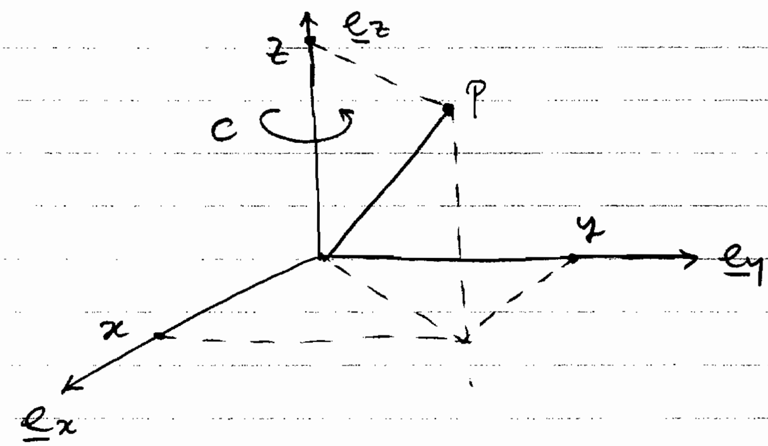
Example: Our first view of the group $C_3$ involved rotations about a z-axis through $2\pi/3$ or $4\pi/3$.




Recall that $C_3 = gp\{c\}$ $(c^3 = e)$. What happens to the point P under the operations of $C_3$, i.e., how is P transformed?

Let P be represented by the cartesian components $(x, y, z)$. Then under $e$, of course, nothing happens, but under c we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{a matrix } D(c) \text{ that represents } c} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

a matrix $D(c) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ that represents c

Under $c^2$ the relevant matrix can easily be seen to be

$$D(c^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and you should check that the composition rule

$$D(c^2) = D(c) \cdot D(c)$$ is obeyed.

ordinary matrix multiplication

We see that the group $C_3$ can be represented by the matrices

$e \to D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad c \to D(c)$ See above, $\quad c^2 \to D(c^2)$ See above

End example.

More generally, in physical applications the elements of a group are genuine spatial rotations (at least in many common settings) through some restricted set of angles (eg $C_3$, $D_4$). In such cases, points are transformed into new points (and their coordinates are mapped into new values) in ordinary space. Such transformations of coordinates are accomplished via 3x3 matrices associated with the rotations in question such that

- each element of the group is associated with a matrix
- the matrix associated with a product of group elements equals the product of matrices associated with each of the elements
- these matrices constitute a group representation.

Formal definition: A representation of dimension $n$ of the abstract group $G$ is a homomorphism $D$ from $G$ to $GL(n, \mathbb{C})$, the group of nonsingular $n \times n$ matrices with complex entries.

det $\neq 0$ — guarantees invertibility (ie inverses exist)

But what is a (group) homomorphism? A mapping from one group to another that preserves some structure (ie does not contradict the group multiplication structure) — then the image of a product equals the product of the images.

Faithful representation — the homomorphism is an isomorphism, so no information is lost by it — image matrices are only equal if the corresponding group elements are equal.

Look at our 3x3 representation of $C_3$. Notice that it does not mix $z$ with $(x,y)$ — rotations about $\hat{e}_z$ preserve $z$.

This shows up in the apparent block-diagonal structure of all three matrices $D(e)$, $D(c)$, $D(c^2)$, namely

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{2\times 2}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1\times 1 \end{array}\right)$$

Under matrix multiplication the "algebras" of these blocks do not mix — we may break the representation into two pieces

| | $D(e)$ | $D(c)$ | $D(c^2)$ | |
|---|---|---|---|---|
| 2x2 | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -\frac{1}{2} & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$ | $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$ | faithful |
| 1x1 | $(1)$ | $(1)$ | $(1)$ | unfaithful |

The set of 2x2 matrices obey the same multiplication table as the abstract group $C_3$ (and is an isomorphism of $C_3$): the matrices are in 1:1 correspondence with the group elements.

