddenly came to exist on a spherical Earth, all the water would like huge tides run towards the equator. Since there *is* land at equator, we may conclude that even the massive Earth must over time have flowed into the centrifugal valley. The difference between the equatorial and polar radii is in fact 21.4 km [2], and coincidentally, this is roughly the same as the difference between the highest mountain top and the deepest ocean trench on Earth.



Exaggerated sketch of the change in shape of the Earth due to rotation.

The flattening of the Earth due to rotation has like the tides been a problem attracting the best minds of the past centuries [9]. We shall here consider the simplest possible model, which nevertheless captures all the relevant features for slowly rotating planets.

Rigid spherical planet



Polar and azimuthal angles.

If a spherical planet rotates like a stiff body, the gravitational potential above the surface will be composed of the gravitational potential of planet and the centrifugal potential. In spherical coordinates we have,

$$\Phi_0 = -g_0 \frac{a^2}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta \ . \tag{7-28}$$

from which we get the vertical and horizontal components of surface gravity,

$$g_r = -\left.\frac{\partial \Phi_0}{\partial r}\right|_{r=a} = -g_0(1 - q\sin^2\theta) \ , \tag{7-29a}$$

$$g_\theta = -\frac{1}{r}\left.\frac{\partial \Phi_0}{\partial \theta}\right|_{r=a} = g_0\, q \sin\theta\cos\theta \ , \tag{7-29b}$$

where $q = \Omega^2 a/g_0$ is the "levitation parameter" defined in (7-18). This confirms that the magnitude of vertical gravity is reduced, and that horizontal gravity points towards the equator. For Earth the changes are all of relative magnitude $q \approx 1/291$.

Fluid planet

Suppose now the planet is made from a heavy fluid which given time will adapt its shape to an equipotential surface of the form,

$$r = a + h(\theta) \ , \tag{7-30}$$

with a small radial displacement, $|h|(\theta) \ll a$. Assuming that the displaced material is incompressible we must require,

$$\int_0^\pi h(\theta)\sin\theta\, d\theta = 0 \ . \tag{7-31}$$

If we naively disregard the extra gravitational field created by the displacement of material, the potential is given by (7-28). On the displaced surface this becomes to first order in the small quantities h and q,

$$\Phi_0 \approx g_0 h - g_0 a(1 + \frac{q}{2}\sin^2\theta) \ . \tag{7-32}$$

Demanding that it be constant, it follows that

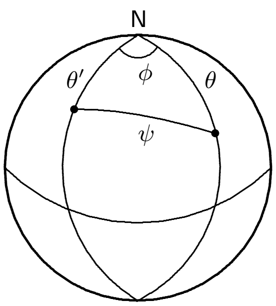
$$h = h_0\left(\sin^2\theta - \frac{2}{3}\right) , \tag{7-33}$$

with $h_0 = \frac{1}{2}aq$. The $-2/3$ in the parenthesis has been chosen such that (7-31) is fulfilled. For Earth we find $h_0 = 11$ km, which is only half the expected result.

Including the self-potential

The preceding result shows that the gravitational potential of the shifted material must play an important role. Assuming that the shifted material has constant density ρ_1, the extra gravitational potential due to the shifted material is calculated from (3-24) by integrating over the (signed) volume ΔV occupied by the shifted material,

$$\Phi_1 = -G\rho_1 \int_{\Delta V} \frac{dV'}{|\boldsymbol{x} - \boldsymbol{x}'|} \ . \tag{7-34}$$



Spherical triangle formed by the various angles.

Since the shifted material is a thin layer of thickness h, the volume element is $dV' \approx h(\theta')dS'$ where dS' is the surface element of the original sphere , $|\boldsymbol{x}'| = a$. There are of course corrections but they will be of higher order in h. The square of the denominator may be written as $|\boldsymbol{x} - \boldsymbol{x}'|^2 = r^2 + a^2 - 2ra\cos\psi$ where ψ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}'$. Consequently we have to linear order in h

$$\Phi_1 = -G\rho_1 \oint_S \frac{h(\theta')}{\sqrt{r^2 + a^2 - 2ra\cos\psi}} dS' \ , \tag{7-35}$$

where $\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi$ and $dS' = a^2\sin\theta' d\theta' d\phi'$. Notice that this is exactly the same potential as would have been obtained from a surface distribution of mass with surface density $\rho_1 h(\theta)$.

There are various ways to do this integral. We shall use a wonderful theorem about Legendre polynomials, which says that a mass distribution with an angular dependence given by a Legendre polynomial, creates a potential with exactly the same angular dependence. So if we assume that the surface shape is of the form (7-33) with angular dependence proportional to $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{2}(2 - 3\sin^2\theta)$, the effective surface mass distribution will be proportional to $P_2(\cos\theta)$, implying that the self-potential will be of precisely the same shape (see problem 7.11),

$$\Phi_1(r,\theta) = F(r)\left(\sin^2\theta - \frac{2}{3}\right) \ . \tag{7-36}$$

The radial function may now be determined from the integral (7-36) by taking $\theta = 0$. Since now $\psi = \theta'$, all the difficult integrals disappear and we obtain

$$\begin{aligned} F(r) = -\frac{3}{2}\Phi_1(r,0) &= \frac{3}{2}G\rho_1 h_0 a^2 2\pi \int_0^\pi \frac{\sin\theta'\left(\sin^2\theta' - \frac{2}{3}\right)}{\sqrt{r^2 + a^2 - 2ra\cos\theta'}} d\theta' \\ &= \frac{3}{2}G\rho_1 a^2 h_0 2\pi \int_{-1}^{1} \frac{\frac{1}{3} - u^2}{\sqrt{r^2 + a^2 - 2rau}} du \ . \end{aligned}$$

The integral is now standard, and we find

$$F(r) = -\frac{4\pi}{5}\rho_1 G a h_0 \frac{a^3}{r^3} = -\frac{3}{5} g_0 h_0 \frac{\rho_1}{\rho_0}\frac{a^3}{r^3} \ . \tag{7-37}$$

In the last step we have used that $g_0 = GM_0/a^2 = \frac{4}{3}\pi G a \rho_0$ where ρ_0 is the average density of the planet.

