7

Hydrostatic shapes

It is primarily the interplay between gravity and contact forces that shapes the macroscopic world around us. The seas, the air, planets and stars all owe their shape to gravity, and even our own bodies bear witness to the strength of gravity at the surface of our massive planet. What physics principles determine the shape of the surface of the sea? The sea is obviously horizontal at short distances, but bends below the horizon at larger distances following the planet's curvature. The Earth as a whole is spherical and so is the sea, but that is only the first approximation. The Moon's gravity tugs at the water in the seas and raises tides, and even the massive Earth itself is flattened by the centrifugal forces of its own rotation.

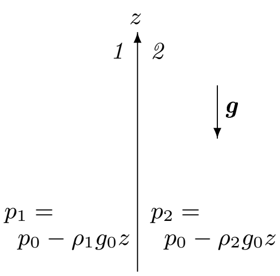
Disregarding surface tension, the simple answer is that in hydrostatic equilibrium with gravity, an interface between two fluids of different densities, for example the sea and the atmosphere, must coincide with a surface of constant potential, an equipotential surface. Otherwise, if an interface crosses an equipotential surface, there will arise a tangential component of gravity which can only be balanced by shear contact forces which a fluid at rest is unable to supply. An iceberg rising out of the sea does not obey this principle because it is solid, not fluid. Neither is the principle valid for fluids in motion. Waves in the sea are in fact "waterbergs" that normally move along the surface, but under special circumstances are able stay in one place, as for example in a river flowing past a big stone.

In this chapter the influence of gravity on the shape of large bodies of fluid is analyzed, the primary goal being the calculation of the size and shape of the tides. Centrifugal forces give rise to a gravity-like field, which shapes all rotating fluid bodies with open surfaces, for example a bucket of water. Surface tension only plays a role for small bodies of fluid and will be discussed in chapter 8.

7.1 Fluid interfaces in hydrostatic equilibrium

The intuitive argument about the impossibility of creating a hydrostatic "waterberg" must in fact follow from the equations of hydrostatic equilibrium. We shall now show that hydrostatic equilibrium implies that the interface between two fluids with different densities ρ_1 and ρ_2 must be an equipotential surface.

Since the gravitational field is the same on both sides of the interface, hydrostatic balance $\boldsymbol{\nabla} p = \rho \boldsymbol{g}$ implies that there is a jump in the pressure gradient across the interface, because on one side $(\boldsymbol{\nabla} p)_1 = \rho_1 \boldsymbol{g}$ and on the other $(\boldsymbol{\nabla} p)_2 = \rho_2 \boldsymbol{g}$. If the field of gravity has a component tangential to the interface, there will consequently be a jump in the tangential pressure gradient. Even if the pressures should be equal in one point, they must therefore be different a little distance away along the surface. Newton's third law, however, requires pressure to be continuous everywhere, also across an interface (as long as there is no surface tension), so this problem can only be avoided if the tangential component of gravity vanishes everywhere at the interface, implying that it is an equipotential surface.

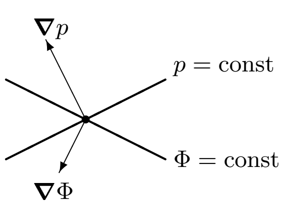


An impossible vertical interface between two fluids at rest with different densities. Even if the hydrostatic pressures on the two sides are the same for $z = 0$ they will be different everywhere else.

If, on the other hand, the fluid densities are the exactly same on both sides of the interface but the fluids themselves are different, the interface is not forced to follow an equipotential surface. This is, however, an unusual and highly unstable situation. The smallest deviation from equality in density on the two sides will call gravity in to make the interface horizontal. Stable vertical interfaces between fluids are simply not seen.

Isobars and equipotential surfaces

Surfaces of constant pressure, satisfying $p(\boldsymbol{x}) = p_0$, are called *isobars*. Through every point of space runs one and only one isobar, namely the one corresponding to the pressure in that point. The gradient of the pressure is everywhere normal to the local isobar surface, like gravity, $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, is everywhere normal to the local equipotential surface, defined by $\Phi(\boldsymbol{x}) = \Phi_0$. Local hydrostatic equilibrium, $\boldsymbol{\nabla} p = \rho \boldsymbol{g} = -\rho \boldsymbol{\nabla}\Phi$, tells us that the normal to the isobar is everywhere parallel with the normal to the equipotential surface. This can only be the case if *isobars coincide with equipotential surfaces in hydrostatic equilibrium.* For if an isobar crossed an equipotential surface anywhere at a finite angle the two normals could not be parallel.



If isobars and equipotential surfaces cross, hydrostatic balance $\boldsymbol{\nabla} p + \rho \boldsymbol{\nabla}\Phi = \boldsymbol{0}$ becomes impossible.

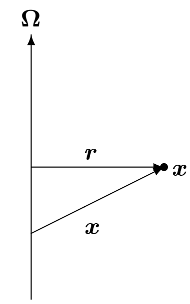
Since the curl of a gradient trivially vanishes, $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = \boldsymbol{0}$, it follows from hydrostatic equilibrium that

$$\boldsymbol{0} = \boldsymbol{\nabla} \times (\rho \boldsymbol{g}) = \boldsymbol{\nabla}\rho \times \boldsymbol{g} + \rho \boldsymbol{\nabla} \times \boldsymbol{g} = -\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}\Phi \ . \tag{7-1}$$

This implies that $\boldsymbol{\nabla}\rho \sim \boldsymbol{\nabla}\Phi$ so that the surfaces of constant density must also coincide with the equipotential surfaces in hydrostatic equilibrium.