The "set of 1x1 matrices" (ie the number 1) does not violate the table inasmuch as (e.g.)

$$D(c)\cdot D(c) = 1\cdot 1 = D(c^2)$$
$$D(c)\cdot D(c^2) = 1\cdot 1 = D(e)$$

but "resolution" is lost because the mapping from $G$ to the set of matrices is 3:1.

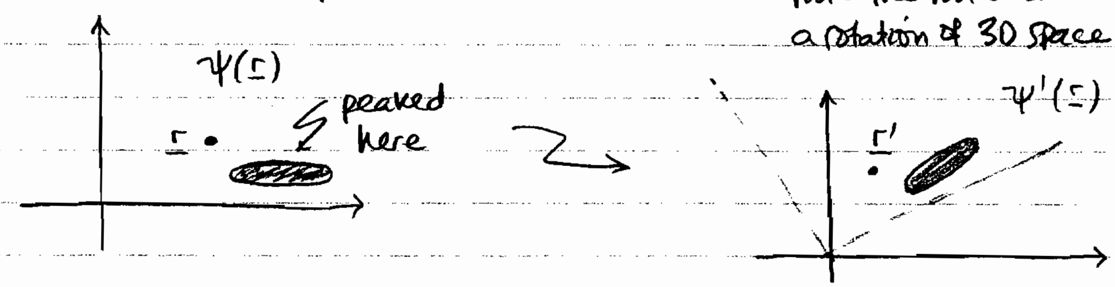
# Induced transformations of quantum mechanical states

We have begun to see explicit representations emerging from contexts in which the group was originally defined (ie rotations of points in three-dimensional space). Such settings will be interesting when we come to examine normal modes of oscillation of classical systems. But let us first turn to the setting of quantum mechanics, which provides us with far more intricate and interesting representations.

Consider the quantum mechanics of a single spinless particle moving in three-dimensional space. Its QM state is described by a (complex-valued) wave function $\psi(\underline{r})$.

position in 3D space

Let us ask what wavefunction $\psi'(\underline{r})$ replaces $\psi(\underline{r})$ if the state (or the apparatus preparing the state) is rotated in 3D space by the rotation $\underline{\underline{R}}$? ~~note the notation:~~ a rotation of 3D space




Under this rotation, the point $\underline{r}$ moves to the point $\underline{r}' = \underline{\underline{R}}\cdot\underline{r}$. So the new wavefunction can be found from the old one via

$$\psi'(\underline{r}') = \psi(\underline{r}) \quad \left\{ \begin{array}{l} \text{amplitude at } \underline{r}' \text{ is} \\ \text{what it was at } \underline{r} \end{array} \right.$$

Using $\underline{r}' = \underline{\underline{R}}\cdot\underline{r}$ we find $\psi'(\underline{\underline{R}}\cdot\underline{r}) = \psi(\underline{r})$, and changing the free variable from $\underline{r}$ to $\underline{\underline{R}}^{-1}\cdot\underline{r}$ we find

$$\boxed{\psi'(\underline{r}) = \psi(\underline{\underline{R}}^{-1}\cdot\underline{r})}$$

Is this transformation rule consistent with the group property of the rotations?

Consider 2 rotations, first $\underline{\underline{R}}$ then $\underline{\underline{S}}$, under which

$$\underline{r} \xrightarrow{R} \underline{r}' \xrightarrow{S} \underline{r}''$$

$$\underline{r}' = \underline{\underline{R}}\cdot\underline{r} \qquad \underline{r}'' = \underline{\underline{S}}\cdot\underline{r}' = \underline{\underline{S}}\cdot\underline{\underline{R}}\cdot\underline{r}$$

Then $\psi \xrightarrow{R} \psi' \xrightarrow{S} \psi''$ with

$$\left.\begin{aligned} \psi'(\underline{r}) &= \psi(\underline{\underline{R}}^{-1}\cdot\underline{r}) \\ \psi''(\underline{r}) &= \psi'(\underline{\underline{S}}^{-1}\cdot\underline{r}) \end{aligned}\right\} \Rightarrow \psi''(\underline{r}) = \psi(\underline{\underline{R}}^{-1}\cdot\underline{\underline{S}}^{-1}\cdot\underline{r}) = \psi((\underline{\underline{S}}\cdot\underline{\underline{R}})^{-1}\cdot\underline{r}) \checkmark$$

