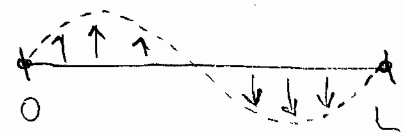
Example: (Normal modes of a string)



Consider the 1-D wave equation:

$$\frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial x^2} \; ; \quad \Psi(0,t) = \Psi(L,t) = 0$$

$$v^2 = \frac{\text{string tension}}{\text{linear density}}$$

The normal modes are $\Psi(x,t) = u(x)e^{i\omega t}$,

that is all points on the string oscillate transversely with the same frequency ω. Substitute into PDE:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left(u(x)e^{i\omega t}\right) = \frac{\partial^2}{\partial x^2}\left(u(x)e^{i\omega t}\right)$$

$$\Rightarrow \quad -\frac{\omega^2}{v^2}u(x)\cancel{e^{i\omega t}} = u''(x)\cancel{e^{i\omega t}}$$

Boundary value problem:

$$\begin{cases} u'' = -\dfrac{\omega^2}{v^2}u & \leftarrow \text{2}^{\text{nd}} \text{ order PDE} \\ u(0) = u(L) = 0. & \leftarrow \text{boundary conditions} \end{cases}$$

$\hookrightarrow$ $\mathcal{L}[u] = \lambda_n w u_n$ with $w(x) = 1$, because

$\mathcal{L} = \frac{\partial^2}{\partial x^2}$ – formally self-adjoint (Hermitian)

The eigenvalues are quantized due to the b.c.:

$$u(x) = A\cos\left(\frac{\omega}{v}x\right) + B\sin\left(\frac{\omega}{v}x\right)$$

$u(0) = 0 \Rightarrow A = 0$

$u(L) = 0 \Rightarrow \frac{\omega}{v}L = \pi n, \quad n = \cancel{0}, \pm 1, \pm 2, \ldots$

$\Rightarrow \omega_n = \frac{v}{L}\pi n$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{eigenfunction}}}{u_n(x)} = \sin\left(\frac{\pi n}{L}x\right), \quad \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\lambda_n} = \frac{\mathcal{L}[u_n]}{u_n} = -\left(\frac{\pi n}{L}\right)^2$$

Example: (Normal modes of a round drum head)

"2D string" or membrane $\Rightarrow$ 2D wave equation:

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \quad \text{with } \psi = 0 \text{ for } r = a$$

Boundary conditions w/cylindrical symmetry $\Rightarrow$ use polar coordinates to write normal mode:

$$\psi(r,\varphi,t) = u(r,\varphi)\, e^{i\omega t}$$

$$\Rightarrow \underbrace{\frac{1}{v^2} u(r,\varphi) \frac{\partial^2}{\partial t^2}\left(e^{i\omega t}\right)}_{-\frac{\omega^2}{v^2} u(r,\varphi) e^{i\omega t}} = e^{i\omega t}\, \nabla^2 u(r,\varphi)$$

$$\Rightarrow \nabla^2 u + \frac{\omega^2}{v^2} u = 0$$

Helmholtz equation in 2D!

Separation of variables in polar coordinates:

$$u(r,\varphi) = R(r)\,\Phi(\varphi) \Rightarrow \begin{cases} \Phi'' + c\,\Phi = 0 \\ (rR')' + \left(\frac{\omega^2}{v^2} r - \frac{c}{r}\right) R = 0. \end{cases}$$

$$\Phi = A\cos\sqrt{c}\,\varphi + B\sin\sqrt{c}\,\varphi, \quad \Phi(2\pi+\varphi) = \Phi(\varphi) \rightarrow c = m^2,\ m = 0, 1, 2, \ldots$$

$$\Rightarrow \begin{cases} (rR')' + \left(\frac{\omega^2}{v^2} r - \frac{m^2}{r}\right) R = 0 & \leftarrow \text{Bessel's eq.}/r \\ R(a) = 0 \quad (\text{and } R(0) < \infty) \end{cases}$$

Rewrite Bessel's eq. as an eigenfunction problem:

$$\underbrace{(rR')' - \frac{m^2}{r} R}_{\mathcal{L}[R]} = \underbrace{-\frac{\omega^2}{v^2}}_{\lambda} \cdot \underbrace{r}_{w(r)} \cdot R$$