7.2 Shape of rotating fluids

Newton's second law of motion is only valid in *inertial* coordinate systems, where free particles move on straight lines with constant velocity. In rotating, or otherwise accelerated, non-inertial coordinate systems, one may formally write the equation of motion in their usual form, but the price to be paid is the inclusion of certain force-like terms that do not have any obvious connection with material bodies, but derive from the overall motion of the coordinate system (see chapter 26 for a more detailed analysis). Such terms are called *fictitious forces*, although they are by no means pure fiction, as one becomes painfully aware when standing up in a bus that suddenly stops. A more reasonable name might be *inertial* forces, since they arise as a consequence of the inertia of material bodies.



The geometry of a rotating system is characterized by a rotation vector $\boldsymbol{\Omega}$ directed along the axis of rotation with magnitude equal to the angular velocity. The vector $\boldsymbol{r}$ is directed orthogonally out from the axis to a point $\boldsymbol{x}$.

Antigravity of rotation

A material particle at rest in a coordinate system rotating with constant angular velocity Ω in relation to an inertial system will experience only one fictitious force, the *centrifugal force*. We all know it from carroussels. It is directed perpendicularly outwards from the axis of rotation and of magnitude $M\,r\,\Omega^2$, where r is the shortest distance to the axis.

In a rotating coordinate system placed with its origin on the rotation axis, and z-axis coincident with it, the shortest vector to a point $\boldsymbol{x} = (x, y, z)$ is $\boldsymbol{r} = (x, y, 0)$. The centrifugal force is proportional to the mass of the particle and thus mimics a gravitational field

$$\boldsymbol{g}_{\text{centrifugal}}(\boldsymbol{r}) = \boldsymbol{r}\Omega^2 = (x, y, 0)\Omega^2 \ . \tag{7-2}$$

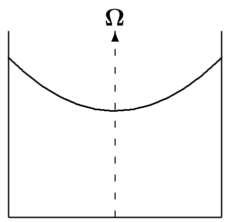
This fictitious gravitational field may be derived from a (fictitious) potential

$$\Phi_{\text{centrifugal}}(\boldsymbol{r}) = -\frac{1}{2}r^2\Omega^2 = -\frac{1}{2}\Omega^2(x^2 + y^2) \ . \tag{7-3}$$

Since the centrifugal field is directed away from the axis of rotation the centrifugal field is a kind of *antigravity* field, which will try to split things apart and lift objects off a rotating planet. The antigravity field of rotation is, however, cylindrical in shape rather than spherical and has consequently the greatest influence at the equator of Earth. If our planet rotated once in a little less than $1\frac{1}{2}$ hours, people at the equator could (and would) actually levitate!

Newton's bucket

A bucket of water on a rotating plate is an example going right back to Newton himself. Internal friction (viscosity) in the water will after some time bring it to rest relative to the bucket and plate, and the whole thing will end up rotating as a solid body. In a rotating coordinate system with z-axis along the axis of rotation, the total gravitational field becomes $\boldsymbol{g} = (\Omega^2 x, \Omega^2 y, -g_0)$, including



The water surface in rotating bucket as a parabolic shape because of centrifugal forces.

both "real" gravity and the "fictitious" centrifugal force. Correspondingly, the total gravitational potential is,

$$\Phi = -\boldsymbol{g} \cdot \boldsymbol{x} = g_0 z - \frac{1}{2}\Omega^2(x^2 + y^2) , \tag{7-4}$$

and the pressure

$$p = p_0 - \rho_0 \Phi = p_0 - \rho_0 g_0 z + \frac{1}{2}\rho_0 \Omega^2 \left(x^2 + y^2\right) \tag{7-5}$$

where p_0 is the pressure at the origin of the coordinate system. It grows towards the rim, reflecting everywhere the change in height of the water column.

The isobars and equipotential surfaces are in this case rotation paraboloids,

$$z = z_0 + \frac{\Omega^2}{2g_0}(x^2 + y^2) , \tag{7-6}$$

where z_0 is a constant. In a bucket of diameter 20 cm rotating once per second the water stands 2 cm higher at the rim than in the center.

Example 7.2.1: An ultracentrifuge of radius 10 cm contains water and rotates at $\Omega = 60,000$ rpm ≈ 6300 s^{-1}. The centrifugal acceleration becomes $400,000\ g_0$ and the maximal pressure close to $2,000$ atm, which is the double of the pressure at the bottom of the deepest abyss in the sea. At such pressures, the change in water density is about 10%.

∗ Stability of rotating bodies

Including the centrifugal field (7-2) in the fundamental field equation (6-6), the divergence of the total acceleration field $\boldsymbol{g} = \boldsymbol{g}_{\text{gravity}} + \boldsymbol{g}_{\text{centrifugal}}$ becomes,

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G \rho + 2\Omega^2 . \tag{7-7}$$

Effectively, centrifugal forces create a negative mass density $-\Omega^2/2\pi G$. This is, of course, a purely formal result, but it nevertheless confirms the "antigravity" aspect of centrifugal forces, which makes gravity effectively repulsive wherever $\Omega^2/2\pi G\rho > 1$.

