Singular Perturbation Expansions for ODEs

Example (overdamped pendulum)

$$\begin{cases} y'' + by' + \omega_0^2 y = 0, & y' = \frac{dy}{d\tau} \\ y(0) = y_0 \\ y'(0) = u_0 \end{cases}$$

In the overdamped limit $(b \gg \omega_0)$ inertia is negligible

$\Rightarrow$ y'' term becomes vanishingly small

$\Rightarrow$ expect singular perturbation expansion

Rescale variables: $y \to Ax$, $\tau \to \Omega t$

$$y'' + by' + \omega_0^2 y = \underbrace{\Omega^2 A}_{\varepsilon} \ddot{x} + \underbrace{b\Omega A}_{1} \dot{x} + \underbrace{\omega_0^2 A}_{1} x = 0, \quad \dot{x} = \frac{dx}{dt}$$

Solving obtain: $A = \omega_0^{-2}$, $\Omega = 1/(bA) = \omega_0^2/b$,

$$\varepsilon = \Omega^2 A = \frac{\omega_0^4}{b^2} \frac{1}{\omega_0^2} = \left(\frac{\omega_0}{b}\right)^2 \ll 1.$$

Rescaled equation: $\varepsilon \ddot{x} + \dot{x} + x = 0$

Initial conditions: $x(0) = \frac{1}{A} y(0) = \omega_0^2 y_0 \equiv \hat{x}_0$

$\dot{x}(0) = (\Omega A)^{-1} y'(0) = b u_0 \equiv \hat{v}_0$

Seek solution in the form of a perturbation expansion:

$x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$

$$\varepsilon(\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots) + (\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \dots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) =$$

$$= (\dot{x}_0 + x_0) + \varepsilon(\ddot{x}_0 + \dot{x}_1 + x_1) + \varepsilon^2(\ddot{x}_1 + \dot{x}_2 + x_2) + \dots = 0$$

At ε^0: $\dot{x}_0 + x_0 = 0 \Rightarrow x_0 = Ae^{-t}$

ε^1: $\dot{x}_1 + x_1 = -\ddot{x}_0 = -Ae^{-t} \Rightarrow x_1 = (B - At)e^{-t}$

ε^2: $\dot{x}_2 + x_2 = -\ddot{x}_1 = (A + B - At)e^{-t} \Rightarrow x_2 = (C - (B+2A)t + \frac{A}{2}t^2)e^{-t}$

Collecting everything:

$$x(t) = \left[A + \varepsilon(B - At) + \varepsilon^2\left(C - (B+2A)t + \tfrac{A}{2}t^2\right) + \cdots\right] e^{-t}$$

Boundary conditions: $x(0) = A + \varepsilon B + \varepsilon^2 C + \cdots = \hat{x}_0 \;, \; \forall \varepsilon$

$$\Rightarrow A = \hat{x}_0 \;, \; B = C = 0$$

$$\dot{x}(0) = \underbrace{-A - \varepsilon(A+B) - \varepsilon^2(2A+B+C) + \cdots}_{-\hat{x}_0(1+\varepsilon-2\varepsilon^2+\cdots) \neq \hat{v}_0 \,!} = \hat{v}_0$$

Problem: Cannot satisfy both initial conditions at once.

(Of course, the unperturbed problem ($\varepsilon = 0$) is 1st order ODE!)

Same as in algebraic equations: small coeff. at highest order.

Non-uniform convergence: $\lim\limits_{\varepsilon \to 0} \dot{x}(t,\varepsilon) = \dot{x}(t,0) \;, \; t > 0$

$$\lim_{\varepsilon \to 0} \dot{x}(0,\varepsilon) \neq \dot{x}(0,0)$$

Goal: try to build an asymptotic expansion, such that

1) $\lim\limits_{\varepsilon \to 0} x(t,\varepsilon) = x(t,0)$ 2) $\lim\limits_{\varepsilon \to 0} \dot{x}(t,\varepsilon) = \dot{x}(t,0)$, $\forall t \geq 0$