Total potential and strength of gravity

The total potential now becomes

$$\Phi = \Phi_0 + \Phi_1 = -g_0 \frac{a^2}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta - \frac{3}{5} g_0 h(\theta) \frac{\rho_1}{\rho_0}\frac{a^3}{r^3} \tag{7-38}$$

Inserting $r = a + h$ and expanding to lowest order in h and q, we finally obtain,

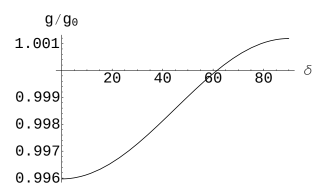
$$\boxed{h_0 = \frac{\frac{1}{2}qa}{1 - \frac{3}{5}\frac{\rho_1}{\rho_0}} \ .} \tag{7-39}$$

For Earth, the average density of the mantle material is $\rho_1 \approx 4.5$ g/cm^3 whereas the average density is $\rho_0 \approx 5.5$ g/cm^3. With these densities one gets $h_0 = 21.5$ km in close agreement with the quoted value [2]. In the same vein, we may also calculate the influence of the self-potential on the tidal range. Since the density of water is $\rho_1 \approx 1.0$ g/cm^3, the tidal range (7-25) is increased by a factor 1.12.

From the total potential we calculate gravity at the displaced surface,

$$g_r = -\left.\frac{\partial\Phi}{\partial r}\right|_{r=a+h} = -g_0\left(1 - q\sin^2\theta - 2\frac{h(\theta)}{a} + \frac{9}{5}\frac{\rho_1}{\rho_0}\frac{h(\theta)}{a}\right) , \tag{7-40a}$$

$$g_\theta = -\frac{1}{r}\left.\frac{\partial\Phi}{\partial\theta}\right|_{r=a+h} = g_0\left(q + \frac{6}{5}\frac{h_0}{a}\frac{\rho_1}{\rho_2}\right)\sin\theta\cos\theta \ . \tag{7-40b}$$



The variation of g/g_0 with latitude δ.

Finally, projecting on the local normal and tangent we find to first order in h and q,

$$g_\perp \approx g_r \ , \qquad g_\parallel = g_\theta + g_r \frac{1}{a}\frac{dh(\theta)}{d\theta} \approx 0 \ . \tag{7-41a}$$

The field of gravity is orthogonal to the equipotential surface, as it should be.

Notice that the three correction terms to the vertical field are due to the centrifugal force, to the change in gravity from the change in height, and to the displacement of material. All three contributions are of the same order of magnitude, q, because they all ultimately derive from the centrifugal force.

Example 7.4.1 (Olympic games): The dependence of gravity on polar angle (or latitude) given in (7-40a) has practical consequences. In 1968 the Olympic games were held in Mexico City at latitude $\delta = 19^\circ$ north whereas in 1980 they were held in Moscow at latitude $\delta' = 55^\circ$ north. To compare record heights in jumps (or throws), it is necessary to correct for the variation in gravity due to the centrifugal force, the geographical difference in height, and air resistance. Assuming that the initial velocity is the same, the height h attained in Mexico city would correspond to a height h' in Moscow , related to h by $v^2 = 2gh = 2g'h'$. Using (7-40a) we find $h/h' = g'/g = 1.00296$. This shows that a correction of -0.3% due to variation in gravity (among other corrections) would have to be applied to the Mexico City heights before they were compared with the Moscow heights.

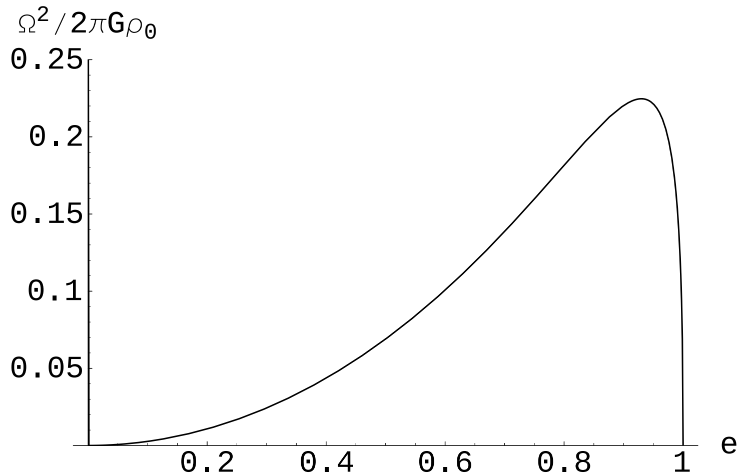


Figure 7.2: *The MacLaurin function (right hand side of eq. (7-42)). The maximum 0.225 is reached for* $e = 0.93$.

Fast rotating planet

One of the main assumptions behind the calculations in this section was that the planet should be slowly rotating, meaning that the deformation of the planet due to rotation be small, or $|h(\theta)| \ll a$. Intuitively, it is fairly obvious, that if the rate of rotation of the planet is increased, the flattening increases until it reaches a point, where the "antigravity" of rotation overcomes the "true" gravity of planetary matter as well as cohesive forces. Then the planet becomes unstable with dramatic change of shape or even breakup as a consequence.

Colin MacLaurin (1698–1746). *Scottish mathematician who developed and extended Newton's work on calculus and gravitation.*

The study of the possible forms of rotating planets was initiated very early by Newton and in particular by MacLaurin. It was found that oblate ellipsoids of rotation are possible allowed shapes for rotating planets with constant matter density, ρ_0. An oblate ellipsoid of rotation is characterized by equal-size major axes, $a = b$ and a smaller minor axis $c < a$, about which it rotates.

MacLaurin found that the angular rotation rate is related to the eccentricity $e = \sqrt{1 - c^2/a^2}$ through the formula

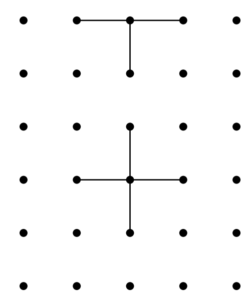
$$\frac{\Omega^2}{2\pi G\rho_0} = \frac{1}{e^3}\left(\sqrt{1-e^2}\left(3-2e^2\right)\arcsin e - 3e\left(1-e^2\right)\right) . \tag{7-42}$$

The right hand side is shown in Fig. 7.2 and has a maximum 0.225 for $e = 0.93$, implying that stability can only be maintained for $\Omega^2/2\pi G\rho_0 < 0.225$. Actually, various other shape instabilities set in at even lower values of the eccentricity (see ref. [21] for a thorough discussion of these instabilities and their astrophysical consequences). For small e, the MacLaurin(!) expansion of the right hand side of (7-42) becomes $4e^2/15$. Since $e^2 \approx 2h_0/a$, we obtain $h_0 = 15\Omega^2 a/16\pi G\rho_0 = 5\Omega^2 a^2/4g_0$, in complete agreement with (7-39) for $\rho_1 = \rho_0$.