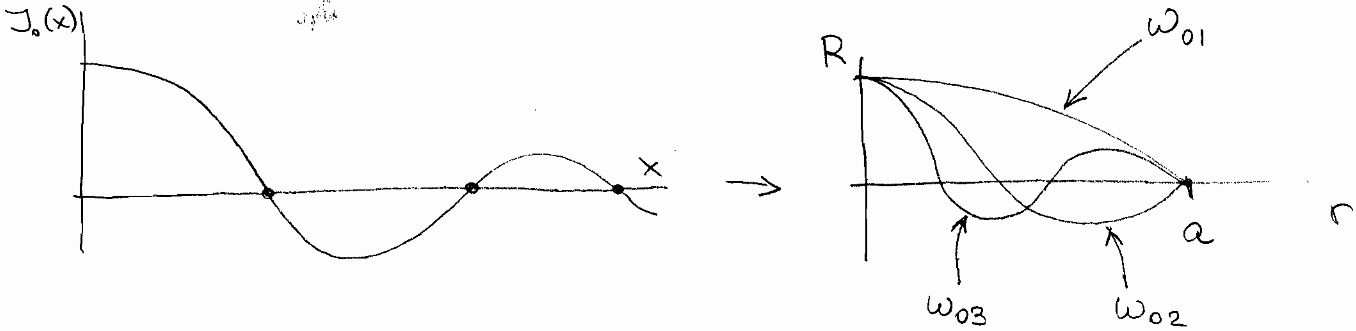
Note: $w(r) = r \neq 1$! (why?)

For a given m there exists a set of eigenvalues $\lambda_{m,n}$ correspond. to different oscillation frequencies $\omega_{m,n} = v\sqrt{-\lambda_{m,n}}$.

The eigenfunctions are the Bessel functions:

$$R_{m,n}(r) = J_m\left(\frac{\omega}{v} r\right), \quad m = 0, 1, 2, \dots$$

$$R_{m,n}(a) = 0 \Rightarrow \boxed{J_m\left(\frac{a}{v}\omega_{m,n}\right) = 0}$$



For each m we get a sequence $\omega_{m0}, \omega_{m1}, \omega_{m2}, \dots = \{\omega_{m,n}\}$

To find $\omega_{m,n}$ explicitly need to solve $\boxed{\text{eq.}}$:

$J_0(x) = 0$: $\quad x = 2.40,\ 5.52,\ 8.65, \dots$

$J_1(x) = 0$: $\quad x = 3.83,\ 7.02,\ 10.17, \dots$

...

(More Bessel function stuff to come later in the course)

$\Rightarrow \omega_{01} = (2.40)\frac{v}{a}, \quad \omega_{02} = (5.52)\frac{v}{a}, \dots$ and so on.

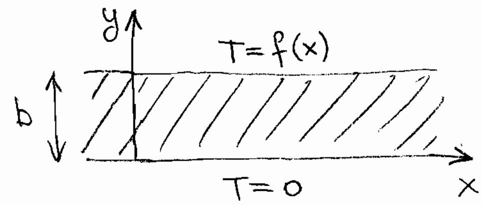
Summarizing:

normal modes are $\Psi_{m,n}(r,\varphi,t) = J_m\left(\frac{\omega_{mn}}{v} r\right) \cdot \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} e^{i\omega_{m,n} t}$

<u>Example</u> (Temperature distribution in a slab)

In steady state $\partial_t T = \varkappa \nabla^2 T = 0$

$\Rightarrow$ Laplace's eq: $\nabla^2 T = \partial_x^2 T + \partial_y^2 T = 0$



Separate variables: $T(x,y) = X(x)Y(y)$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow X'' + k^2 X = 0, \quad Y'' - k^2 Y = 0$$

i.e. we need to solve an eigenvalue equation:

$$\frac{d^2}{dx^2}\Psi_n = \lambda_n \Psi_n. \quad (w(x)=1, \text{ because } \frac{d^2}{dx^2} \text{ is self-adjoint})$$

The eigenfunctions are
$$\Psi_n = e^{\pm i\sqrt{-\lambda_n}\,x}, \quad \lambda_n < 0$$
$$= e^{\pm\sqrt{\lambda_n}\,x}, \quad \lambda_n > 0$$

Eigenvalues have continuum spectrum $-\infty < \lambda < \infty$ <u>in the absense of boundary conditions.</u>

In our case $\lambda_n = -k^2 < 0$ for x-coord.

$\lambda_n = +k^2 > 0$ for y-coord

Eigenfunctions satisfying boundary conditions are

$$\begin{cases} X_k(x) = e^{ikx}, & -\infty < k < \infty \\ Y_k(y) = A_k \sinh(ky) + B_k \cosh(ky), & B_k = 0 \quad (T(x,0)=0 \Rightarrow Y_k(0)=0) \end{cases}$$

Collecting everything together: $T(x,y) = \int A_k \sinh(ky)\, e^{ikx}\, dk$

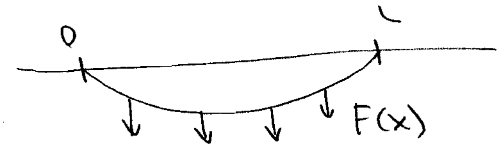
Taking Fourier transform of b.c. at $y=b$:

$$2\pi f_q = \int e^{-iqx} f(x)\,dx = \int e^{-iqx}\, T(x,b)\,dy = \int dk \underbrace{\int dx\, e^{i(k-q)x}}_{2\pi\delta(k-q)} A_k \sinh(kb)$$

$$= 2\pi A_q \sinh(qb)$$

$$\Rightarrow A_k = \frac{f_k}{\sinh(kb)} \Rightarrow T(x,y) = \int f_k \frac{\sinh(ky)}{\sinh(kb)} e^{ikx}\, dk$$

<u>Example</u> (Loaded string)



Force $F(x)$ per unit length of the string (wind, gravity, etc.)

