Problem 1.

$$A=\begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

The secular equation is: $\det|A-\lambda I| = (2-\lambda)^3$

$\Rightarrow$ The eigenvalues are: $\lambda_1=\lambda_2=\lambda_3=2$ (degenerate)

The eigenvalue equation yields:

$$x+y=0; \quad z=0 \quad ; \quad x+y+z=0$$

$\Rightarrow$ $\vec{e} = x\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ (only one eigenvector).

For convenience, we choose: $\vec{e}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\Rightarrow$ $\dim(\ker(A-\lambda I)) = 1$

$\ker(A-\lambda I)$ is the eigenspace for the eigenvalue λ.

The generalized λ-eigenspace is $\ker[(A-\lambda I)^n]$ with the corresponding secular equation: $\det|A-\lambda I| = (t-\lambda)^n g(t)$.

$\Rightarrow$ $\ker(A-\lambda I) \subset \ker[(A-\lambda I)^2] \subset \cdots \subset \ker[(A-\lambda I)^n]$

For this case, we cannot have $\dim(\ker[(A-\lambda I)^2]) = 3$, so extend $\vec{e}_1$ to a basis of $\ker[(A-\lambda I)^2]$ given by $\vec{e}_1, \vec{e}_2$ (i.e. find such $\vec{e}_2$).

Let $\vec{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Now, $\ker[(A-\lambda I)^3]$ must be $\mathbb{R}^3$.

so extend $\vec{e}_1$ and $\vec{e}_2$ to a basis of $\mathbb{R}^3$ by, for example,

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now modify this basis by replacing $\vec{e}_2$ by

$$\vec{e}_2{}' = (A-\lambda I)\vec{e}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and by replacing $\vec{e}_1$ with

$$\vec{e}_1{}' = (A-\lambda I)\vec{e}_2{}' = (A-\lambda I)^2\vec{e}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

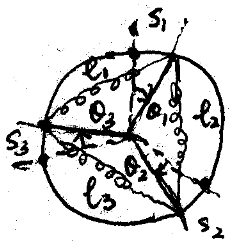
$\Rightarrow$ $\vec{e}_1{}'$ still gives a basis of $\ker(A-\lambda I)$ and $\vec{e}_1{}', \vec{e}_2{}'$ together still give a basis of $\ker[(A-\lambda I)^2]$. So $\vec{e}_1{}', \vec{e}_2{}'$ and $\vec{e}_3$ give a basis of $\mathbb{R}^3$.

Form the matrix: $P = (\vec{e}_1{}' \ \vec{e}_2{}' \ \vec{e}_3) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

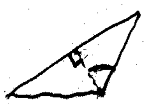
$$\Rightarrow P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 2.



S_1, S_2, S_3 are the coordinates of three beads along the circle. θ_1, θ_2 and θ_3 are the corresponding angle coordinates.



$$\Rightarrow S_1 = a\theta_1, \quad S_2 = a\theta_2, \quad S_3 = a\theta_3.$$

$$\Delta l_1 = l_1 - l_0 = 2a\left[\sin\left(\frac{\pi}{3} + \frac{\theta_1 - \theta_3}{2}\right) - \sin\frac{\pi}{3}\right]$$

$$\approx 2a\left[\sin\frac{\pi}{3} + \cos\frac{\pi}{3} \cdot \frac{(\theta_1 - \theta_3)}{2} - \sin\frac{\pi}{3}\right]$$

$$= -\frac{(\theta_3 - \theta_1)}{2} a = -\frac{S_3 - S_1}{2}$$

$$\Delta l_2 = -\frac{(\theta_1 - \theta_2)}{2} a = -\frac{(S_1 - S_2)}{2}, \quad \Delta l_3 = -\frac{(\theta_2 - \theta_3)}{2} a = -\frac{(S_2 - S_3)}{2}$$

$\Rightarrow$ The kinetic energy of the system is:

$$T = \frac{m}{2}\left(\dot{S}_1^2 + \dot{S}_2^2 + \dot{S}_3^2 \frac{\mu}{m}\right)$$

The potential energy is:

$$V = \frac{k}{2}\left[(\Delta l_1)^2 + (\Delta l_2)^2 + (\Delta l_3)^2\right] = \frac{k}{8}\left[(S_3 - S_1)^2 + (S_1 - S_2)^2 + (S_2 - S_3)^2\right]$$

$$= \frac{k}{2}\left[\frac{S_1^2}{2} + \frac{S_2^2}{2} + \frac{S_3^2}{2} - \frac{S_1 S_2}{2} - \frac{S_2 S_3}{2} - \frac{S_3 S_1}{2}\right]$$

