

## Adjoint Operators

Action of matrices is most conveniently described by using their eigenvalues and eigenvectors. Same is true of operators, with eigenvectors  $\rightarrow$  eigenfunctions:

$$\mathcal{L}\Psi_n(x) = \lambda_n\Psi_n(x)$$

The properties of  $\Psi_n(x)$  depend on the properties of  $\mathcal{L}$ , in particular, on whether  $\mathcal{L}$  is "symmetric".

$$\langle v | \mathcal{L} | u \rangle = \int_a^b v(x) \{ P_0(x) u''(x) + P_1(x) u'(x) + P_2(x) u(x) \} dx$$

Integrating by parts:

$$\textcircled{1} \int_a^b v(x) P_1(x) u'(x) dx = [v P_1 u]_a^b - \int_a^b [v P_1]' u dx$$

$$\begin{aligned} \textcircled{2} \int_a^b v(x) P_0(x) u''(x) dx &= [v P_0 u']_a^b - \int_a^b [v P_0]' u' dx \\ &= [v P_0 u']_a^b - [(v P_0)' u]_a^b + \int_a^b [v P_0]'' u dx \end{aligned}$$

so that

$$\begin{aligned} \langle v | \mathcal{L} | u \rangle &= \int_a^b [(P_0 v)'' u - (P_1 v)' u + (P_2 v) u] dx \\ &\quad + [v P_1 u + v P_0 u' - (v P_0)' u]_a^b \end{aligned}$$

Factor out  $u$  inside the integral and note that

$$-(v P_0)' u = -v P_0' u - v' P_0 u$$

such that

$$\langle v | \mathcal{L} | u \rangle = \langle \mathcal{L}^\dagger v | u \rangle + [v P_1 u + v P_0 u' - v' P_0 u - v P_0' u]_a^b$$

where

$$\mathcal{L}^\dagger v \equiv (P_0 v)'' + (P_1 v)' + P_2 v \quad \text{is an adjoint of } \mathcal{L}. \\ \text{(formal adjoint)}$$

# Self-adjoint operators

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Under what conditions  $\mathcal{L}^\dagger = \mathcal{L}$ ?

$$\mathcal{L}u = P_0 u'' + P_1 u' + P_2 u$$

$$\begin{aligned}\mathcal{L}^\dagger u &= (P_0 u)'' - (P_1 u)' + P_2 u = \\ &= P_0'' u + 2P_0' u' + P_0 u'' - P_1' u - P_1 u' + P_2 u \\ &= P_0 u'' + (2P_0' - P_1) u' + (P_0'' - P_1' + P_2) u\end{aligned}$$

Now,  $\mathcal{L}u = \mathcal{L}^\dagger u$  for arbitrary  $u$ , provided

$$\begin{cases} P_1 = 2P_0' - P_1 \\ P_0'' - P_1' = 0 \end{cases} \Leftrightarrow \boxed{P_1 = P_0'}$$

Definition:  $\mathcal{L}$  is formally self-adjoint (Hermitian), if  $\mathcal{L}^\dagger = \mathcal{L}$ .

What about the boundary value problem? For it to be self-adjoint not only  $\mathcal{L}^\dagger = \mathcal{L}$  is required, but bound. cond. have to be considered.

Remember,

$$\langle v | \mathcal{L}u \rangle = \langle \mathcal{L}^\dagger v | u \rangle + \underbrace{[\cancel{\sigma P_1 u} + \cancel{\sigma P_0 u'} - \sigma' P_0 u - \cancel{\sigma P_0' u}]_a^b}_{-\sigma P_1 u}$$

$$\Rightarrow \text{Need } [\sigma P_0 u' - \sigma' P_0 u]_a^b = 0.$$

If the b.c. on  $u$  are homogeneous, the entire BVP can be self-adjoint. For example, suppose

$$u(a) = u'(b) = 0$$

$$\begin{aligned}\text{Then } [\sigma P_0 u' - \sigma' P_0 u]_a^b &= \cancel{\sigma(b) P_0(b) u'(b)} - \sigma'(b) P_0(b) u(b) - \\ &\quad - \sigma(a) P_0(a) \cancel{u'(a)} + \sigma'(a) P_0(a) \cancel{u(a)}\end{aligned}$$

so that the choice  $\sigma(a) = \sigma'(b) = 0$  eliminates the remaining terms, and

$$\langle \sigma | \mathcal{L}u \rangle = \langle \mathcal{L}^\dagger \sigma | u \rangle \text{ exactly.}$$