Need to solve: $\tau \Psi''(x) = F(x)$, $\Psi(0) = \Psi(L) = 0$

Can "easily" solve by direct integration:

$$\Psi(x) = \int_0^x dy \int_0^y dz \frac{1}{\tau} F(z) \quad (\Psi(L) \stackrel{?}{=} 0)$$

However, let's illustrate the eigenfunction approach:

Pretend we don't know the eigenfunctions $\varphi_n(x)$ of our differential operator $\mathcal{L} = \frac{d^2}{dx^2}$. We do know that $\mathcal{L}^+ = \mathcal{L}$, so the weight $w(x) = 1$, and

$$\frac{d^2}{dx^2}\varphi_n = \lambda_n \varphi_n \;, \quad \int_0^L \varphi_n(x)\varphi_m(x)\,dx = \delta_{nm}$$

Find generalized Fourier coeff's of $F(x)$:

$$F(x) = \sum_n c_n \varphi_n(x) \;, \quad c_n = \int_0^L \varphi_n(x) F(x)\,dx$$

Let $\psi = \sum_n a_n \varphi_n(x)$:

$$\tau \frac{d^2\psi}{dx^2} = \tau \frac{d^2}{dx^2}\sum_n a_n \varphi_n = \tau \sum_n \lambda_n a_n \varphi_n = \sum_n c_n \varphi_n = F$$

$$\Rightarrow \tau a_n \lambda_n = c_n \Rightarrow a_n = \frac{c_n}{\tau \lambda_n} \quad \Rightarrow \boxed{\psi(x) = \frac{1}{\tau}\sum_n \frac{c_n}{\lambda_n}\varphi_n(x)}$$

We have put off solving for $\lambda_n, \varphi_n(x)$ as long as we could! Now assume $F(x) = F_0 = \text{const}$ and solve:

$$\varphi_n(x) = D_n \sin\left(\sqrt{-\lambda_n}\,x\right), \quad \varphi_n(0) = \varphi_n(L) = 0 \Rightarrow \lambda_n = -\left(\frac{\pi n}{L}\right)^2$$

$$\int_0^L \varphi_n^2(x)\,dx = D_n^2 \int_0^L \sin^2\left(\frac{\pi n}{L}x\right)dx = D_n^2 \frac{L}{2} = 1 \Rightarrow D_n = \sqrt{\frac{2}{L}}$$

$$c_n = \int_0^L F(x)\varphi_n(x)\,dx = \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n}{L}x\right)F_0\,dx = \sqrt{\frac{2}{L}}F_0\left(-\frac{L}{\pi n}\right)\cos\left(\frac{\pi n}{L}x\right)\Bigg|_0^L =$$

$$= \sqrt{\frac{2}{L}}F_0\left(-\frac{L}{\pi n}\right)\left(\cos\pi n - \cos 0\right) = -\frac{\sqrt{2L}\,F_0}{\pi n}\left[(-1)^n - 1\right] = \begin{cases} +2F_0\dfrac{\sqrt{2L}}{\pi n}, & n-\text{odd} \\ 0, & n-\text{even} \end{cases}$$

$$\Rightarrow \psi(x) = \frac{1}{\tau}\sum_{n=0}^{\infty}\frac{c_{2n+1}}{\lambda_{2n+1}}\varphi_{2n+1}(x) =$$

$$= -2\frac{F_0\sqrt{2L}}{\tau}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\pi}\left(\frac{L}{(2n+1)\pi}\right)^2\sqrt{\frac{2}{L}}\sin\frac{(2n+1)\pi}{L}x$$

$$= -\frac{4F_0}{L\tau}\sum_{n=0}^{\infty}\left(\frac{L}{(2n+1)\pi}\right)^3\sin\frac{(2n+1)\pi}{L}x$$

More generally: $\quad \underbrace{L|\psi\rangle = \sum_n |\varphi_n\rangle\lambda_n\langle\varphi_n|}_{L}\psi\rangle = |f\rangle \Rightarrow |\psi\rangle = \underbrace{\sum_n |\varphi_n\rangle\lambda_n^{-1}\overbrace{\langle\varphi_n|f\rangle}^{c_n}}_{L^{-1}}$