For a spherical planet stability against levitation at the equator requires the centrifugal force at the equator $\Omega^2 a$ to be smaller than surface gravity, $g_0 = GM/a^2 = \frac{4}{3}\pi\rho_0 G a$, which leads to the stronger condition,

$$q = \frac{\Omega^2 a}{g_0} = \frac{3}{2}\frac{\Omega^2}{2\pi G \rho_0} < 1 . \tag{7-8}$$

Inserting the parameters of the Earth we find $q \approx 1/291$. At the end of section 7.4 the influence of the deformation caused by rotation is also taken into account, leading to an even stricter stability condition.

7.3 The Earth, the Moon and the tides

Kepler thought that the Moon would influence the waters of Earth and raise tides, but Galilei found this notion of Kepler's completely crazy and compared it to common superstition. After Newton we know that the Moon's gravity acts on everything on Earth, also on the water in the sea, and attempts to pull it out of shape, thereby creating the tides. But since high tides occur roughly at the same time at antipodal points of the Earth, and twice a day, the explanation is not simply that the Moon lifts the sea towards itself but a little more sophisticated.

Johannes Kepler (1580–1635). *German mathematician and astronomer. Discovered that planets move in elliptical orbits and that their motion obeys mathematical laws.*

> Galileo wrote about Kepler: "But among all the great men who have philosophized about this remarkable effect, I am more astonished at Kepler than at any other. Despite his open and acute mind, and though he has at his fingertips the motions attributed to the earth, he has nevertheless lent his ear and his assent to the moon's dominion over the waters, and to occult properties, and to such puerilities." (see [25, p. 145]).

The best natural scientists and mathematicians of the eighteenth and nineteenth centuries worked on the dynamics of the tides, but here we shall only consider the simplest possible case of a quasistatic Moon. For a more complete discussion, including the dynamics of tidal waves, see for example Sir Horace Lamb's classical book [9] or ref. [24] for a modern account.

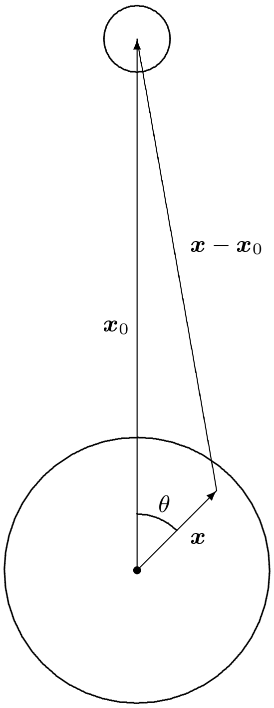
The Earth

We shall limit ourselves to study the Moon's influence on a liquid surface layer of the Earth. The solid parts of the Earth will of course also react to the Moon's field, but the effects are somewhat smaller and are due to elastic deformation rather than flow. This deformation has been indirectly measured to a precision of a few percent in the daily 0.1 ppm variations in the strength of gravity (see fig. 7.1 on page 117). There are also tidal effects in the atmosphere, but they are dominated by other atmospheric motions.

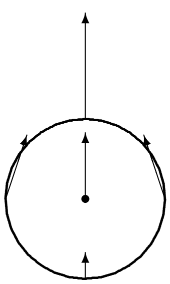
We shall furthermore disregard the changes to the Earth's own gravitational potential due to the shifting waters of the tides themselves, as well as the centrifugal antigravity of Earth's rotation causing it to deviate from a perfect sphere (which increases the tidal range by slightly more than 10 %, see section 7.4). Under all these assumptions the gravitational potential at a height h over the surface of the Earth is to first order in h given by

$$\boxed{\Phi_{\text{Earth}} = g_0 h \ ,} \tag{7-9}$$

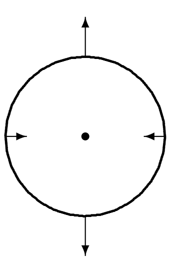
where g_0 is the magnitude of the surface gravity.



Geometry of the Earth and the Moon (not to scale)



How the Moon's gravity varies over the Earth (exaggerated).



The Moon's gravity with common acceleration cancelled. This also explains why the variations in tidal height have a semi-diurnal period.

The Moon

The Moon is not quite spherical, but nevertheless so small and far away that we may approximate its potential across the Earth with that of a point particle $-Gm/|\boldsymbol{x}-\boldsymbol{x}_0|$ situated at the Moon's position $\boldsymbol{x}_0$ with the Moon's mass m. Choosing a coordinate system with the origin at the center of the Earth and the z-axis in the direction of the Moon, we have $\boldsymbol{x}_0 = (0,0,D)$ where $D = |\boldsymbol{x}_0|$ is the Moon's distance. Since the Moon is approximately 60 Earth radii (a) away, *i.e.* $D \approx 60a$, the Moon's potential across the Earth (for $r = |\boldsymbol{x}| \leq a$) may conveniently be expanded in powers of $\boldsymbol{x}/D$, and we find to second order

$$\begin{aligned}
\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|} &= \frac{1}{\sqrt{x^2+y^2+(z-D)^2}} = \frac{1}{\sqrt{D^2-2zD+r^2}} \\
&= \frac{1}{D}\frac{1}{\sqrt{1-\dfrac{2z}{D}+\dfrac{r^2}{D^2}}} \\
&\approx \frac{1}{D}\left(1-\frac{1}{2}\left(-\frac{2z}{D}+\frac{r^2}{D^2}\right)+\frac{3}{8}\left(-\frac{2z}{D}\right)^2\right) \\
&= \frac{1}{D}\left(1+\frac{z}{D}+\frac{3z^2-r^2}{2D^2}\right) .
\end{aligned}$$

The first term in this expression leads to a constant potential $-Gm/D$, which may of course be ignored. The second term corresponds to a constant gravitational field in the direction towards the moon $g_z = Gm/D^2 \approx 30\ \mu\text{m/s}^2$, which is precisely cancelled by the centrifugal force due to the Earth's motion around the common center-of-mass of the Earth-Moon system (an effect we shall return to below). Spaceship Earth is therefore completely unaware of the two leading terms in the Moon's potential, and these terms cannot raise the tides. Galilei was right to leading non-trivial order, and that's actually not so bad.