Self-adjoint operators are of special interest for the same reason self-adjoint (Hermitian) matrices are:

- 1) They have special & useful properties
- 2) They often arise in physical problems

Any 2<sup>nd</sup> order linear ODE can be transformed into a self-adjoint form by multiplying by  $\sigma(x) = \exp\left[\int \frac{P_1 - P_0'}{P_0} dx\right]$ :

$$\mathcal{L}u = p \Rightarrow \underbrace{\sigma \mathcal{L}}_{\text{new diff. operator, which is self-adjoint}} u = \sigma p$$

$$(\sigma \mathcal{L})^+ = \sigma \mathcal{L} \Leftrightarrow \sigma p_1 = (\sigma p_0)' \Rightarrow \frac{\sigma'}{\sigma} = \frac{P_1 - P_0'}{P_0}$$

Example: Bessel's eq. is not formally self-adjoint:

$$x^2 u'' + x u' + (Ax^2 - B)u = 0$$

$$\rightarrow P_0' = (x^2)' = 2x \neq x = P_1$$

Multiplying by  $\sigma = 1/x$  converts the ODE into self-adjoint form:

$$x u'' + u' + (Ax - \frac{B}{x})u = 0$$

$$\rightarrow P_0' = (x)' = 1 = P_1$$

Alternative representation:

Any self-adjoint operator can be written in another form:

$$\begin{aligned} \mathcal{L}u &= p_0 u'' + p_1 u' + p_2 u = p_0 u'' + p_0' u' + p_2 u = (p_0 u')' + p_2 u \\ &= \frac{d}{dx} \left( p_0(x) \frac{d}{dx} u(x) \right) + p_2(x) u(x) \end{aligned}$$

Example: Bessel's eq. was originally obtained in the form

$$p \left( \partial_p'(p \partial_p R) + (pK^2 - \frac{1}{p}V)R \right) = 0 \quad - \text{just multiply by } p^{-1} \text{ to convert into alternative representation}$$

Example: (Legendre's eq.)

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$$(1-x^2)u'' - 2xu' + l(l+1)u = 0 \quad (x = \cos \theta)$$

$$P_0' = (1-x^2)' = -2x = P_1 \Rightarrow \text{formally self-adjoint!}$$

Alternative form:

$$\begin{aligned} (1-x^2)u'' + (1-x^2)'u' + l(l+1)u &= \\ &= ((1-x^2)u')' + l(l+1)u = 0. \end{aligned}$$

Sturm-Liouville (eigenvalue) problem

The eigenvalues of an arbitrary (not necessarily self-adjoint) operator  $\tilde{\mathcal{L}}$  are defined via

$$\tilde{\mathcal{L}} \Psi_n(x) = \lambda_n \Psi_n(x)$$

If we construct a self-adjoint operator  $\mathcal{L} = \sigma \tilde{\mathcal{L}}$ , this e-value problem is equivalent to  $\mathcal{L} \Psi_n(x) = \lambda_n w(x) \Psi_n(x)$   
 $\uparrow$  "weight function"  $w = \sigma(x)$

Self-adjoint operators were defined with respect to a scalar product

$$\langle u | \mathcal{L} | v \rangle = \int_a^b u(x) \mathcal{L} v(x) dx$$

Non-self-adjoint operators  $\tilde{\mathcal{L}} = \frac{1}{\sigma(x)} \mathcal{L}$  will satisfy