7.5 Surface tension

At the interface between two materials physical properties change rapidly over distances comparable to the molecular separation scale L_{mol} given in (1-1). From a macroscopic point of view, the transition layer is an infinitely thin sheet coinciding with the interface between the two materials. Although the transition layer in the continuum limit thus appears to be a mathematical surface, it may nevertheless possess macroscopic physical properties, such as energy.

Molecular estimate of surface energy density



Two-dimensional cross section of a primitive three-dimensional model of a material interfacing to vacuum. A molecule at the surface has only five bonds compared to the six that tie a molecule in the interior.

The apparent paradox that a mathematical surface with no volume can possess energy may be resolved by considering a primitive three-dimensional model of a material in which the molecules are placed in a cubic grid with grid length L_{mol}. Each molecule in the interior has six bonds to its neighbors with a total binding energy of $-\epsilon$, but a surface molecule will only have five bonds when the material is interfacing to vacuum. The (negative) binding energy of the missing bond is equivalent to an extra positive energy $\epsilon/6$ for a surface molecule relative to an interior molecule, and thus an extra surface energy density,

$$\alpha \approx \frac{1}{6} \frac{\epsilon}{L_{\mathsf{mol}}^2} . \tag{7-43}$$

The binding energy may be estimated from the specific enthalpy of evaporation H of the material as $\epsilon \approx H M_{\mathsf{mol}}/N_A$. Notice that the unit for surface tension is $\mathsf{J/m^2} = \mathsf{kg/s^2}$.

Example 7.5.1: For water the specific evaporation enthalpy is $H \approx 2.2 \times 10^6$ J/kg, leading to the estimate $\alpha \approx 0.12$ J/m^2. The measured value of the surface energy for water/air interface is in fact $\alpha \approx 0.073$ J/m^2 at room temperature. Less than a factor of 2 wrong is not a bad estimate at all!

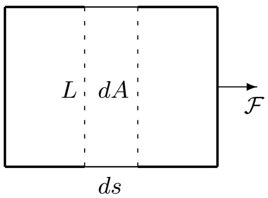
Definition of surface tension

Increasing the area of the interface by a tiny amount dA, takes an amount of work equal to the surface energy contained in the extra piece of surface,

$$\boxed{dW = \alpha\, dA\ .} \tag{7-44}$$

This is quite analogous to the mechanical work $dW = -p\, dV$ performed against pressure when the volume of the system is expanded by dV. But where a volume expansion under positive pressure takes negative work, increasing the surface area takes positive work. This resistance against extension of the surface shows that the interface has a permanent internal tension, called *surface tension*[1] which we shall now see equals the energy density α.

[1] There is no universally agreed-upon symbol for surface tension which is variously denoted α, γ, σ, S, Υ and even T. We shall use α, even if it collides with other uses, for example the thermal expansion coefficient.



An external force $\mathcal{F}$ performs the work $dW = \mathcal{F}\,ds$ to stretch the surface by ds. Since the area increase is $dA = Lds$, the force is $\mathcal{F} = \alpha\,L$. The force per unit of length, $\alpha = \mathcal{F}/L$, is the surface tension.

	α[mN/m]
Water	*72*
Methanol	*22*
Ethanol	*22*
Bromine	*41*
Mercury	*485*

Surface tension of some liquids against air at 1 atm *and* 25°C *in units of millinewton per meter (from [3]).*

Formally, surface tension is defined as the force per unit of length that acts orthogonally to an imaginary line drawn on the interface. Suppose we wish to stretch the interface along a straight line of length L by a uniform amount ds. Since the area is increased by $dA = Lds$, it takes the work $dW = \alpha Lds$, implying that the force acting orthogonally to the line is $\mathcal{F} = \alpha L$, or $\mathcal{F}/L = \alpha$. Surface tension is thus identical to the surface energy density. This is also reflected in the equality of the natural units for the two quantities, $\mathsf{N/m} = \mathsf{J/m^2}$.

Since the interface has no macroscopic thickness, it may be viewed as being locally flat everywhere, implying that the energy density cannot depend on the macroscopic curvature, but only on the microscopic properties of the interface. If the interfacing fluids are homogeneous and isotropic — as they normally are — the value of the surface energy density will be the same everywhere on the surface, although it may vary with the local temperature. Surface tension depends on the physical properties of both of the interfacing materials, which is quite different from other material constants that usually depend only on the physical properties of just one material.

Fluid interfaces in equilibrium are usually quite smooth, implying that α must always be positive. For if α were negative, the system could produce an infinite amount of work by increasing the interface area without limit. The interface would fold up like crumbled paper and mix the two fluids thoroughly, instead of separating them. Formally, one may in fact view the rapid dissolution of ethanol in water as due to negative interfacial surface tension between the two liquids. The general positivity of α guarantees that fluid interfaces seek towards the minimal area consistent with the other forces that may be at play, for example pressure forces and gravity. Small raindrops and champagne bubbles are for this reason nearly spherical. Larger raindrops are also shaped by viscous friction, internal flow, and gravity, giving them a much more complicated shape.

Pressure excess in a sphere

Consider a spherical ball of liquid of radius a, for example hovering weightlessly in a spacecraft. Surface tension will attempt to contract the ball but is stopped by the build-up of an extra pressure Δp inside the liquid. If we increase the radius by an amount da we must perform the work $dW_1 = \alpha\, dA = \alpha d(4\pi a^2) = \alpha 8\pi a\, da$ against surface tension. This work is compensated by the thermodynamic work against the pressure excess $dW_2 = -\Delta p\, dV = -\Delta p\, 4\pi a^2\, da$. In equilibrium there should be nothing to gain, $dW_1 + dW_2 = 0$, leading to,

$$\Delta p = \frac{2\alpha}{a} \ . \tag{7-45}$$

The pressure excess is inversely proportional to the radius of the sphere.