Tidal effects come from the *variation* in the gravitational field across the Earth, to leading order given by the third term in the expansion of the potential. Introducing the angle θ between the direction to the Moon and the observation point on Earth, we have $z = r\cos\theta$, and the Moon's potential becomes (after dropping the two first terms)

$$\boxed{\Phi_{\text{Moon}} = -\frac{1}{2}(3\cos^2\theta-1)\left(\frac{r}{D}\right)^2\frac{Gm}{D}} . \tag{7-10}$$

This expansion may of course be continued indefinitely to higher powers of r/D. The coefficients $P_n(\cos\theta)$ are called Legendre polynomials (here $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta-1)$).

The gravitational field of the Moon is found from the gradient of the potential. It is simplest to convert to Cartesian coordinates, writing $(3\cos^2\theta-1)r^2 =$

$2z^2 - x^2 - y^2$, before calculating the gradient. In the xz-plane we get at the surface $r = a$

$$\boldsymbol{g}_{\text{Moon}} = (-\sin\theta, 0, 2\cos\theta)\frac{aGm}{D^3} \ .$$

Projecting on the local normal $\boldsymbol{e}_r = (\sin\theta, 0, \cos\theta)$ and tangent $\boldsymbol{e}_\theta = (\cos\theta, 0, -\sin\theta)$ to the Earth's surface, we finally obtain the vertical and horizontal components of the gravitational field of the Moon at any point of the Earth's surface

$$g^{\perp}_{\text{Moon}} = \boldsymbol{g}_{\text{Moon}} \cdot \boldsymbol{e}_r = (3\cos^2\theta - 1)\,\frac{aGm}{D^3} \ , \tag{7-11}$$

$$g^{\parallel}_{\text{Moon}} = \boldsymbol{g}_{\text{Moon}} \cdot \boldsymbol{e}_\theta = -\sin 2\theta\,\frac{3aGm}{2D^3} \ . \tag{7-12}$$

The magnitude of the horizontal component is maximal for $\theta = 45^o$ (and of course also 135^o because of symmetry).

Concluding, we repeat that tide-generating forces arise from variations in the Moon's gravity across the Earth. As we have just seen, the force is generally not vertical, but has a horizontal component of the same magnitude. From the sign and shape of the potential as a function of angle, we see that effectively the Moon lowers the gravitational potential just below its position, and at the antipodal point on the opposite side of the Earth, exactly as if there were shallow "valleys" at these places. Sometimes these places are called the Moon and anti-Moon positions.

And the tides

If the Earth did not rotate and the Moon stood still above a particular spot, water would rush in to fill up these "valleys", and the sea would come to equilibrium with its open surface at constant total gravitational potential. The total potential near the surface of the Earth is

$$\Phi = \Phi_{\text{Earth}} + \Phi_{\text{Moon}} = g_0 h - \frac{1}{2}(3\cos^2\theta - 1)\left(\frac{a}{D}\right)^2 \frac{Gm}{D} \ , \tag{7-13}$$

Requiring this potential to be constant we find the tidal height

$$\boxed{h = h_0 + \frac{1}{2}(3\cos^2\theta - 1)\left(\frac{a}{D}\right)^2 \frac{Gm}{g_0 D}} \ , \tag{7-14}$$

where h_0 is a constant. Since the average over the sphere of the second term is,

$$\frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} \sin\theta\, d\phi\, (3\cos^2\theta - 1) = \frac{1}{2}\int_{-1}^{+1} (3z^2 - 1) dz = 0 \ ,$$

we conclude that h_0 is the average water depth.

Tidal range

The maximal difference between high and low tides, called the tidal range, occurs between the extreme positions at $\theta = 0$ and $\theta = 90°$,

$$\boxed{H_0 = \frac{3}{2}\left(\frac{a}{D}\right)^2 \frac{Gm}{g_0 D} = \frac{3}{2} a \frac{m}{M}\left(\frac{a}{D}\right)^3}\,, \tag{7-15}$$

where the last equation is obtained using $g_0 = GM/a^2$ with M being the Earth's mass. Inserting the values for the Moon we get $H_0 \approx 54$ cm. Interestingly, the range of the tides due to the Sun turns out to be half as large, about 25 cm. This makes spring tides when the Sun and the Moon cooperate almost three times higher than neap tides when they don't.

For the tides to reach full height, water must move in from huge areas of the Earth as is evident from the shallow shape of the potential. Where this is not possible, for example in lakes and enclosed seas, the tidal range becomes much smaller than in the open oceans. Local geography may also influence tides. In bays and river mouths funnelling can cause tides to build up to huge values. Spring tides in the range of 15 meters have been measured in the Bay of Fundy in Canada.

∗ Quasistatic tidal cycles

The rotation of the Earth cannot be neglected. If the Earth did not rotate, or if the Moon were in a geostationary orbit, it would be much harder to observe the tides, although they would of course be there (problem 7.5). It is, after all, the cyclic variation in the water level observed at the coasts of seas and large lakes, which makes the tides observable. Since the axis of rotation of the Earth is neither aligned with the direction to the Moon nor orthogonal to it, the tidal forces acquire a diurnal cycle superimposed on the 'natural' semidiurnal one (see Fig. 7.1).