This is consistent with the group property

Our transformation of space, $\underline{r} \to \underline{\underline{R}}\cdot\underline{r}$, has *induced* a transformation in the space of quantum-mechanical wavefunctions. To see how this leads to intricate matrix representations, recall the idea of introducing a complete orthonormal basis of wavefunctions and representing arbitrary wavefunctions as linear combinations:

$\{\phi_j(\underline{r})\}^{(\#)}$, infinite set, with quantum numbers $j$

orthonormality

$$\int d^3r\, \phi_j(\underline{r})^* \phi_k(\underline{r}) = \delta_{jk},$$

$$\delta(\underline{r}-\underline{r}') = \sum_j \phi_j(\underline{r})\, \phi_j(\underline{r}')^*.$$

completeness

arbitrary wavefunction

basis functions

Then $$\psi(\underline{r}) = \sum_j \psi_j\, \phi_j(\underline{r})$$

amplitudes (or expansion coefficients)

Extract via $$\psi_j = \int d^3r\, \phi_j(\underline{r})^* \psi(\underline{r})$$

(#) Eg the eigenfunctions of some hamiltonian operator

Under our rotation $\underline{\underline{R}}$, what happens to our basis functions?

$$\phi_j(\underline{r}) \longrightarrow \phi_j(\underline{\underline{R}}^{-1}\cdot\underline{r}) = \sum_k \underbrace{R_{kj}}_{\text{note order}} \phi_k(\underline{r})$$

depends on $\underline{\underline{R}}$

new linear combination of $\phi$'s

The amplitudes $R_{kj}$ (one set for each $j$) can be computed as

$$R_{kj} = \int d^3r\, \phi_k(\underline{r})^* \phi_j(\underline{\underline{R}}^{-1}\cdot\underline{r}).$$

They form an infinite-dimensional representation of the group of rotations, with the product rule

$$(SR)_{kj} = \int d^3r\, \phi_k(\underline{r})^* \phi_j((\underline{\underline{S}}\underline{\underline{R}})^{-1}\cdot\underline{r})$$

$$= \int d^3r\, \phi_k(\underline{r})^* \phi_j(\underline{\underline{R}}^{-1}\cdot \underbrace{\underline{\underline{S}}^{-1}\cdot\underline{r}})$$

$$= \int d^3r\, \phi_k(\underline{r})^* \sum_l R_{lj}\, \phi_l(\underline{\underline{S}}^{-1}\cdot\underline{r})$$

$$= \int d^3r\, \phi_k(\underline{r})^* \sum_{lm} R_{lj}\, S_{ml}\, \phi_m(\underline{r})$$

$$= \sum_{lm} S_{ml} R_{lj} \underbrace{\int d^3r\, \phi_k(\underline{r})^* \phi_m(\underline{r})}_{\delta_{km}}$$

$$= \sum_l S_{kl} R_{lj}.$$

And they prescribe how general amplitudes transform under the rotation of the corresponding physical state

$$\psi_j = \int d^3r\, \phi_j(\underline{r})^* \psi(\underline{r}) \longrightarrow \psi_j' = \int d^3r\, \phi_j(\underline{r})^* \psi'(\underline{r})$$

$$= \int d^3r\, \phi_j(\underline{r})^* \psi(\underline{\underline{R}}^{-1}\cdot\underline{r}) = \int d^3r\, \phi_j(\underline{r})^* \Sigma_k \psi_k\, \phi_k(\underline{\underline{R}}^{-1}\cdot\underline{r})$$

$$= \int d^3r\, \phi_j(\underline{r})^* \Sigma_k \psi_k \Sigma_l R_{lk}\, \phi_l(\underline{r}) = \sum_k R_{jk} \psi_k .$$

gives $\delta_{jl}$

In bra-ket notation we have

$$|\underline{r}\rangle \xrightarrow{R} |\underline{\underline{R}}\cdot\underline{r}\rangle$$

position eigenket — Eigenket at rotated position

and we may introduce the (Hilbert space) operator $\hat{R}$ that accomplishes this :

$$\hat{R}|\underline{r}\rangle = |\underline{\underline{R}}\cdot\underline{r}\rangle.$$

