Problem 3: Choose system of coordinates, such that axes x, y, z are orthogonal to the 6 flat surfaces, and let

$-a/2 < x < a/2$, $-b/2 < y < b/2$, $-c/2 < z < c/2$.

The tensor of inertia of the board prior to being dropped:

$$I^0_{xx} = \int_{-\frac{c}{2}}^{\frac{c}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{M}{abc}(y^2+z^2)\,dx\,dy\,dz =$$

$$= \frac{M}{abc}\left(ac\frac{2}{3}\left(\frac{b}{2}\right)^3 + ab\frac{2}{3}\left(\frac{c}{2}\right)^3\right) = \frac{M}{12}(b^2+c^2)$$

by symmetry $I^0_{yy} = \frac{M}{12}(a^2+c^2)$, $I_{zz} = \frac{M}{12}(b^2+a^2)$

$$I^0_{xy} = -\int_{-\frac{c}{2}}^{\frac{c}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{M}{abc}(xy)\,dx\,dy\,dz = 0$$

$$I^0_{yx} = I^0_{xz} = I^0_{zx} = I^0_{yz} = I^0_{zy} = 0$$

$$\Rightarrow I^0 = \frac{M}{12}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & b^2+a^2 \end{pmatrix} \equiv M\cdot A$$

The principal moments of inertia $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the eigenvectors of I^0, so $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.

The corresponding moment of inertia are eigenvalues,

$I^0_1 = \frac{M}{12}(b^2+c^2)$, $I^0_2 = \frac{M}{12}(a^2+c^2)$, $I^0_3 = \frac{M}{12}(b^2+a^2)$

After we clipped the corner, the new tensor of inertia is given by $I^0 - I^1 = MA + mB = M(A + \frac{m}{M}B) = M(A + \varepsilon B)$, where $\varepsilon = \frac{m}{M}$, $B = -\frac{1}{m} I'$, and I' is the tensor of inertia of the missing corner, at $(x, y, z) = (a/2, b/2, c/2)$:

$$I' = m \begin{pmatrix} \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 & -\frac{a}{2}\frac{b}{2} & -\frac{a}{2}\frac{c}{2} \\ -\frac{a}{2}\frac{b}{2} & \left(\frac{a}{2}\right)^2 + \left(\frac{c}{2}\right)^2 & -\frac{b}{2}\frac{c}{2} \\ -\frac{a}{2}\frac{c}{2} & -\frac{b}{2}\frac{c}{2} & \left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \end{pmatrix} = \frac{m}{4} \begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & b^2+a^2 \end{pmatrix}$$

$$\Rightarrow B = \frac{1}{4} \begin{pmatrix} -b^2-c^2 & ab & ac \\ ab & -a^2-c^2 & bc \\ ac & bc & -b^2-a^2 \end{pmatrix}$$

The new principal axes and principal moments are

$$\left.\begin{aligned} \vec{x}_i &= \vec{e}_i + \varepsilon \vec{x}_i^{(1)} + \varepsilon^2 \vec{x}_i^{(2)} + \cdots \\ I_i &= I_i^0 + \varepsilon I_i^{(1)} + \varepsilon^2 I_i^{(2)} + \cdots \end{aligned}\right\} \quad i = 1, 2, 3$$

with $I_i = \lambda_i$ and $I_i^{(0)} = a_i$ in notations used in class

Now, just plug in the $\vec{e}_i$'s and B into the formulas for the 1st and 2nd order corrections to eigenvectors and eigenvalues and you are done!

Note that, when you calculate 2nd order corrections, the double sum $\sum\limits_{\substack{j \neq i \\ k \neq i,j}}$ reduces to a single sum because there are just three different indices: 1, 2, 3.