For a fixed position on the surface of the Earth, the dominant variation in the lunar zenith angle θ is due to Earth's diurnal rotation with angular rate $\Omega = 2\pi/24$ hours $\approx 7 \times 10^{-5}$ radians per second. On top of that, there are many other sources of periodic variations in the lunar angle [24], which we shall ignore here.

The dominant such source is the lunar orbital period of a little less than a month. Furthermore, the orbital plane of the Moon inclines about 5° with respect to the ecliptic (the orbital plane of the Earth around the Sun), and precesses with this inclination around the ecliptic in a little less than 19 years. The Earth's equator is itself inclined about 23° to the ecliptic, and precesses around it in about 25,000 years. Due to lunar orbit precession, the angle between the equatorial plane of the Earth and the plane of the lunar orbit will range over $23 \pm 5°$, *i.e.* between 18° and 28°, in about 9 years.

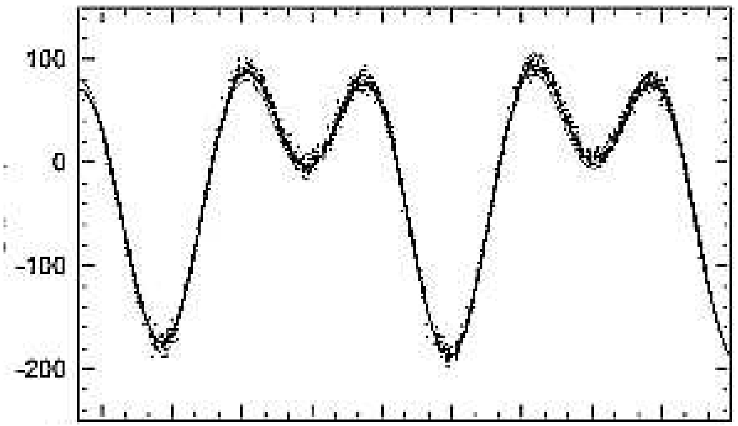
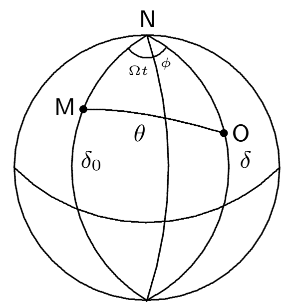


Figure 7.1: *Variation in the vertical gravitational acceleration over a period of 56 hours in units of* $10^{-9}g_0$ *(measured in Stanford, California, on Dec. 8–9, 1996 [19], reproduced here with the permission of the authors (to be obtained)). The semidiurnal as well as diurnal tidal variations are prominently visible as dips in the curves. Modelling the Earth as a solid elastic object and taking into account the effects of ocean loading, the measured data is reproduced to within a few times* $10^{-9}g_0$.



Spherical triangle formed by Moon (M), observer (O) and north pole (N).

Let the fixed observer position at the surface of the Earth have (easterly) longitude ϕ and (northerly) latitude δ. The lunar angle θ is then calculated from the spherical triangle formed by the north pole, the lunar position and the observer's position,

$$\cos\theta = \sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos(\Omega t + \phi) \ . \qquad (7\text{-}16)$$

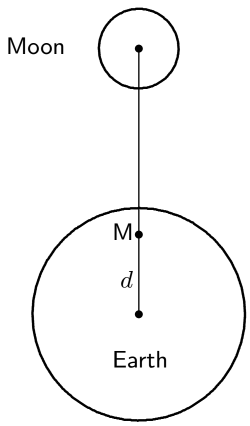
Here δ_0 is the latitude of the lunar position and the origin of time has been chosen such that the Moon at $t = 0$ is directly above the meridian $\phi = 0$.

Inserting this into the static expression for the tidal height (7-14), we obtain the quasistatic height variation with time at the observer's place, which becomes the sum of a diurnal and a semidiurnal cycle

$$\boxed{h = \langle h\rangle + h_1\cos(\Omega t + \phi) + h_2\cos 2(\Omega t + \phi)} \ . \qquad (7\text{-}17)$$

Here $\langle h\rangle$ is the time-averaged height, and $h_1 = \frac{1}{2}H_0\sin 2\delta\sin 2\delta_0$ and $h_2 = \frac{1}{2}H_0\cos^2\delta\cos^2\delta_0$ are the diurnal and semidiurnal tidal amplitudes. The full tidal range is not quite $2h_1 + 2h_2$, because the two cosines cannot simultaneously take the value -1 (see problem 7.6).

To go beyond the quasistatic approximation, the full theory of fluid dynamics on a rotating planet becomes necessary. The tides will then be controlled not only by the tide-generating forces, but also by the interplay between the inertia of the moving water and friction forces opposing the motion. High tides will no more be tied to the Moon's instantaneous position, but may both be delayed and advanced relative to it.



The center of mass of the Earth-Moon system lies below the surface of the Earth.

∗ Influence of the Earth-Moon orbital motion

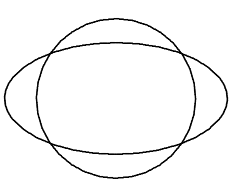
A question is sometimes raised concerning the role of centrifugal forces from the Earth's motion around the center-of-mass of the Earth-Moon system. This point lies a distance $d = Dm/(m + M)$ from the center of the Earth, which is actually about 1700 km below the surface, and during a lunar cycle the center of the Earth and the center of the Moon move in circular orbits around it. Were the Earth (like the Moon) in bound rotation so that it always turned the same side towards the Moon, one would in the corotating coordinate system, where the Moon and the Earth have fixed positions, have to add a centrifugal potential to the previously calculated potential (7-13), and the tidal range (see problem 7.7) would become about 14 m!