These operators combine as follows : for all $\underline{r}$ we have

$$\widehat{SR}|\underline{r}\rangle = |(\underline{\underline{S}}\cdot\underline{\underline{R}})\cdot\underline{r}\rangle = |\underline{\underline{S}}\cdot(\underline{\underline{R}}\cdot\underline{r})\rangle$$

$$= \hat{S}|\underline{\underline{R}}\cdot\underline{r}\rangle = \hat{S}\hat{R}|\underline{r}\rangle$$

the Hilbert space operator corresponding to the "first R then S" real space rotation

$$\Rightarrow \widehat{SR} = \hat{S}\hat{R}.$$

So the operators corresponding to rotations compose as operator products.

We can also see that, not surprisingly, our matrix representation $R_{jk}$ is the matrix element of $\hat{R}$ in the $\phi_j(\underline{r})$ basis :

$$\langle\phi_j|\hat{R}|\phi_k\rangle = \int d^3r\,\langle\phi_j|\hat{R}|\underline{r}\rangle\langle\underline{r}|\phi_k\rangle$$

$$= \int d^3r\,\langle\phi_j|\underline{\underline{R}}\cdot\underline{r}\rangle\langle\underline{r}|\phi_k\rangle$$

$$\left.\begin{matrix}\underline{r}' = \underline{\underline{R}}\cdot\underline{r} \\ d^3r' = d^3r\end{matrix}\right\} \quad J = |\det\underline{\underline{R}}| = +1$$

$$= \int d^3r'\,\langle\phi_j|\underline{r}'\rangle\langle\underline{\underline{R}}^{-1}\cdot\underline{r}'|\phi_k\rangle$$

$$= \int d^3r\,\phi_j(\underline{r})^*\,\phi_k(\underline{\underline{R}}^{-1}\cdot\underline{r}) = R_{jk}.$$

Ex: Spinless particle confined to the surface of a unit sphere

Wavefunctions $\psi(\underline{n})$ ← unit vector, $\underline{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

Convenient complete set : Spherical harmonic functions $\{Y_{\ell m}(\underline{n})\}$
$\ell = 0, 1, 2, \ldots \; ; \; -\ell \le m \le \ell$

Eigenfunctions of total angular momentum operator $L^2$ :

$$L^2 Y_{\ell m}(\underline{n}) = \hbar^2 \ell(\ell+1) Y_{\ell m}(\underline{n}),$$

and z-component $L_z$ :

$$L_z Y_{\ell m}(\underline{n}) = \hbar m Y_{\ell m}(\underline{n}).$$

Eigenfunction expansion: $\psi(\underline{n}) = \sum_{\ell m} \psi_{\ell m} Y_{\ell m}(\underline{n})$.

where $\psi_{\ell m} = \int d^2 n \; Y_{\ell m}(\underline{n})^* \psi(\underline{n})$

$$\int d^2n = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi$$

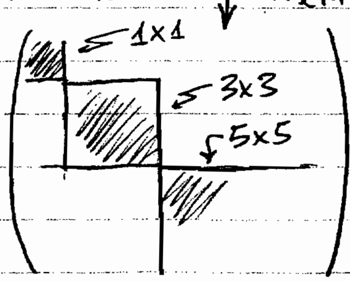
Infinite-dimensional matrix representation of rotations

$$R_{\ell m, \ell' m'} = \int d^2 n \; Y_{\ell m}(\underline{n})^* \; Y_{\ell' m'}(\underline{R}^{-1} \cdot \underline{n})$$