$\langle u | \tilde{\mathcal{L}} | v \rangle = \langle \tilde{\mathcal{L}}^+ u | v \rangle$ , if we redefine scalar product:

$$\langle u | \tilde{\mathcal{L}} | v \rangle \equiv \int_a^b u^*(x) \tilde{\mathcal{L}} v(x) w(x) dx = \int_a^b u^*(x) \frac{1}{\sigma(x)} \mathcal{L} v(x) \cancel{w(x)} dx$$

$$= \int_a^b \mathcal{L}^+ u^*(x) \cdot v(x) dx = \int_a^b \mathcal{L}^+ u^*(x) v(x) \frac{w(x)}{\sigma(x)} dx =$$

$$= \int_a^b \left( \frac{1}{\sigma(x)} \mathcal{L} \right)^+ u^*(x) v(x) w(x) dx = \langle \tilde{\mathcal{L}}^+ u | v \rangle$$

$$\Rightarrow \boxed{\langle u | v \rangle = \int_a^b u^*(x) v(x) w(x) dx, \quad w(x) = \sigma(x)}$$

If  $\mathcal{L}$  is self-adjoint and  $w(x)=1$  (or  $w(x)=\delta(x)$ ), then

1) eigenvalues  $\lambda_n$  are real:

$$\langle \psi_n | \mathcal{L} \psi_n \rangle = \langle \psi_n | \lambda_n w \psi_n \rangle = \lambda_n \langle \psi_n | w \psi_n \rangle = \lambda_n \int_a^b \psi_n^* \psi_n w dx$$

$$\langle \mathcal{L}^+ \psi_n | \psi_n \rangle = \langle \lambda_n^* w \psi_n | \psi_n \rangle = \lambda_n^* \int_a^b \psi_n^* \psi_n w dx \Rightarrow \lambda_n^* = \lambda_n$$

↑  
real!

2) eigenfunctions are orthogonal

a)  $\lambda_n \neq \lambda_m$ :

$$\langle \psi_m | \mathcal{L} \psi_n \rangle = \lambda_n \int_a^b \psi_m^* \psi_n w dx$$

$$\langle \mathcal{L}^+ \psi_m | \psi_n \rangle = \lambda_m^* \int_a^b \psi_m^* \psi_n w dx = \lambda_m \int_a^b \psi_m^* \psi_n w dx$$

b)  $\lambda_m = \lambda_n$ ,  $m \neq n$  can find linear combinations of  $\psi_m, \psi_n$  which are orthogonal.

3) the full set of eigenfunctions is complete, so that for "any"  $f(x)$  we can find coeffs.  $c_n$ , such that

$$f(x) = \sum_n c_n \psi_n(x)$$

Note: This is just an  $\infty$ -dim. analog of the fact that an  $N \times N$  Hermitian matrix has  $N$  eigenvectors spanning  $\mathbb{R}^N$ .

Remember the Gram-Schmidt orthogonalization procedure we have developed for matrices? It works here just as well!

Let us find the coefficients  $c_n$ :

$$\int_a^b \psi_m(x) f(x) w(x) dx = \int_a^b \psi_m(x) \sum_n c_n \psi_n(x) w(x) dx =$$

$$= \sum_n c_n \underbrace{\int_a^b \psi_m(x) \psi_n(x) w(x) dx}_{\delta_{nm}} = c_m$$

$\delta_{nm}$  (if  $\psi_n, \psi_m$  are also normalized)

$$c_m = \int_a^b \psi_m(x) f(x) w(x) dx \quad \leftarrow \text{These are often called } \underline{\text{generalized Fourier coefficients of } f(x)}$$

Many similar properties, such as Parseval's identity:

$$\int_a^b f(x) g(x) w(x) dx = \int_a^b \sum_n F_n \psi_n(x) \sum_m G_m \psi_m(x) w(x) dx =$$

$$= \sum_{n,m} F_n G_m \int_a^b \psi_n(x) \psi_m(x) w(x) dx = \sum_n F_n G_n$$

$$\Rightarrow \int_a^b |f(x)|^2 w(x) dx = \sum_n |F_n|^2$$

Completeness relation (for eigenfunctions of self-adjoint operators)

For any  $f(x)$ :

$$f(x) = \sum_n F_n \psi_n(x) = \sum_n \int_a^b \psi_n(y) f(y) dy \psi_n(x) =$$

$$= \int_a^b \sum_n \psi_n(y) \psi_n(x) f(y) dy = \int_a^b \delta(x-y) f(y) dy$$

$$\Rightarrow \delta(x-y) = \sum_n \psi_n(x) \psi_n(y)$$