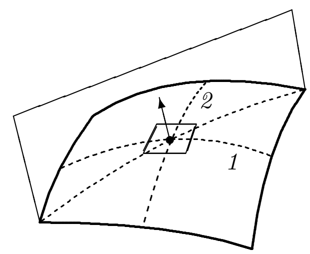
It should be emphasized that the pressure excess is equally valid for a spherical raindrop in air and a spherical air bubble in water. A spherical soap bubble of radius a has two spherical surfaces, one from air to soapy water and one from soapy water to air. Each gives rise to a pressure excess of $2\alpha/a$, such that the total pressure inside a soap bubble is $4\alpha/a$ larger than outside.

Example 7.5.2: A spherical raindrop of diameter 1 mm has an excess pressure of only about 300 Pa, which is tiny compared to atmospheric pressure (10^5 Pa). A spherical air bubble the size of a small bacterium with diameter 1 μm acquires a pressure excess due to surface tension a thousand times larger, about 3 atm.

When can we disregard the influence of gravity on the shape of a raindrop? For a spherical air bubble or raindrop of radius a, the condition must be that the change in hydrostatic pressure across the drop should be negligible compared to the pressure excess due to surface tension, *i.e.* $\rho_0 g_0 2a \ll 2\alpha/a$, or

$$a \ll R_c = \sqrt{\frac{\alpha}{\rho_0 g_0}} \, . \tag{7-46}$$

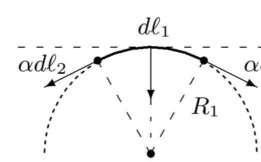
The critical radius R_c is called the *capillary constant* or *capillary radius* and equals 2.7 mm for water and 1.9 mm for mercury. The dimensionless ratio of the pressure excess due to surface tension and the hydrostatic pressure variation due to gravity, $\mathsf{Bo} = (2\alpha/a)/(\rho_0 g_0 2a) = (R_c/a)^2$, is often called the *Bond number*[2].



A plane containing the normal in a point intersects the surface in a planar curve with a signed radius of curvature in the point. The extreme values of the signed radii of curvature define the principal directions. The small rectangle has sides parallel with the principal directions.

Pressure discontinuity due to surface tension

A smooth surface may in a given point be intersected with an infinity of planes containing the normal to the surface. In each normal plane the intersection is a smooth planar curve which at the given point may be approximated by a circle centered on the normal. The center of this circle is called the *center of curvature* and its radius the *radius of curvature* of the intersection. Usually the radius of curvature is given a sign, depending on which side of the surface the center of curvature is situated. As the intersection plane is rotated, the center of curvature moves up and down the normal between extreme values R_1 and R_2 of the signed radius of curvature, called the *principal radii of curvature.* It may be shown [**?**] that the corresponding principal intersection planes are orthogonal, and that the radius of curvature along any other normal intersection may be calculated from the principal radii.



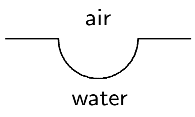
The rectangular piece of the surface of size $d\ell_1 \times d\ell_2$ is exposed to two tension forces along the 1-direction resulting in a normal force pointing towards the center of the circle of curvature. The tension forces in the 2-direction also contribute to the normal force.

Consider now a small rectangle $d\ell_1 \times d\ell_2$ with its sides aligned with the principal directions, and let us to begin with assume that R_1 and R_2 are positive. In the 1-direction surface tension acts with two nearly opposite forces of magnitude $\alpha d\ell_2$, but because of the curvature of the surface there will be a resultant force in the direction of the center of the principal circle of curvature. Each of the tension forces forms an angle $d\ell_1/2R_1$ with the tangent, and projecting both on the normal we obtain the total inwards force $2\alpha d\ell_2 \times d\ell_1/2R_1$. Since the force is proportional to the area $d\ell_1 d\ell_2$ of the rectangle, it represents an excess in pressure $\Delta p = \alpha/R_1$ on the side of the surface containing the center of curvature. Finally, adding the contribution from the 2-direction we obtain the *Young-Laplace* law

[2]Unfortunately it has not been possible for this author to determine the origin of the name.

for the pressure discontinuity due to surface tension,

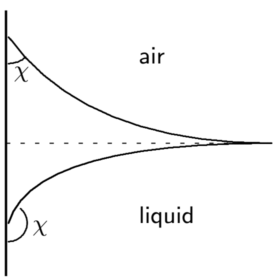
$$\boxed{\Delta p = \alpha \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \ . \tag{7-47}$$



Sketch of the meniscus formed by evaporation of water from the surface of a plant leaf, resulting in a high negative pressure in the water, capable of lifting the sap to great heights.

For the sphere we have $R_1 = R_2 = a$ and recover the preceding result (7-45). The Young-Laplace law may be extended to signed radii of curvature, provided it is remembered that a contribution to the discontinuity is always positive on the side of the surface containing the center of curvature, and otherwise negative.

Example 7.5.3 (How sap rises in plants): Plants evaporate water through tiny pores on the surface of the leaves. This creates a hollow air-to-water surface in the shape of a half-sphere of the same diameter as the pore. Both radii of curvature are negative $R_1 = R_2 = -a$ because the center of curvature lies outside the water, leading to a negative pressure excess in the water. For a pore of diameter $2a \approx 1\ \mu\text{m}$ the excess pressure inside the water will be about $\Delta p \approx -3$ atm, capable of lifting sap through a height of 30 m. In practice, the lifting height is considerably smaller because of resistance in the xylem conduits of the plant through which the sap moves. Taller plants and trees need correspondingly smaller pore sizes to generate sufficient negative pressures, even down to -100 atm! Recent research has confirmed this astonishing picture (see M. T. Tyree, Nature **423**, 923 (2003)).

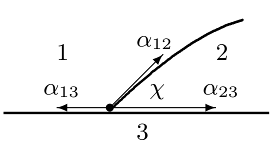


An air/liquid interface meeting a wall. The upper curve makes an acute contact angle, like water, whereas the lower curve makes an obtuse contact angle, like mercury.