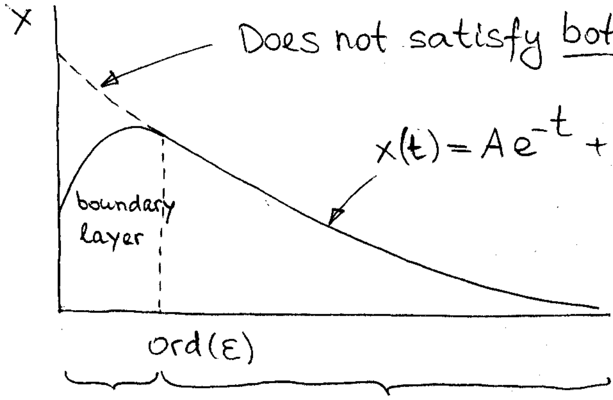
Boundary layers

First, find the characteristic time scales:

$$\varepsilon \ddot{x} + \dot{x} + x = 0 \iff \begin{cases} \dot{x} = v & : \quad x, v = \mathrm{ord}(1) \Rightarrow t = \mathrm{ord}(1), \\ \dot{v} = -\frac{1}{\varepsilon}(x+v) & \Rightarrow t = \mathrm{ord}(\varepsilon) \end{cases}$$

(unless $x + v = \mathrm{ord}(\varepsilon)$)

Generically, $x(0) + v(0) = x(0) + \dot{x}(0) = \hat{x}_0 + \hat{v}_0 = \mathrm{ord}(1)$



Does not satisfy both boundary conditions

$x(t) = Ae^{-t} + O(\varepsilon)$, such that $x + \dot{x} = \text{ord}(\varepsilon)$

"inner solution" (fast transient) "outer solution" (slow relaxation)

<u>outer solution:</u> $t = \text{ord}(1)$: $\varepsilon\ddot{x} + \dot{x} + x = 0$ – already solved!

Not valid for small t $\Rightarrow$ No boundary conditions

<u>inner solution:</u> $t = \text{ord}(\varepsilon) \Leftrightarrow t = \varepsilon T$, $T = \text{ord}(1)$

$$\varepsilon\ddot{x} + \dot{x} + x = \varepsilon\frac{1}{\varepsilon^2}\frac{d^2}{dT^2}X + \frac{1}{\varepsilon}\frac{d}{dT}X + X = \frac{1}{\varepsilon}\left(\frac{d^2X}{dT^2} + \frac{dX}{dT} + \varepsilon X\right) = 0$$

$$\begin{cases} \dfrac{d^2X}{dT^2} + \dfrac{dX}{dT} + \varepsilon X = 0 \\ X(0) = \hat{X}_0 \\ \dfrac{dX}{dT}(0) = \varepsilon\dot{x}(0) = \varepsilon\hat{V}_0 \end{cases}$$ – regular 2nd order problem!

$$X(T,\varepsilon) = X_0(T) + \varepsilon X_1(T) + \varepsilon^2 X_2(T) + \cdots$$

a) $\hat{X}_0 = X(0) = X_0(0) + \varepsilon X_1(0) + \varepsilon^2 X_2(0) + \cdots \Rightarrow X_0(0) = \hat{X}_0,$
$X_1(0) = X_2(0) = \cdots = 0$

b) $\varepsilon\hat{V}_0 = \frac{dX}{dT}(0) = \frac{dX_0}{dT}(0) + \varepsilon\frac{dX_1}{dT}(0) + \varepsilon^2\frac{dX_2}{dT}(0) + \cdots \Rightarrow \frac{dX_1}{dT}(0) = \hat{V}_0$
$\frac{dX_0}{dT}(0) = \frac{dX_2}{dT}(0) = \cdots = 0$

Substitute into the equation:

$$\frac{d^2X_0}{dT} + \varepsilon\frac{d^2X_1}{dT} + \varepsilon^2\frac{d^2X_2}{dT} + \cdots + \frac{dX_0}{dT} + \varepsilon\frac{dX_1}{dT} + \varepsilon^2\frac{dX_2}{dT} + \cdots + \varepsilon X_0 + \varepsilon^2 X_1 + \cdots = 0$$