Luckily, this is not the case. The Earth's own rotation is fixed with respect to the inertial system of the fixed stars (disregarding the precession of its rotation axis). A truly non-rotating Earth would, in the corotating system, rotate backwards in synchrony with the lunar cycle, cancelling the centrifugal potential. Seen from the inertial system, the circular orbital motion imparts the same centripetal acceleration $\Omega^2 d$ (along the Earth-Moon line) to all parts of the Earth. This centripetal acceleration must equal the constant gravitational attraction, Gm/D^2, coming from the linear term in the Moon's potential, and equating the two, one obtains $\Omega^2 = Gm/D^2 d = GM/D^3$, which is the well-known Kepler equation relating the Moon's period of revolution to its mass and distance.

The Moon always turns the same side towards Earth and the bound rotation adds in fact a centrifugal component on top of the tidal field from Earth. Over time these effects have together deformed the Moon into its present egg-like shape.

∗ 7.4 Shape of a rotating fluid planet

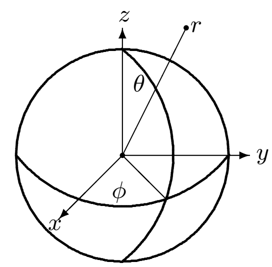
On a rotating planet, centrifugal forces will add a component of "antigravity" to the gravitational acceleration field, making the road from the pole to equator slightly downhill. At Earth's equator the centrifugal acceleration amounts to only $q \approx 1/291$ of the surface gravity, so a first guess would be that there is a centrifugal "valley" at the equator with a depth of 1/291 of the Earths radius, which is about 22 km. If such a difference suddenly came to exist on a spherical Earth, all the water would like huge tides run towards the equator. Since there *is* land at equator, we may conclude that even the massive Earth must over time have flowed into the centrifugal valley. The difference between the equatorial and polar radii is in fact 21.4 km [2], and coincidentally, this is roughly the same as the difference between the highest mountain top and the deepest ocean trench on Earth.



Exaggerated sketch of the change in shape of the Earth due to rotation.

The flattening of the Earth due to rotation has like the tides been a problem attracting the best minds of the past centuries [9]. We shall here consider the simplest possible model, which nevertheless captures all the relevant features for slowly rotating planets.

Rigid spherical planet



Polar and azimuthal angles.

If a spherical planet rotates like a stiff body, the gravitational potential above the surface will be composed of the gravitational potential of planet and the centrifugal potential. In spherical coordinates we have,

$$\Phi_0 = -g_0 \frac{a^2}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta \ . \tag{7-18}$$

from which we get the vertical and horizontal components of surface gravity,

$$g_r = -\left.\frac{\partial \Phi_0}{\partial r}\right|_{r=a} = -g_0(1 - q\sin^2\theta) \ , \tag{7-19a}$$

$$g_\theta = -\frac{1}{r}\left.\frac{\partial \Phi_0}{\partial \theta}\right|_{r=a} = g_0\, q \sin\theta\cos\theta \ , \tag{7-19b}$$

where $q = \Omega^2 a/g_0$ is the "levitation parameter" defined in (7-8). This confirms that the magnitude of vertical gravity is reduced, and that horizontal gravity points towards the equator. For Earth the changes are all of relative magnitude $q \approx 1/291$.

Fluid planet

Suppose now the planet is made from a heavy fluid which given time will adapt its shape to an equipotential surface of the form,

$$r = a + h(\theta) \ , \tag{7-20}$$

with a small radial displacement, $|h|(\theta) \ll a$. Assuming that the displaced material is incompressible we must require,

$$\int_0^\pi h(\theta)\sin\theta\, d\theta = 0 \ . \tag{7-21}$$

If we naively disregard the extra gravitational field created by the displacement of material, the potential is given by (7-18). On the displaced surface this becomes to first order in the small quantities h and q,

$$\Phi_0 \approx g_0 h - g_0 a(1 + \frac{q}{2}\sin^2\theta) \ . \tag{7-22}$$

Demanding that it be constant, it follows that

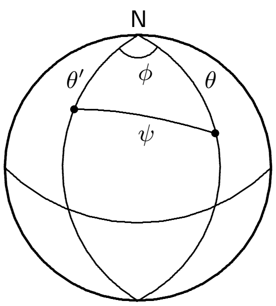
$$h = h_0\left(\sin^2\theta - \frac{2}{3}\right) \ , \qquad h_0 = \frac{1}{2}aq \ . \tag{7-23}$$

The $-\frac{2}{3}$ in the parenthesis has been chosen such that (7-21) is fulfilled. For Earth we find $h_0 = 11$ km, which is only half the expected result.

Including the self-potential

The preceding result shows that the gravitational potential of the shifted material must play an important role. Assuming that the shifted material has constant density ρ_1, the extra gravitational potential due to the shifted material is calculated from (3-24) by integrating over the (signed) volume ΔV occupied by the shifted material,

$$\Phi_1 = -G\rho_1 \int_{\Delta V} \frac{dV'}{|\boldsymbol{x} - \boldsymbol{x}'|} \; . \tag{7-24}$$



Spherical triangle formed by the various angles.