Contact angle

An interface between two fluids is a two-dimensional surface which makes contact with a solid wall along a one-dimensional curve. Locally the plane of the fluid interface typically forms a certain *contact angle* χ with the wall. For the typical case of a liquid/air interface, χ is normally defined as the angle inside the liquid. Water and air against glass meet in a small acute contact angle, $\chi \approx 0$, whereas mercury and air meets glass at an obtuse contact angle of $\chi \approx 140°$. Due to its small contact angle, water is very efficient in *wetting* many surfaces, whereas mercury has a tendency form pearls. It should be emphasized that the contact angle is extremely sensitive to surface properties, such as waxing, and to fluid composition and additives.

In the household we regularly use surfactants that are capable of making dish-water wet greasy surfaces which otherwise would create separate droplets. After washing our cars we apply a wax which makes rainwater pearl and prevents it from wetting the surface, thereby diminishing rust and corrosion.



Two fluids meeting at a solid wall in a line orthogonal to the paper. The tangential component of surface tension must vanish.

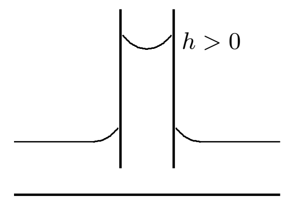
The contact angle is a material constant which depends on the properties of all three materials coming together. Whereas material adhesion can sustain a tension normal to the wall, the tangential tension has to vanish. This yields an equilibrium relation between the three surface tensions,

$$\alpha_{13} = \alpha_{23} + \alpha_{12} \cos\chi \ , \tag{7-48}$$

This relation is, however, not particularly useful because of the sensitivity of χ to surface properties, and it is better to view χ as an independent material constant.

Capillary effect

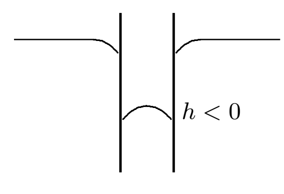
Water has a well-known tendency to rise above the ambient water level in a narrow vertical glass tube which is lowered into the liquid. Closer inspection reveals that the surface inside the tube is concave. This is called the *capillary effect* and is caused by the acute contact angle of water in conjunction with its surface tension which creates a negative pressure just below the liquid surface, balancing the weight of the raised water column. Mercury with its obtuse contact angle displays instead a convex surface shape, creating a positive pressure just below the surface which forces the liquid down to a level where the pressure equals the pressure at the ambient level.



Water rises above the ambient level in a glass tube and displays a concave surface inside the tube. Mercury behaves oppositely and sinks with a convex surface.

Let us first calculate the effect for an acute angle of contact. At the center of the tube the radii of curvature are equal, and since the center of curvature lies outside the liquid, they are also negative, $R_1 = R_2 = -R_0$ where R_0 is positive. Hydrostatic balance at the center of the tube then takes the form, $\rho_0 g_0 h = 2\alpha/R_0$, or

$$h = \frac{2\alpha}{\rho_0 g_0 R_0} = 2\frac{R_c^2}{R_0} \ . \tag{7-49}$$



Mercury sinks below the general level in a capillary glass tube.

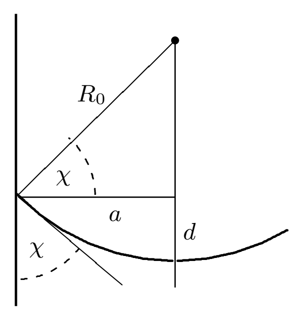
Notice that this is an exact relation which does not depend on the surface being spherical. It also covers the case of an obtuse contact angle by taking R_0 to be negative.

Assuming now that the surface is in fact spherical, which should be the case for $a \lesssim R_c$ where gravity has no effect on the shape, a simple geometric construction shows that $a = R_0 \cos\chi$, and thus,

$$\boxed{h = 2\frac{R_c^2}{a}\cos\chi \ .} \tag{7-50}$$

It is as expected positive for acute and negative for obtuse contact angles. From the same geometry it also follows that the depth of the central point is,

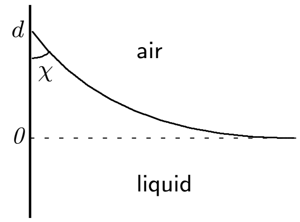
$$\boxed{d = a\frac{1-\sin\chi}{\cos\chi} \ ,} \tag{7-51}$$



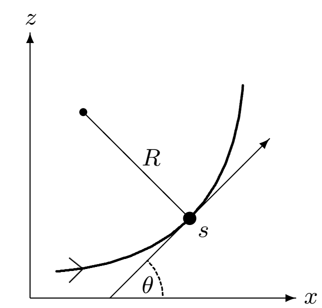
A spherical surface of radius R in a tube of radius a satisfies $a = R_0 \cos\chi$ where χ is the contact angle. The depth is $d = a(1-\sin\chi)/\cos\chi$.

Both of these expressions are modified for larger radius, $a \gtrsim R_c$ where the surface flattens such that $ha \to 0$, and $d/a \to 0$.

Example 7.5.4: In a capillary tube of diameter $2a = 1$ mm For water with $\chi \approx 0$ rises $h = +30$ mm with a surface depth $d = +0.5$ mm . Mercury with contact angle $\chi \approx 140^\circ$ sinks on the other hand to $h = -11$ mm and $d = -0.2$ mm under the same conditions.



The interface at a vertical wall with an acute angle of contact.



The geometry of a planar curve. The curve is parameterized by the arc length s along the curve. A small change in s generates a change in the elevation angle θ determined by the local radius of curvature. Here the radius of curvature is positive.

7.6 Planar capillary effect

In the limit of infinite tube radius there will be no capillary rise although a liquid surface will still rise or sink at a plane vertical wall to accommodate a finite contact angle. This is an exactly solvable case which nicely illustrates the mathematics of planar curved surfaces.

The surface equations

Taking the x-axis orthogonal to the wall, the surface shape is independent of y and described by a simple curve in the xz-plane. The best way to handle the geometry of a planar curve is to use two auxiliary parameters: the arc length s along the interface curve, and the elevation angle θ between the x-axis and the oriented tangent to the curve. From this definition of θ we obtain immediately,

$$\frac{dx}{ds} = \cos\theta , \qquad \frac{dz}{ds} = \sin\theta . \tag{7-52}$$

The radius of curvature may conveniently be defined as,

$$R = \frac{ds}{d\theta} . \tag{7-53}$$

Evidently this *geometric radius of curvature* is positive if s is an increasing function of θ, otherwise negative.