ε^0: $\frac{d^2X_0}{dT} + \frac{dX_0}{dT} = 0$, $X_0(0) = \hat{X}_0$, $\frac{dX_0}{dT}(0) = 0 \Rightarrow X_0(T) = \hat{X}_0$

ε^1: $\frac{d^2X_1}{dT^2} + \frac{d^2X_1}{dT^2} = -X_0(T) = -\hat{X}_0$

$$x_1(0)=0, \quad \frac{dx_1}{dT}(0)=V_0 \Rightarrow x_1(T)=(\hat{x}_0+\hat{v}_0)(1-e^{-T})-\hat{x}_0 T$$

ε^2: $$\frac{d^2x_2}{dT^2}+\frac{dx_2}{dT}=-x_1(T)$$

$$x_2(0)=0, \quad \frac{dx_2}{dT}(0)=0 \Rightarrow x_2(T)=3\hat{x}_0+2\hat{v}_0-(2\hat{x}_0+\hat{v}_0)T+\frac{1}{2}\hat{x}_0T^2$$
$$-\left(3\hat{x}_0+2\hat{v}_0-(\hat{x}_0+\hat{v}_0)T\right)e^{-T}$$

Substitute $T=t/\varepsilon$, collect terms of different order in ε:

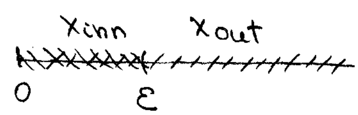
Inner solution: (valid for $t < O(\varepsilon)$)

$$x_{inn}(t,\varepsilon)=x_0\left(1-t+\frac{1}{2}t^2\right)+\varepsilon\left(\hat{x}_0(1-2t)+\hat{v}_0(1-t)-(\hat{x}_0+\hat{v}_0)(1+t)e^{-t/\varepsilon}\right)+$$
$$+\varepsilon^2(3\hat{x}_0+2\hat{v}_0)(1-e^{-t/\varepsilon})+\dots$$

Outer solution: (valid for $t>0$)

$$x_{out}(t,\varepsilon)=Ae^{-t}+\varepsilon(B-At)e^{-t}+$$
$$+\varepsilon^2\left(C-(B+2A)t+\frac{1}{2}At^2\right)e^{-t}+\dots$$

Matched Asymptotic Expansions



$$\lim_{\varepsilon\to 0}\left(x_{out}(t,\varepsilon)-x_{inn}(t,\varepsilon)\right)=0$$
$$\lim_{\varepsilon\to 0}\left(\dot{x}_{out}(t,\varepsilon)-\dot{x}_{inn}(t,\varepsilon)\right)=0$$
for t such that both expansions are valid

Taylor-expand both x_{out} and x_{inn} for small t ($t<O(\varepsilon)$):

$$x_{inn}(t,\varepsilon)=x_0\left(1-t+\frac{1}{2}t^2\right)+\varepsilon\left(\hat{x}_0+\hat{v}_0-(2\hat{x}_0+\hat{v}_0)t\right)+\varepsilon^2(3\hat{x}_0+2\hat{v}_0)+\dots$$
$$x_{out}(t,\varepsilon)=A\left(1-t+\frac{1}{2}t^2\right)+\varepsilon\left(B-(A+B)t\right)+\varepsilon^2C+\dots$$

(Discard terms of order $\varepsilon^3, \varepsilon^2t, \varepsilon t^2, t^3$)

$$\Rightarrow A=\hat{x}_0, \quad B=\hat{x}_0+\hat{v}_0, \quad C=3\hat{x}_0+2\hat{v}_0$$

Coefficients of x_{out} are found through matching, rather than by using boundary conditions:

$$x_{out}(t,\varepsilon)=\left(\hat{x}_0+\varepsilon(\hat{x}_0+\hat{v}_0-\hat{x}_0t)+\varepsilon^2\left(3\hat{x}_0+2\hat{v}_0-(3\hat{x}_0+\hat{v}_0)t+\frac{1}{2}\hat{x}_0t^2\right)+\dots\right)e^{-t}$$

Piecewise approximation:

$$x(t,\varepsilon) = \begin{cases} x_{inn}(t,\varepsilon), & t < O(\varepsilon) \\ x_{out}(t,\varepsilon), & t > O(\varepsilon) \end{cases}$$

Can we find a single continuous approximation?