Since the shifted material is a thin layer of thickness h, the volume element is $dV' \approx h(\theta')dS'$ where dS' is the surface element of the original sphere , $|\boldsymbol{x}'| = a$. There are of course corrections but they will be of higher order in h. The square of the denominator may be written as $|\boldsymbol{x} - \boldsymbol{x}'|^2 = r^2 + a^2 - 2ra\cos\psi$ where ψ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}'$. Consequently we have to linear order in h

$$\Phi_1 = -G\rho_1 \oint_S \frac{h(\theta')}{\sqrt{r^2 + a^2 - 2ra\cos\psi}} dS' \; , \tag{7-25}$$

where $\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi$ and $dS' = a^2\sin\theta' d\theta' d\phi'$. Notice that this is exactly the same potential as would have been obtained from a surface distribution of mass with surface density $\rho_1 h(\theta)$.

There are various ways to do this integral. We shall use a wonderful theorem about Legendre polynomials, which says that a mass distribution with an angular dependence given by a Legendre polynomial, creates a potential with exactly the same angular dependence. So if we assume that the surface shape is of the form (7-23) with angular dependence proportional to $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{2}(2 - 3\sin^2\theta)$, the effective surface mass distribution will be proportional to $P_2(\cos\theta)$, implying that the self-potential will be of precisely the same shape (see problem 7.8),

$$\Phi_1(r,\theta) = F(r)\left(\sin^2\theta - \frac{2}{3}\right) \; . \tag{7-26}$$

The radial function may now be determined from the integral (7-26) by taking $\theta = 0$. Since now $\psi = \theta'$, all the difficult integrals disappear and we obtain

$$F(r) = -\frac{3}{2}\Phi_1(r,0) = \frac{3}{2}G\rho_1 h_0 a^2 2\pi \int_0^\pi \frac{\sin\theta'\left(\sin^2\theta' - \frac{2}{3}\right)}{\sqrt{r^2 + a^2 - 2ra\cos\theta'}} d\theta'$$
$$= \frac{3}{2}G\rho_1 a^2 h_0 2\pi \int_{-1}^{1} \frac{\frac{1}{3} - u^2}{\sqrt{r^2 + a^2 - 2rau}} du \; .$$

The integral is now standard, and we find

$$F(r) = -\frac{4\pi}{5}\rho_1 G a h_0 \frac{a^3}{r^3} = -\frac{3}{5} g_0 h_0 \frac{\rho_1}{\rho_0}\frac{a^3}{r^3} \; . \tag{7-27}$$

In the last step we have used that $g_0 = GM_0/a^2 = \frac{4}{3}\pi G a \rho_0$ where ρ_0 is the average density of the planet.

Total potential and strength of gravity

The total potential now becomes

$$\Phi = \Phi_0 + \Phi_1 = -g_0 \frac{a^2}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta - \frac{3}{5} g_0 h(\theta) \frac{\rho_1}{\rho_0} \frac{a^3}{r^3} \tag{7-28}$$

Inserting $r = a + h$ and expanding to lowest order in h and q, we finally obtain,

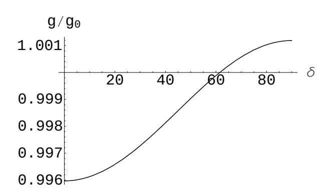
$$\boxed{h_0 = \frac{\frac{1}{2} q a}{1 - \frac{3}{5}\frac{\rho_1}{\rho_0}} \ .} \tag{7-29}$$

For Earth, the average density of the mantle material is $\rho_1 \approx 4.5$ g/cm^3 whereas the average density is $\rho_0 \approx 5.5$ g/cm^3. With these densities one gets $h_0 = 21.5$ km in close agreement with the quoted value [2]. In the same vein, we may also calculate the influence of the self-potential on the tidal range. Since the density of water is $\rho_1 \approx 1.0$ g/cm^3, the tidal range (7-15) is increased by a factor 1.12.

From the total potential we calculate gravity at the displaced surface,

$$g_r = -\left.\frac{\partial \Phi}{\partial r}\right|_{r=a+h} = -g_0 \left(1 - q\sin^2\theta - 2\frac{h(\theta)}{a} + \frac{9}{5}\frac{\rho_1}{\rho_0}\frac{h(\theta)}{a}\right) , \tag{7-30a}$$

$$g_\theta = -\frac{1}{r}\left.\frac{\partial \Phi}{\partial \theta}\right|_{r=a+h} = g_0 \left(q + \frac{6}{5}\frac{h_0}{a}\frac{\rho_1}{\rho_2}\right)\sin\theta\cos\theta \ . \tag{7-30b}$$



The variation of g/g_0 with latitude δ.

Finally, projecting on the local normal and tangent we find to first order in h and q,

$$g_\perp \approx g_r \ , \qquad g_\parallel = g_\theta + g_r \frac{1}{a}\frac{dh(\theta)}{d\theta} \approx 0 \ . \tag{7-31a}$$

The field of gravity is orthogonal to the equipotential surface, as it should be.

Notice that the three correction terms to the vertical field are due to the centrifugal force, to the change in gravity from the change in height, and to the displacement of material. All three contributions are of the same order of magnitude, q, because they all ultimately derive from the centrifugal force.