But, in fact, this greatly simplifies down to finite-dimensional building blocks because the rotation changes only the components of the angular momentum, not the total,

so $$Y_{\ell m}(\underline{R}^{-1}\cdot\underline{n}) = \sum_{m''=-\ell}^{\ell} Y_{\ell m''}(\underline{n}) \, d^{(\ell)}_{m'' m}(R)$$ ← defines the linear comb's

and hence $$R_{\ell m, \ell' m'} = \delta_{\ell, \ell'} \, d^{(\ell)}_{m m'}(R)$$




block-diagonal

different $\ell$'s do not mix, Constitute $(2\ell+1) \times (2\ell+1)$ dimensional representations

$-l \le m \le +l$

7/90

We see that for fixed $l = 0, 1, 2 \ldots$ the functions $\{Y_{lm}(\Omega)\}$ form bases for "disconnected" representations of the group of rotations

$$\text{abstract group element } R \xrightarrow{\text{homom.}} (2l+1) \times (2l+1) \text{ matrix } d^{(l)}_{mm'}(R)$$

In fact, as we shall see later, these representations constitute the irreducible representations of the 3D rotation group.

Our general task will be to

- classify and enumerate the possible representations
- to combine representations (cf addition of angular momentum)
- to relate representations of subgroups to those of the original group (cf splitting of levels by perturbations)

Equivalence of representations, Character of representations 

Consider a group $G$, and let $D^{(1)}$ and $D^{(2)}$ be 2 $n \times n$ representations.

independent of $g$

Suppose that, for all $g \in G$, we have $D^{(1)}(g) = S\, D^{(2)}(g)\, S^{-1}$

Then we say that the representations $D^{(1)}$ and $D^{(2)}$ are <u>equivalent</u>.

In this case, $D^{(1)}$ and $D^{(2)}$ are essentially the same, differing only because different bases (coordinates, eigenfunctions, ...) were used to construct them.

It is useful to regard equivalent representations as <u>not</u> being distinct from one another.

Notice that the similarity transformation induced by $S$ is consistent with the group property:

$n \times n$ matrix multiplication

$$
\begin{aligned}
\underbrace{S\, D^{(2)}(gg')\, S^{-1}}_{D^{(1)}(gg')} &= S\, D^{(2)}(g)\, D^{(2)}(g')\, S^{-1} \\
&= S\, D^{(2)}(g)\, S^{-1}\, S\, D^{(2)}(g')\, S^{-1} \\
&= D^{(1)}(g) \quad D^{(1)}(g') .
\end{aligned}
$$

In order to proceed in a basis-independent way, and hence to be blind to the differences between equivalent reps (ie representations), it is very useful to focus on the <u>character of a rep</u> (ie the invariant aspect of a rep) $\chi$:

$$\chi \equiv \{\chi(g) \mid g \in G\} \quad \text{where} \quad \chi(g) = \textstyle\sum_j D_{jj}(g)$$

string of complex numbers, one for each $g$

character of element $g$ in this rep.

trace of matrix $D_{jk}(g)$

The character $\chi$ is a function that depends only on what equivalence class a representation resides, because it is insensitive to similarity transformations (owing to the trace):

$$\chi'(g) = \Sigma_j \, D'_{jj}(g)$$

$$= \Sigma_{jk\ell} \, S_{jk} D_{k\ell}(g)(S^{-1})_{\ell j}$$

$$= \sum_{k\ell} \Big( \sum_j (S^{-1})_{\ell j} S_{jk} \Big) D_{k\ell}(g)$$

$$= \Sigma_{k\ell} \, \delta_{\ell k} D_{k\ell}(g)$$

$$= \Sigma_k \, D_{kk}(g) = \chi(g)$$ (ie traces are invariant under cyclic permutations)

So, if $D^{(1)}$ and $D^{(2)}$ are equivalent representations then $\chi^{(1)} = \chi^{(2)}$.

Later we shall see that if $\chi^{(1)} = \chi^{(2)}$ for two representations, then the representations are equivalent.

- On what do our representation matrices act?

Certainly our rep. matrices act on one another, e.g., when we compose rep. matrices via matrix multiplication

Example: $G = C_3 = \{e, c, c^2\}$, $c^3 = e$

$e \to D(e)$, $c \to D(c)$, $c^2 \to D(c^2)$

$D$'s are 3x3 matrices representing the elements $\{e, c, c^2\}$ of $C_3$.