Notice that $O(1)$ terms in $x_{inn}(t,\varepsilon)$ and $x_{out}(t,\varepsilon)$ are the same:

$$x_0(1-t+\tfrac{1}{2}t^2) = x_0 e^{-t} + O(t^3)$$

Uniform approximation:

taylor series for $x_{out}(t,\varepsilon)$

$$x_{unif}(t,\varepsilon) = x_{out}(t,\varepsilon) + x_{inn}(t,\varepsilon) - x_{match}(t,\varepsilon) =$$

$$= (\hat{x}_0 + \varepsilon(\hat{x}_0 + \hat{v}_0 - \hat{x}_0 t) + \dots)e^{-t} + (\varepsilon(\hat{x}_0+\hat{v}_0)(-1-t) + \dots)e^{-t/\varepsilon}$$

Exact Solution:

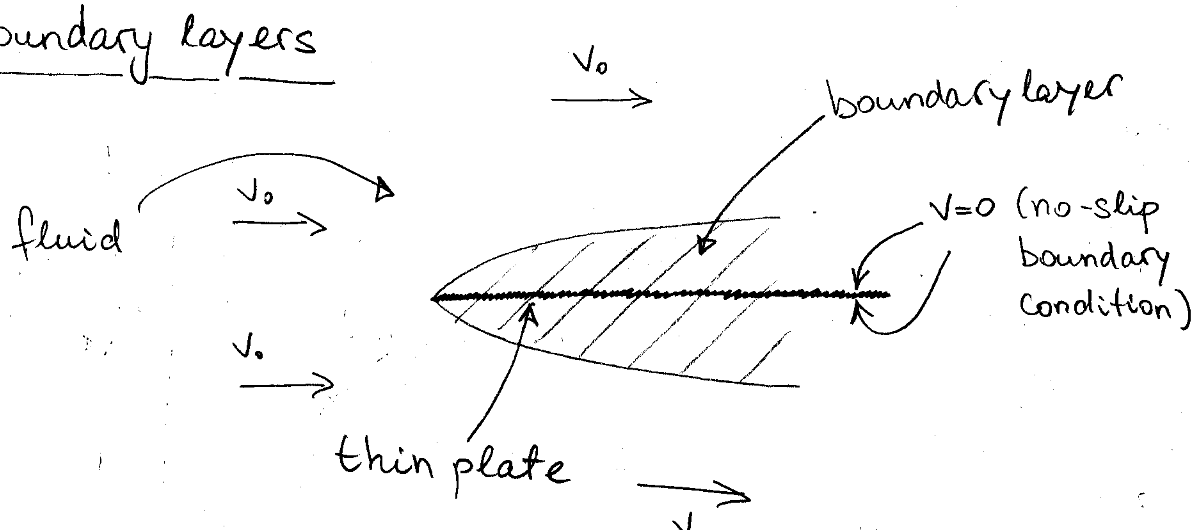
$$\varepsilon\ddot{x} + \dot{x} + x = 0, \quad x = e^{\lambda t} \Rightarrow \varepsilon\lambda^2 + \lambda + 1 = 0$$