Example 7.4.1 (Olympic games): The dependence of gravity on polar angle (or latitude) given in (7-30a) has practical consequences. In 1968 the Olympic games were held in Mexico City at latitude $\delta = 19°$ north whereas in 1980 they were held in Moscow at latitude $\delta' = 55°$ north. To compare record heights in jumps (or throws), it is necessary to correct for the variation in gravity due to the centrifugal force, the geographical difference in height, and air resistance. Assuming that the initial velocity is the same, the height h attained in Mexico city would correspond to a height h' in Moscow , related to h by $v^2 = 2gh = 2g'h'$. Using (7-30a) we find $h/h' = g'/g = 1.00296$. This shows that a correction of -0.3% due to variation in gravity (among other corrections) would have to be applied to the Mexico City heights before they were compared with the Moscow heights.

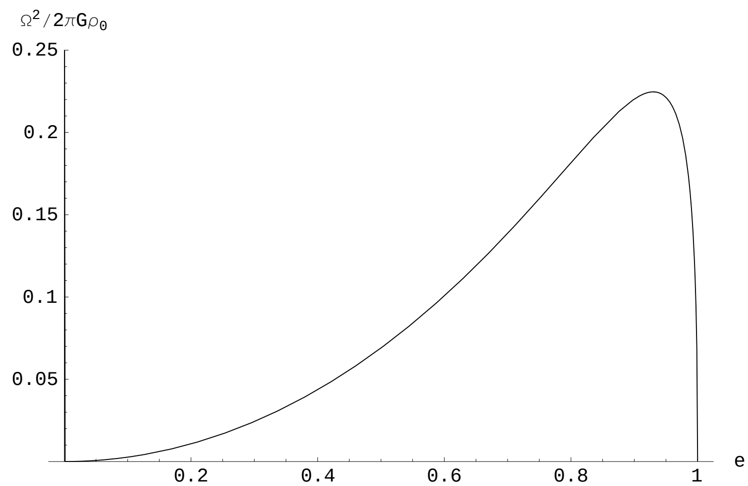


Figure 7.2: *The MacLaurin function (right hand side of eq. (7-32)). The maximum 0.225 is reached for* $e = 0.93$.

Fast rotating planet

One of the main assumptions behind the calculations in this section was that the planet should be slowly rotating, meaning that the deformation of the planet due to rotation be small, or $|h(\theta)| \ll a$. Intuitively, it is fairly obvious, that if the rate of rotation of the planet is increased, the flattening increases until it reaches a point, where the "antigravity" of rotation overcomes the "true" gravity of planetary matter as well as cohesive forces. Then the planet becomes unstable with dramatic change of shape or even breakup as a consequence.

Colin MacLaurin (1698–1746). *Scottish mathematician who developed and extended Newton's work on calculus and gravitation.*

The study of the possible forms of rotating planets was initiated very early by Newton and in particular by MacLaurin. It was found that oblate ellipsoids of rotation are possible allowed shapes for rotating planets with constant matter density, ρ_0. An oblate ellipsoid of rotation is characterized by equal-size major axes, $a = b$ and a smaller minor axis $c < a$, about which it rotates.

MacLaurin found that the angular rotation rate is related to the eccentricity $e = \sqrt{1 - c^2/a^2}$ through the formula

$$\frac{\Omega^2}{2\pi G\rho_0} = \frac{1}{e^3}\left(\sqrt{1-e^2}\left(3-2e^2\right)\arcsin e - 3e\left(1-e^2\right)\right) . \tag{7-32}$$

The right hand side is shown in Fig. 7.2 and has a maximum 0.225 for $e = 0.93$, implying that stability can only be maintained for $\Omega^2/2\pi G\rho_0 < 0.225$. Actually, various other shape instabilities set in at even lower values of the eccentricity (see ref. [21] for a thorough discussion of these instabilities and their astrophysical consequences). For small e, the MacLaurin(!) expansion of the right hand side of (7-32) becomes $4e^2/15$. Since $e^2 \approx 2h_0/a$, we obtain $h_0 = 15\Omega^2 a/16\pi G\rho_0 = 5\Omega^2 a^2/4g_0$, in complete agreement with (7-29) for $\rho_1 = \rho_0$.

Problems

7.1 The spaceship Rama (from the novel by Arthur C. Clarke) is a hollow cylinder hundreds of kilometers long and tens of kilometers in diameter. The ship rotates so as to create a standard pseudo-gravitational field g_0 on the inner side of the cylinder. Calculate the escape velocity to the center of the cylinder for radius $a = 10$ km.

7.2 Calculate the change in sea level if the air pressure locally rises by $\Delta p = 20$ hPa.

7.3 Calculate the changes in air pressure due to tidal motion of the atmosphere **(a)** over sea, and **(b)** over land?

7.4 How much water is found in the tidal bulge (above average height).

7.5 How heavy must a satellite in geostationary orbit (problem 3.3) be for the tides to be of the same size as the Moon's? **(a)** Assuming the same average density, what would be the apparent size of such a satellite?

7.6 Calculate the mean value $\langle h \rangle$ and the tidal range in the quasistatic approximation (7-17).

7.7 Estimate the tidal range that would result if the Earth were in bound rotation around the center-of-mass of the Earth-Moon system.

∗ **7.8** Show that $\boldsymbol{\nabla}^2[f(r)(3\cos^2\theta - 1)] = g(r)(3\cos^2\theta - 1)$ and determine $g(r)$.

7.9 Show that the integral (7-24) may be written explicitly as

$$\Phi_1(\theta) = -\frac{G\rho_1 a}{\sqrt{2}} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{h(\theta')}{\sqrt{1 - \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\phi'}} \tag{7-33}$$