Let $m = 1$, $k = 1$. We see:

$$[M] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu/m \end{pmatrix}, \quad [K] = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

with $2T = [\dot{S}]^T [M] [\dot{S}]$, $2V = [S]^T [K] [S]$

$\Rightarrow$ The secure equation is

$$\left| [K] - [M]\,\omega^2 \right| = 0$$

With the normal modes: $S_i = S_{i0} e^{i\omega t}$ $\quad i=1,2,3$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}-\omega^2 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2}-\omega^2 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2}-\frac{M}{m}\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{3}{16}\frac{M}{m}\omega^2 - \frac{3}{8}\omega^2 + \left(\frac{1}{2}+\frac{M}{m}\right)\omega^4 - \frac{M}{m}\omega^6 = 0$$

$$\Rightarrow \omega_1^2 = 0, \quad \omega_2^2 = \frac{3}{4}, \quad \omega_3^2 = \frac{1}{4}\left(1+\frac{2m}{\mu}\right) \qquad \left(\text{Unit: } \frac{k}{m}\right)$$

Substituting ω_i $(i=1,2,3)$ into $([K]-[M]\omega^2)[S]=0$:

$$\omega_1^2 = 0: \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{21} \\ S_{31} \end{pmatrix} = 0 \Rightarrow S_{11} = S_{21} = S_{31} = S$$

$$[S]_1 = \begin{pmatrix} S \\ S \\ S \end{pmatrix}$$

$$\omega_2^2 = \frac{3}{4}: \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2}-\frac{3M}{4m} \end{pmatrix} \begin{pmatrix} S_{12} \\ S_{22} \\ S_{32} \end{pmatrix} = 0 \Rightarrow S_{32} = 0, \quad S_{12} = -S_{22} = \beta$$

$$[S]_2 = \begin{pmatrix} \beta \\ -\beta \\ 0 \end{pmatrix}$$

$$\omega_3^2 = \frac{1}{4}\left(1+\frac{2m}{\mu}\right): \begin{pmatrix} \frac{1}{4}-\frac{m}{2\mu} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4}-\frac{m}{2\mu} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{M}{4m} \end{pmatrix} \begin{pmatrix} S_{13} \\ S_{23} \\ S_{33} \end{pmatrix} \Rightarrow S_{13} = S_{23} = \alpha, \quad S_{33} = -\frac{2m}{\mu}\alpha$$

$$\Rightarrow [S] = \begin{bmatrix} s & \beta & \alpha \\ s & -\beta & \alpha \\ s & 0 & -\frac{2m}{\mu}\alpha \end{bmatrix}$$

Applying the normalized orthogonal condition:

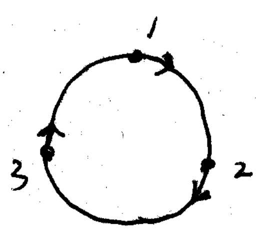
$$[S]^T [M] [S] = I$$

$$\Rightarrow \begin{cases} (2m+\mu)s^2 = 1 \\ 2m\beta^2 = 1 \\ \left(2m - \frac{4m^2}{\mu}\right)\alpha^2 = 1 \end{cases} \Rightarrow \begin{cases} s = \frac{1}{\sqrt{2m+\mu}} \\ \beta = \frac{1}{\sqrt{2m}} \\ \alpha = \frac{1}{\sqrt{2m + \frac{4m^2}{\mu}}} \end{cases}$$

$$\Rightarrow \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2m+\mu}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m+\frac{4m^2}{\mu}}} \\ \frac{1}{\sqrt{2m+\mu}} & -\frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m+\frac{4m^2}{\mu}}} \\ \frac{1}{\sqrt{2m+\mu}} & 0 & -\frac{2m}{\mu\sqrt{2m+\frac{4m^2}{\mu}}} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

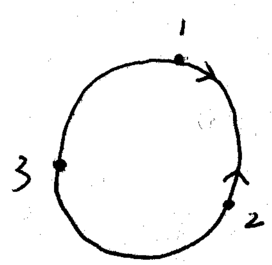
With $\eta_1 = a + bt$; $\eta_2 = C_2 \sin\left(\frac{3k}{4m}t + \alpha_2\right)$; $\eta_3 = C_3 \sin\left[\left(1+\frac{2m}{\mu}\right)\frac{k}{4m}t + \alpha_3\right]$

The normal modes.