They obey, e.g., $D(c) \cdot D(c) = D(c^2)$
$D(e) \cdot D(c^2) = D(c^2)$

↑ 3x3 matrix multiplication

But our rep. matrices also act on column entities, such as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$, which describe the state of a physical system. In acting on such state "vectors", our rep. matrices determine how the state of a system is transformed under the corresponding transformation of the physical entity that prepared the state.

↑ position of a classical particle

↑ state of a quantum system

I hesitate to use the word "state vector" for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$ because I would prefer to reserve this phrase for the true (basis independent) objects

$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$ classical setting

$|\psi\rangle = \sum_i \psi_i |\phi_i\rangle$ quantum setting

Then we should call the column entities

" $n \times 1$ matrices comprising the amplitudes (or components) of the state vector along some basis vector chosen from the stated set of basis vectors "

but this is a little clumsy – so we shall call them "state vectors".

Note that we shall always use orthonormal bases

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} \,, \qquad \langle \phi_i | \phi_j \rangle = \delta_{ij}$$

This is not necessary, but it is easy to make errors otherwise.

• What is the origin of the similarity transformations that we have just discussed?

They result from changes of the basis used to construct the rep. matrices in the first place. (Recall that we used the geometry of rotations to ascertain the transformation of $x, y$ and $z$ under rotations, and we abstracted from this computation the rep. matrices $D(e)$, $D(C)$ and $D(C^2)$.)

So, if we change the basis

e.g. $\quad \underline{e}_i \rightarrow \underline{f}_i = \Sigma_k T_{ki} \underline{e}_k$

$$|\phi_i\rangle \rightarrow |\chi_i\rangle = \Sigma_k Q_{ki} |\phi_k\rangle$$

Then we would arrive at (numerically) different but equivalent representations obtained via similarity transformations of the old representations.

To see this explicitly, consider the following example.

Let the state of the system be described by the position vector

$$\underline{r} = \sum_{k=1}^{3} x_k \underline{e}_k$$

basis vectors

coordinates (or components or amplitudes)

Let $\{\underline{f}_j\}_{j=1}^{3}$ be a new basis, related to the old basis $\{\underline{e}_k\}_{k=1}^{3}$ via

$$\underline{e}_k = \sum_j \sigma_{jk} \underline{f}_j$$

nonsingular (in fact orthogonal, if we would like all bases to be orthonormal)

Relative to the new basis, the coordinates of $\underline{r}$ can be found as follows

$$\underline{r} = \sum_k x_k \underline{e}_k = \sum_j y_j \underline{f}_j$$

Then $\sum_k x_k \underline{e}_k = \sum_k x_k \sum_j \sigma_{jk} \underline{f}_j = \sum_j y_j \underline{f}_j$

So, by the linear independence of the $\underline{f}$'s (necessary for them to constitute a basis) we have

$$y_j = \sum_k \sigma_{jk} x_k$$

Next, let us suppose that $\underline{r}$ is transformed to $\underline{r}' = \sum_k x'_k \underline{e}_k$ by some operation $R$. If $R$ is represented by the matrix $D(R)$ such that, under $R$, we have

$$\underline{e}_j \to \sum_k D_{kj} \underline{e}_k,$$

then, under $R$, we have $\underline{r} = \sum_j x_j \underline{e}_j \to \sum_j x_j \sum_k D_{kj} \underline{e}_k$

$$= \sum_k \left( \sum_j D_{kj} x_j \right) \underline{e}_k$$

ie $x_k \to x'_k = \sum_j D_{kj} x_j$.