One should be aware that this sign convention may not agree with the physical sign convention for the Young-Laplace law (7-47), and that it may be necessary to introduce an explicit sign to get the physics right. Assuming that the air pressure is constant, $p = p_0$, the pressure in the liquid just below the surface is $p = p_0 + \Delta p$ where Δp is given by the Young-Laplace (7-47) law. For an acute angle of contact we have $R_1 = -R$ and $R_2 = \infty$, because the center of curvature lies outside the liquid. Hydrostatic balance then implies that

$$H = g_0 z - \frac{1}{\rho_0}\frac{\alpha}{R} , \tag{7-54}$$

is a constant along the surface. For $x \to \infty$ the curvature has to vanish, $R \to \infty$, and normalizing such that $z \to 0$ for $x \to \infty$ we find that $H = 0$. Eliminating R using (7-53), we obtain

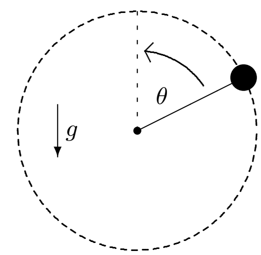
$$\frac{d\theta}{ds} = \frac{z}{R_c^2} , \tag{7-55}$$

where R_c is the capillary radius (7-46). Together with (7-52) we have obtained three equations for x, z, and θ which should be solved with the boundary conditions $x = 0$, $z = d$, and $\theta = \chi - 90°$ for $s = 0$. A quick analysis shows that these equations are equally valid for an obtuse angle of contact.

The pendulum connection

Differentiating the angular derivative (7-55) once more after s and introducing the parameter $\tau = s/R_c$ we obtain,

$$\boxed{\frac{d^2\theta}{d\tau^2} = \sin\theta \ .} \tag{7-56}$$



Inverted mathematical pendulum with angle θ moving towards the unstable equilibrium at $\theta = 0$. This corresponds to the liquid surface falling as one moves away from the plate.

This is the equation of motion for an *inverted mathematical pendulum*. For an acute angle of contact, the boundary conditions correspond to starting the pendulum at a negative angle $\theta = \chi - 90°$ with positive velocity d/R_c^2. The depth d must be adjusted such that the pendulum eventually comes to rest in the unstable equilibrium at $\theta = 0$. If the depth is larger, the pendulum will continue through the unstable equilibrium, corresponding to the liquid surface rising again to meet another vertical wall at a finite distance.

Multiplying the equation of motion with $d\theta/d\tau$ and integrating, we find

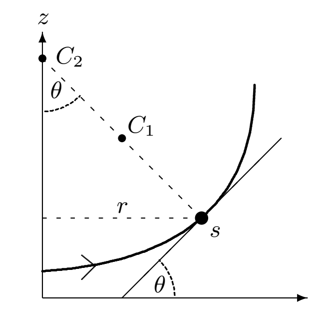
$$\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 = 1 - \cos\theta \ , \tag{7-57}$$

where the constant has been determined by the condition that $d\theta/d\tau = 0$ and $\theta = 0$ for $\tau \to \infty$. This equation may be solved by quadrature (see problem 7.13), but the height d of the rise may in fact be determined without integrating. Using the initial conditions and (7-55) we find $d\theta/d\tau = d/R_c$ and $\theta = \chi - 90°$ for $\tau = 0$, and solving for d we obtain,

$$\boxed{d = \pm R_c\sqrt{2(1 - \sin\chi)} \ ,} \tag{7-58}$$

with positive sign for acute angle of contact. For water with nearly vanishing angle of contact, we find $d \approx \sqrt{2}R_c \approx 3.9$ mm whereas for mercury with $\chi = 140°$ we get $d \approx 1.6$ mm.

7.7 Rotationally invariant shapes



The interface curve is parameterized by the arc length s. A small change in s generates a change in the elevation angle θ determined by the local radius of curvature R_1 with center C_1. The second radius of curvature R_2 lies on the z-axis with center C_2 (see problem 7.14).

Many static interfaces — hanging raindrops and capillary surfaces in circular tubes — are rotationally invariant around the z-axis, allowing us to establish a fairly simple formalism for the shape of the surface, which in cylindrical coordinates is a planar curve in the rz-plane. In terms of the arc length s along the curve and the angle of elevation θ for its slope, we have the geometric relations (see problem 7.14),

$$\frac{dr}{ds} = \cos\theta \ , \qquad \frac{dz}{ds} = \sin\theta \ , \tag{7-59}$$

$$R_1 = \frac{ds}{d\theta} \ , \qquad R_2 = \frac{r}{\sin\theta} \ . \tag{7-60}$$

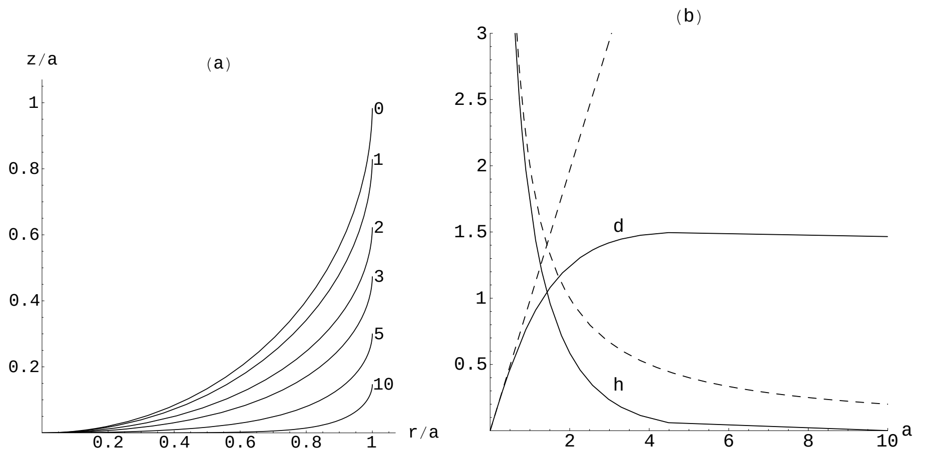
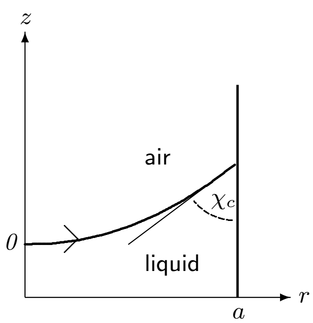


Figure 7.3: **(a)** *Capillary surface shape $z(r)/a$ as a function of r/a for $\chi_c = 1^\circ$ and $a = 0, 1, 2, 3, 5, 10$. The length scale is fixed by setting $R_c = 1$. Notice how the shape becomes gradually spherical as the tube radius $a \to 1$. For $a \lesssim 1$ the shape is constant.* **(b)** *Computed capillary rise h and depth d as functions of a (fully drawn). For $a \gtrsim 1$ the computed values deviate from the spherical surface results (7-50) and (7-51) (dashed).*



For the capillary meniscus with $\chi_c < 90^\circ$ both centers of curvature lie outside the liquid.