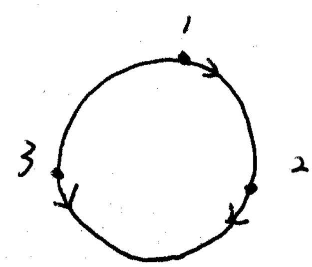


(I)

pure translation



(II)



(III)

(II) The two masses 1 and 2 are moving together. The mass 3 is stationary

(III) The two masses 3 and 2 are moving together. The mass 1 is moving along the same direction as mass 2.

Problem 2

a) $$y'''' + 4y''' + 5y'' + 2y' = 0$$

$$(y''' + 4y'' + 5y' + 2y) = C$$

Because $y(0) = 1$. $y'(0) = -1$. $y''(0) = 2$. $y'''(0) = -5$

$$\Rightarrow C = 0$$

i.e. $$y''' + 4y'' + 5y' + 2y = 0$$

Define

$y = y_1$

$y' = y_2$

$y'' = y_3$

then $\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $\begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ -2y_1 - 5y_2 - 4y_3 \end{pmatrix}$

$$\therefore \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\boxed{\overleftrightarrow{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix}}$$

b)

$$\det\left|\overleftrightarrow{A}-\lambda \mathbb{I}\right| = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -5 & -4-\lambda \end{pmatrix}$$

$$= \lambda^2(-4-\lambda) - 2 - 5\lambda = 0$$

$$\Rightarrow (\lambda+2)(\lambda+1)^2 = 0 \qquad \text{i.e. } \boxed{\lambda_1=\lambda_2=-1. \quad \lambda_3=-2.}$$

when $\lambda_1=-1$.

$$\overleftrightarrow{A}\cdot\vec{y} = -\vec{y} \Rightarrow \boxed{\vec{y}=\begin{pmatrix}1\\-1\\1\end{pmatrix}}$$

when $\lambda_2=-1$.

$$\overleftrightarrow{A}\cdot\vec{y} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + (-1)\vec{y} \Rightarrow \boxed{\vec{y}=\begin{pmatrix}1\\0\\-1\end{pmatrix}}$$

when $\lambda_3=-2$.

$$\overleftrightarrow{A}\cdot\vec{y} = -2\vec{y} \Rightarrow \boxed{\vec{y}=\begin{pmatrix}1\\-2\\4\end{pmatrix}}$$

c). Jordan normal form.

$$\Lambda = S^{-1}AS = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 4 & 1 & -1 \end{pmatrix}$$

d). $\vec{u} = S^{-1}\vec{y}$

$$\Rightarrow \dot{\vec{u}} = S^{-1}\dot{\vec{y}} = S^{-1}A\vec{y} = S^{-1}AS\vec{u} = \Lambda\vec{u}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2u_1 \\ -u_2 + u_3 \\ -u_3 \end{pmatrix}$$

$\therefore \dot{u}_1 = -2u_1 \quad \Rightarrow u_1 = a_1 e^{-2x}$

$\dot{u}_2 = -u_2 + u_3$

$\dot{u}_3 = -u_3 \quad \Rightarrow u_3 = a_3 e^{-x}$

$\therefore \dot{u}_2 = -u_2 + a_3 e^{-x} \quad \Rightarrow u_2 = (a_2 + a_3 x)e^{-x}$

$$\Rightarrow \boxed{\vec{u} = \begin{pmatrix} a_1 e^{-2x} \\ (a_2 + a_3 x)e^{-x} \\ a_3 e^{-x} \end{pmatrix}}$$

e). $\vec{y} = S\vec{u}$

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 e^{-2x} \\ (a_2 + a_3 x)e^{-x} \\ a_3 e^{-x} \end{pmatrix}$$

$$\Rightarrow \quad y_1 = (a_1 e^{-2x}) + (a_2 + a_3 x)e^{-x} + a_3 e^{-x}$$

$$y_2 = -2(a_1 e^{-2x}) - (a_2 + a_3 x)e^{-x}$$

$$y_3 = 4a_1 e^{-2x} + (a_2 + a_3 x)e^{-x} - a_3 e^{-x}$$

Insert initial condition.

$$\left.\begin{aligned} y(0) &= a_1 + a_2 + a_3 = +1 \\ y'(0) &= -2a_1 - a_2 = -1 \\ y''(0) &= 4a_1 + a_2 - a_3 = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= a_3 = 1 \\ a_2 &= -1 \end{aligned}$$

$$\Rightarrow \boxed{y = e^{-2x} + x e^{-x}}$$