So, we have the relationship between the coordinates of $\underline{r}$ and $\underline{r}'$ relative to the $\underline{e}$ basis:

$$x'_k = \Sigma_j D_{kj} x_j .$$

What is the relationship between these coordinates relative to the $\underline{f}$ basis. In other words,

if $\quad \underline{r} = \Sigma_k y_k \underline{f}_k$ and $\underline{r}' = \Sigma_k y'_k \underline{f}_k$

then how are $\{y_k\}$ and $\{y'_k\}$ related? Well, with summation over repeated indices implied, we have

$$\underline{r}' = y'_j \underline{f}_j = \sigma_{jk} x'_k \underline{f}_j = \sigma_{jk} D_{kl} x_l \underline{f}_j$$

$$= \sigma_{jk} D_{kl} (\sigma^{-1})_{lm} y_m \underline{f}_j$$

So, by the linear independence of the $\{\underline{f}_j\}$ we have

$$y'_j = \sum_m \underbrace{\sum_{kl} \sigma_{jk} D(R)_{jl} (\sigma^{-1})_{lm}} y_m$$

ie the matrix $\sigma D \sigma^{-1}$

ie the similarity transform of $D$

Note – if $\sigma$ transforms between orthonormal bases then it is an orthogonal matrix ($\sigma^T = \sigma^{-1}$, ie $\sigma_{ji} = (\sigma^{-1})_{ij}$) for real-component state vectors or, more generally, $\sigma$ is a unitary matrix ($\sigma^+ = \sigma^{-1}$, ie $(\sigma_{ji})^* = (\sigma^{-1})_{ij}$) for complex-component state vectors.

## Reducibility of representations



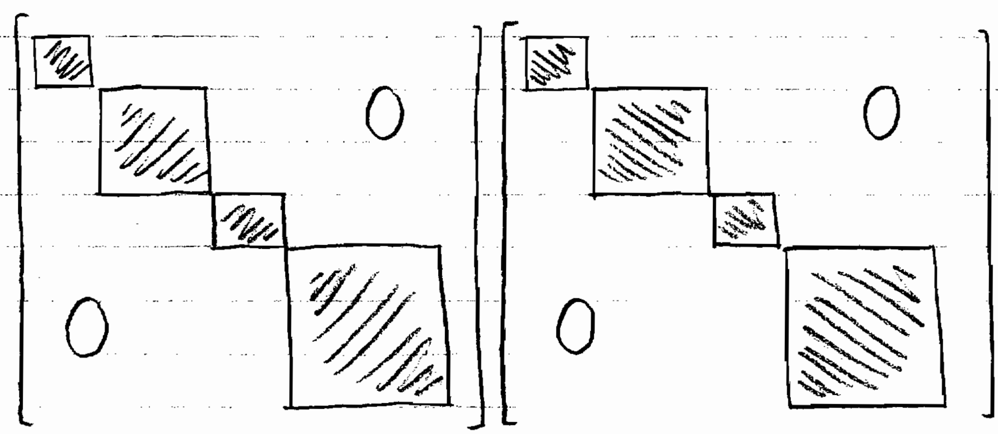
We have seen that in quantum mechanical settings we can easily find ourselves facing matrix representations of groups that are infinite - dimensional (ie involve infinity × infinity matrices).

Fortunately, there ~~make~~ exist choices of basis vectors that render all the matrices* block-diagonal, with the same block structure in all matrices. Then composition of matrices takes place separately, block by block. The blocks are finite matrices.

* in a representation

The task and consequences of choosing bases that render the matrices block-diagonal is the subject of the reducibility and reduction of representations.

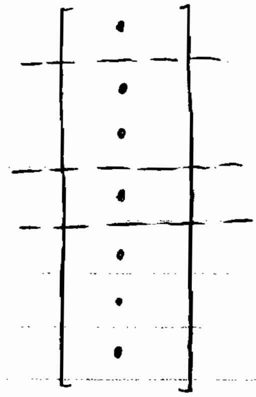
There are various "smallest" blocks that can be obtained. These are called irreducible representations, and they form the building blocks of generic representations. Learning how to reduce a representation into its irreducible components will be one of our main goals.




The product of two block-diagonal matrices is block-diagonal. Composition takes place separately — block by block.

2/170

When acting on state vectors, block diagonal matrices do not mix (ie produce linear combinations of) elements from different blocks.