These are the *geometric* radii of curvature with signs given by their definitions. One should be aware that this sign convention may not agree with the physical sign convention for the Young-Laplace law (7-47), and that it may be necessary to introduce explicit signs to get the physics right.

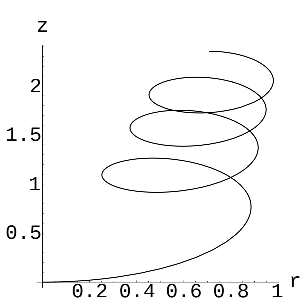
The capillary surface

For the rising liquid/air capillary surface with acute contact angle both geometric radii of curvature are positive. Since both centers of curvature lie outside the liquid, the physical radii are $-R_1$ and $-R_2$ in the Young-Laplace law (7-47). Assuming that the air pressure is constant, hydrostatic balance demands that $g_0 z + \Delta p/\rho_0$ be constant, or

$$g_0 z - \frac{\alpha}{\rho_0}\left(\frac{d\theta}{ds} + \frac{\sin\theta}{r}\right) = -\frac{2\alpha}{R_0} .$$

The value of the constant has been determined from the initial condition that the curve starts in $r = z = \theta = 0$ with equal geometric radii of curvature, $R_1 = R_2 = R_0$. Solving for $d\theta/ds$ we find,

$$\frac{d\theta}{ds} = \frac{2}{R_0} - \frac{\sin\theta}{r} + \frac{z}{R_c^2} , \tag{7-61}$$



The numeric solution for $R_0 = R_c = a = 1$ in the interval $0 < \theta < 6\pi$. The spirals are unphysical. For $\chi_c = 0$ the curve terminates at the first vertical tangent.

where R_c is the capillary constant (7-46). Together with the two equations (7-59) we have obtained three first order differential equations for r, z, and θ. Since s does not occur explicitly, and since θ grows monotonically with s, one may eliminate s and instead use θ as the running parameter.

Unfortunately these equations cannot be solved analytically, but given R_0 they may be solved numerically with the boundary conditions $r = z = 0$ for

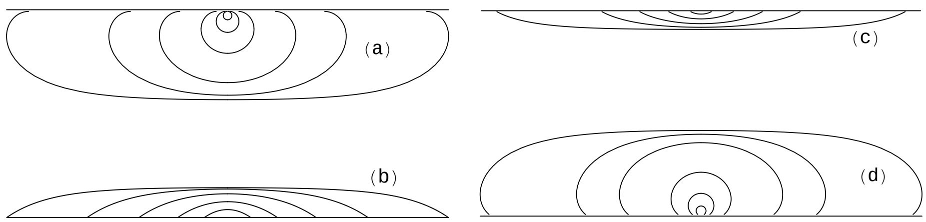
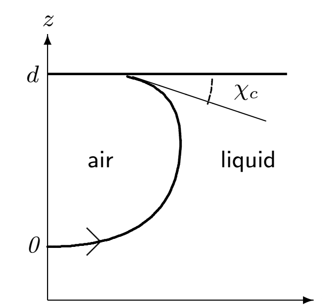
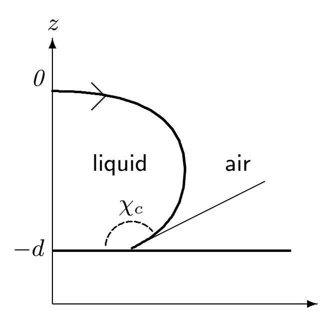


Figure 7.4: *Shapes of stable bubbles and droplets of water ($\chi_c = 1°$) and mercury ($\chi_c = 140°$) in units of the capillary radius R_c.* **(a)** *Air bubbles in water under a lid (to scale). Maximal depth is 2.1.* **(b)** *Water droplet on table plotted with vertical scale enlarged 40 times. Maximal depth is 0.019.* **(c)** *Air bubbles in mercury (to scale). Maximal depth is 0.74 .* **(d)** *Mercury droplets on a table (to scale). Maximal depth 2.0 .*



An air bubble in under a horizontal lid with acute angle of contact.



A droplet on a horizontal plate with an obtuse angle of contact.

$\theta = 0$. The solutions are quasi-periodic curves that spiral upwards forever. The physical solution must however stop at the wall $r = a$ for $\theta = \theta_c = 90° - \chi_c$, and that fixes R_0. The results of integrating the equations are displayed in fig. 7.3.

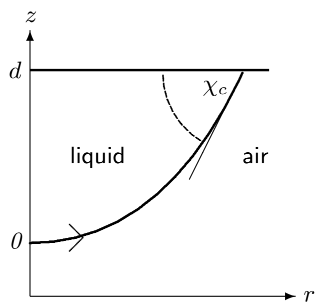
Stable bubbles and droplets

If a horizontal plate is inserted into water, air bubbles may come up against its underside, and remain stably there. The bubbles are pressed against the plate by buoyancy forces that also tend to flatten bubbles larger than the capillary radius. The shape may be obtained from the solution to the capillary effect by continuing to $\theta = 180° - \chi_c$.

It is well-known that mercury on a plate forms small nearly spherical droplets that may be brought to merge and form flat splotches that tend to break up if they become too large. In this case the geometric radii of curvature will both be negative while the physical radii of curvature are both positive because the centers of curvature lie inside the liquid. The formalism is consequently exactly the same as before, except that the central radius of curvature R_0 is now negative. The shapes are nearly the same as for air bubbles, except for the different angles of contact.