We already solved this equation: $\lambda_1 = -1 + O(\varepsilon)$
$\lambda_2 = -\frac{1}{\varepsilon} + O(1)$

$$\Rightarrow x(t,\varepsilon) = Ae^{-t} + Be^{-t/\varepsilon} + \dots$$

slow relaxation   fast transient

Fluid boundary layers



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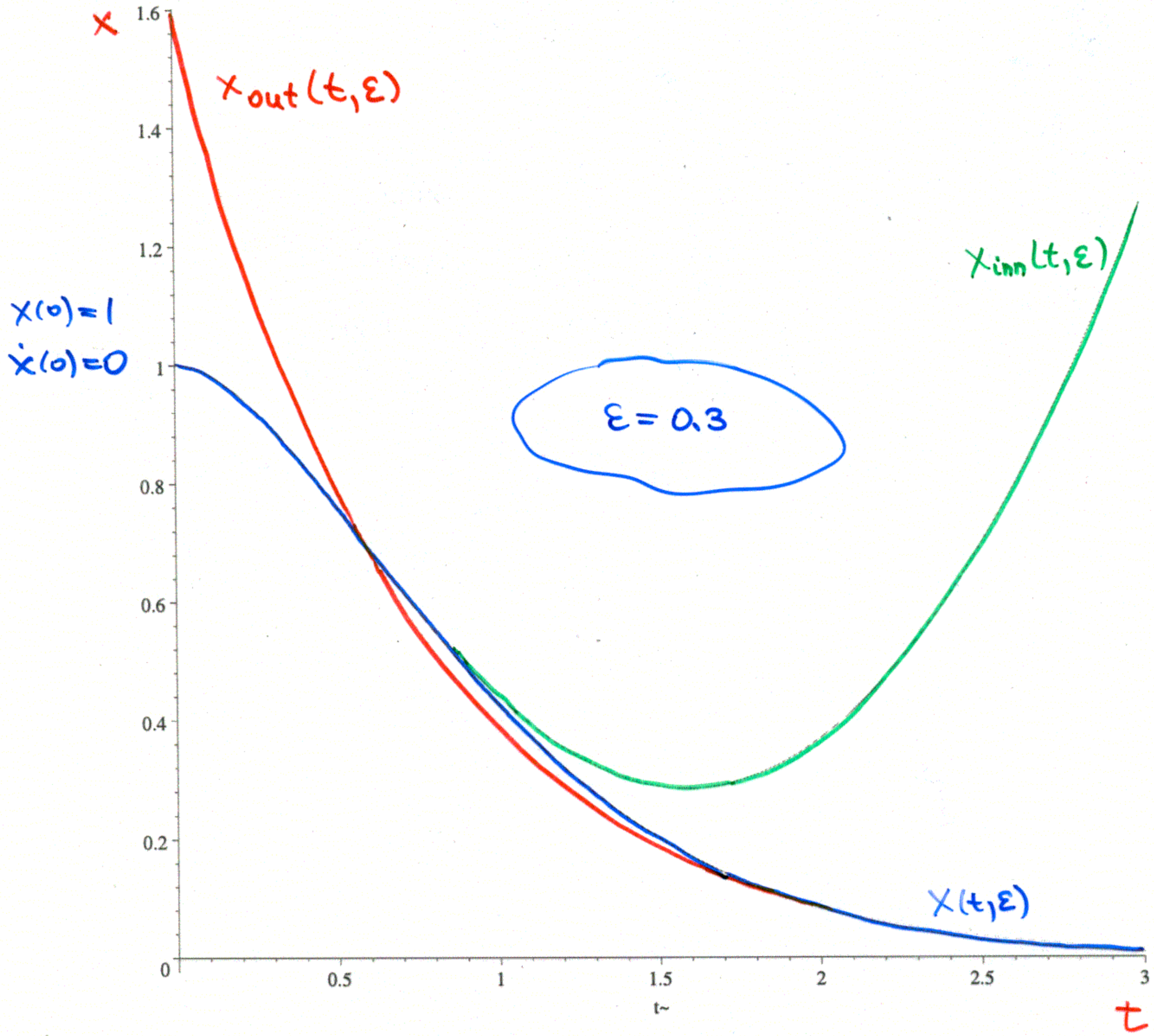