In fig. 7.4 the four stable configurations of bubbles and droplets are displayed. The depth stabilizes in all cases at a maximal value for $R_0 \to \infty$, indicating that larger bubbles are unstable and tend to break up. Notice that the depth of the water droplet (frame 7.4b) is enlarged by a factor 40. If water really has contact angle $\chi_c = 1°$, the maximal depth of a water droplet on a flat surface is only $0.019 R_c = 50$ μm. Due to its small contact angle, water is very efficient in *wetting* many surfaces. It should again be emphasized that the contact angle is extremely sensitive to surface properties, such as waxing, and to fluid composition and additives.

In the household we regularly use surfactants that are capable of making dish-water wet greasy surfaces which otherwise would create separate droplets. After



A liquid drop hanging from a horizontal plate with an acute angle of contact.

washing our cars we apply a wax which makes rainwater perl and prevents it from wetting the surface, thereby diminishing rust and corrosion.

Unstable bubbles and droplets

Rain or humidity create growing drops that eventually fall, while growing air bubbles at the bottom of a soda bottle eventually loose their grip and rise to the surface. We shall now investigate the hydrostatic solutions and attempt to determine the limits to stability. Both the geometric and physical radii of curvature are in this case positive, such that we now get,

$$\frac{d\theta}{ds} = \frac{2}{R_0} - \frac{\sin\theta}{r} - \frac{z}{R_c^2} \,, \tag{7-62}$$

with the opposite sign of z. In this case we may not eliminate s because θ is not a monotonic function of s.

Problems

7.1 The spaceship Rama (from the novel by Arthur C. Clarke) is a hollow cylinder hundreds of kilometers long and tens of kilometers in diameter. The ship rotates so as to create a standard pseudo-gravitational field g_0 on the inner side of the cylinder. Calculate the escape velocity to the center of the cylinder.

7.2 Calculate the increase in buoyancy due to its own gravity for a spherical body in a) a fluid of constant density, b) a fluid of constant compressibility, and c) isentropic gas.

7.3 a) Calculate the repulsion between two identical spheres with constant mass density ρ_1 and radii a, submerged a distance $D \gg a$ apart in a fluid obeying the equation of state $\rho = \rho(p)$. There is no other gravitational field present, the fluid pressure is p_0 in the absence of the spheres, and one may assume that the pressure corrections due to the spheres are everywhere small in comparison with p_0. b) Compare with the gravitational attraction between the spheres. c) Under which conditions will the total force between the spheres vanish.

7.4 Calculate the change in sea level if the air pressure locally rises by 20 millibars.

7.5 Calculate the changes in air pressure due to tidal motion of the atmosphere a) over sea, and b) over land?

7.6 a) Show that for an ideal gas in isentropic balance the quantity

$$H = \frac{\gamma}{\gamma - 1} \frac{RT}{M_{\mathsf{mol}}} + \Phi \tag{7-63}$$

is constant. b) Use this to calculate the temperature gradient in an arbitrary gravitational field.

7.7 How much water is found in the tidal bulge (the water found above average height).

7.8 a) Calculate the height of a geostationary orbit, *i.e.* an orbit where a sattelite would appear to be at rest with respect to a point on the surface of the Earth. b) How heavy must a sattelite in geostationary orbit be for the tides to be of the same size as the Moon's? c) Assuming the same average density, what would be the apparent size of such a sattelite?

7.9 Calculate the mean value $\langle h \rangle$ and the tidal range in the quasistatic approximation (7-27).

7.10 Calculate the tidal range that would result if the Earth were in bound rotation around the center-of-mass of the Earth-Moon system.

* **7.11** Show that $\boldsymbol{\nabla}^2[f(r)(3\cos^2\theta - 1)] = g(r)(3\cos^2\theta - 1)$ and determine $g(r)$.

7.12 Show that the integral (**??**) may be written explicitly as

$$\Phi_1(\theta) = -\frac{G\rho_1 a}{\sqrt{2}} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{h(\theta')}{\sqrt{1 - \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\phi'}} \tag{7-64}$$

7.13 Show that (**??**) and (**??**) for $\lambda \to 0$ become

$$\frac{z}{R_c} = \sqrt{2(1-\cos\theta)} \tag{7-65}$$

$$\frac{x}{R_c} = \log\frac{8}{\lambda} + \log\left|\tan\frac{\theta}{4}\right| - 2\left(1 - \cos\frac{\theta}{2}\right) \tag{7-66}$$

Interpret the solution.

7.14 Determine the radii of curvature in section 7.7 by expanding the shape $z = f(r)$ with $r = \sqrt{x^2+y^2}$ to second order around $x = x_0$, $y = 0$, and $z = z_0$.

7.14 Expanding to second order around $(x, y, z) = (x_0, 0, z_0)$ we find

$$\Delta z = \alpha\Delta x + \frac{1}{2}\beta\Delta x^2 + \frac{\alpha}{2x_0} y^2 \ , \tag{7-67}$$

where $\Delta z = z - z_0$, $\Delta x = x - x_0$, $\alpha = f'(x_0) = \tan\theta$, and $\beta = f''(x_0)$. Introduce a local coordinate system with coordinates ξ and η in $(x_0, 0, z_0)$

$$\Delta x = \xi\cos\theta + \eta\sin\theta \tag{7-68}$$

$$\Delta z = -\xi\sin\theta + \eta\cos\theta \tag{7-69}$$

Substituting and solving for η keeping up to second order terms,

$$\eta = \frac{1}{2}\beta\cos^3\theta\,\xi^2 + \frac{\sin\theta}{2x_0} y^2 \tag{7-70}$$

Hence

$$\frac{1}{R_1} = \frac{\partial^2\eta}{\partial\xi^2} = \beta\cos^3\theta \ , \qquad \frac{1}{R_2} = \frac{\partial^2\eta}{\partial y^2} = \frac{\sin\theta}{x_0} \tag{7-71}$$

But

$$\beta = \frac{d^2 z}{dx^2} = \frac{d\tan\theta}{dx} = \frac{1}{\cos^2\theta}\frac{d\theta}{dx} = \frac{1}{\cos^2\theta}\frac{ds}{dx}\frac{d\theta}{ds} = \frac{1}{\cos^3\theta}\frac{d\theta}{ds} \tag{7-72}$$

proving that $1/R_1 = d\theta/ds$.