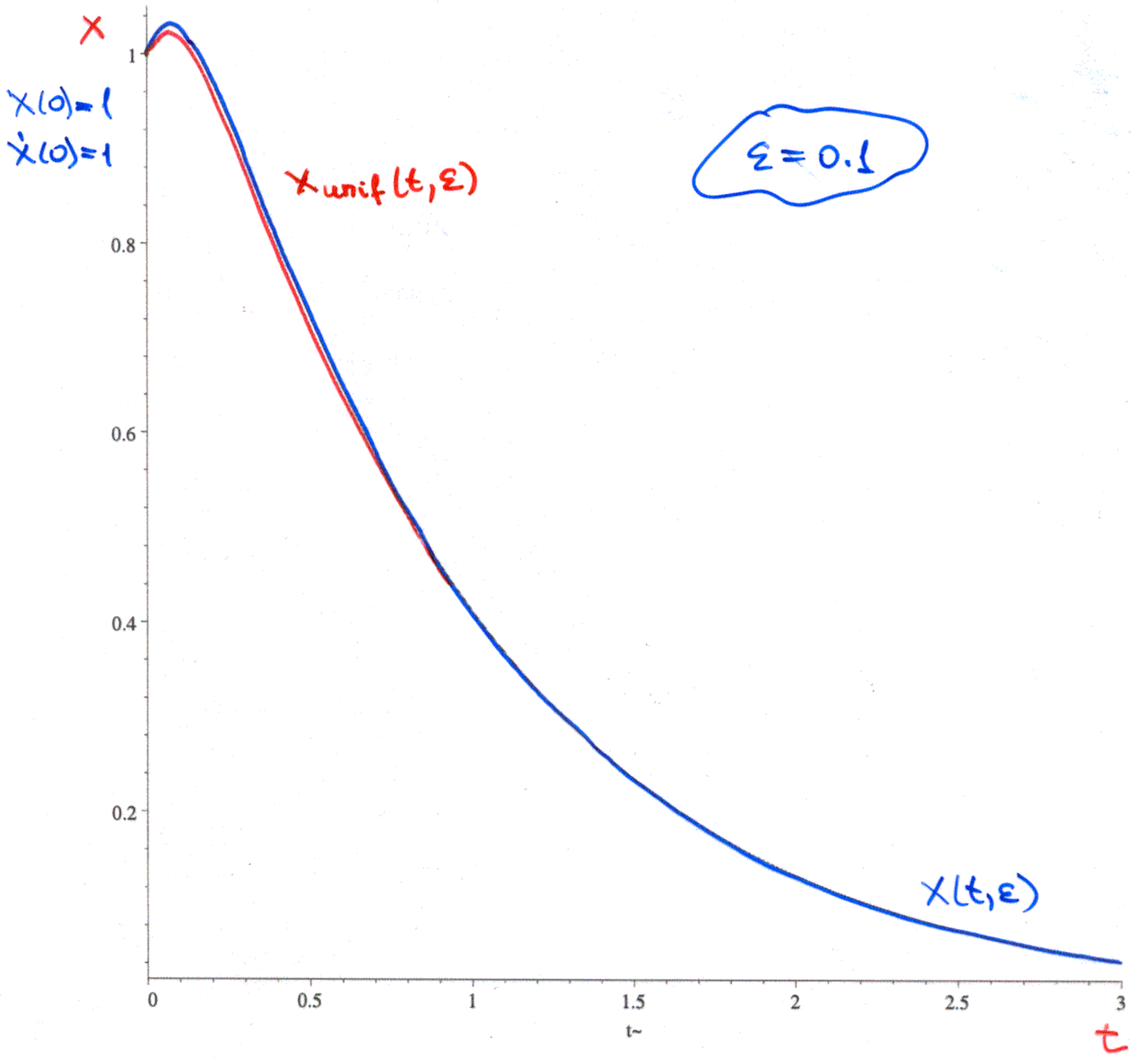

x
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$x_{out}(t,\varepsilon)$
$x_{inn}(t,\varepsilon)$
$x(0)=1$
$\dot{x}(0)=0$
$\varepsilon = 0.3$
$x(t,\varepsilon)$
0.5
1
1.5
2
2.5
3
t~
t


x
1
0.8
0.6
0.4
0.2
x(0)=1
ẋ(0)=1
x_unif(t,ε)
ε=0.1
x(t,ε)
0
0.5
1
1.5
2
2.5
3
t~
t