Thus, the space of vectors decomposes into invariant subspaces (i.s.) each spanned by the basis vectors ~~[illegible]~~ associated with a block

$$|\psi\rangle = +\psi_1|\phi_1\rangle \qquad \text{one i.s.}$$

$$+\{\psi_2|\phi_2\rangle + \psi_3|\phi_3\rangle\} \qquad \text{another i.s.}$$

$$+\ \psi_4|\phi_4\rangle \qquad \text{another}$$

$$+\{\psi_5|\phi_5\rangle + \psi_6|\phi_6\rangle + \psi_7|\phi_7\rangle\} \qquad \text{another}$$

Each line is a vector in one of the invariant subspaces.

The subspaces are called invariant because vectors lying wholly within any one of them remain in that subspace under the action of any of the transformations.

We say that a collection of basis vectors for an invariant subspace form a basis for the associated representation

# Example of a Reducible Representation

Group: $G = C_3 = \{e, c, c^2\}$ with $c^3 = e$

Rep:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} -\frac{1}{2} & +\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$D(e) \qquad\qquad D(c) \qquad\qquad D(c^2)$$

When acting on $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\left.\begin{matrix} x \\ y \end{matrix}\right\}$ mixes these components; $z \}$ leaves this unchanged

$$\underline{r} = \underbrace{x\underline{e}_x + y\underline{e}_y}_{} + \underbrace{z\underline{e}_z}_{}$$

two uncoupled spaces

If we had chosen a basis other than $\{\underline{e}_x, \underline{e}_y, \underline{e}_z\}$ then we would have obtained an equivalent rep, but the reducibility of the rep would not have manifested itself, unless the new basis mixed $\underline{e}_x$ and $\underline{e}_y$ but did not mix $\underline{e}_z$ with ~~either~~ $\underline{e}_x$ and/or $\underline{e}_y$

In fact, irreducible

The two reduced reps are: Called

i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ $\quad D^{(2)}$

ii) $[1] \quad [1] \quad [1]$ $\quad D^{(1)}$

$D(e) \quad D(c) \quad D(c^2)$

Indicates that bases exist in which D are block-diagonal

We indicate the decomposability as $D = D^{(1)} \oplus D^{(2)}$

Definition – A representation of dimension $n+m$ is said to be reducible if at least one basis exists in which, for all $g \in G$, the matrices $D(g)$ take the form

$$D(g) = \left[\begin{array}{c|c} A(g) & C(g) \\ \hline 0 & B(g) \end{array}\right] \begin{array}{l} \updownarrow n \\ \updownarrow m \end{array} \qquad (\#)$$

$$\underset{n \geq 1}{\longleftrightarrow} \; \underset{m \geq 1}{\longleftrightarrow}$$

$A$ is $n \times n$ $\qquad$ $C$ is $n \times m$

$B$ is $m \times m$ $\qquad$ $0$ is $m \times n$ (an array of zeros)

Products: $\begin{pmatrix} A_1 & C_1 \\ 0 & B_1 \end{pmatrix}\begin{pmatrix} A_2 & C_2 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & A_1C_2 + C_1A_2 \\ 0 & B_1B_2 \end{pmatrix}$

Note that the structure is preserved.

For any finite (or any compact – see later) group, $C(g)$ can be taken to be zero, in which case the representation above would be said to be completely reducible (aka decomposible).

Reason: All reps of finite (or compact) groups are equivalent to unitary reps, ie reps for which, for all $g \in G$,

$$D(g)^+ = D(g)^{-1} \quad \text{ie} \quad (D_{ji})^* = (D^{-1})_{ij}. \text{ But}$$

if $D$ (in #) is unitary then $C = 0$, because

$$D(g)^+ = \left[\begin{array}{c|c} A(g)^+ & 0 \\ \hline C(g)^+ & B(g)^+ \end{array}\right] = D(g)^{-1} = D(g^{-1})$$

must have structure (#)

Why only finite and compact groups? Proof requires working with a suitable orthonormal basis, which one can always build for such groups via a certain group-averaging procedure.

[Cf Affine group example for homework]

often omit this qualification

When a rep is completely reducible we write 2/200

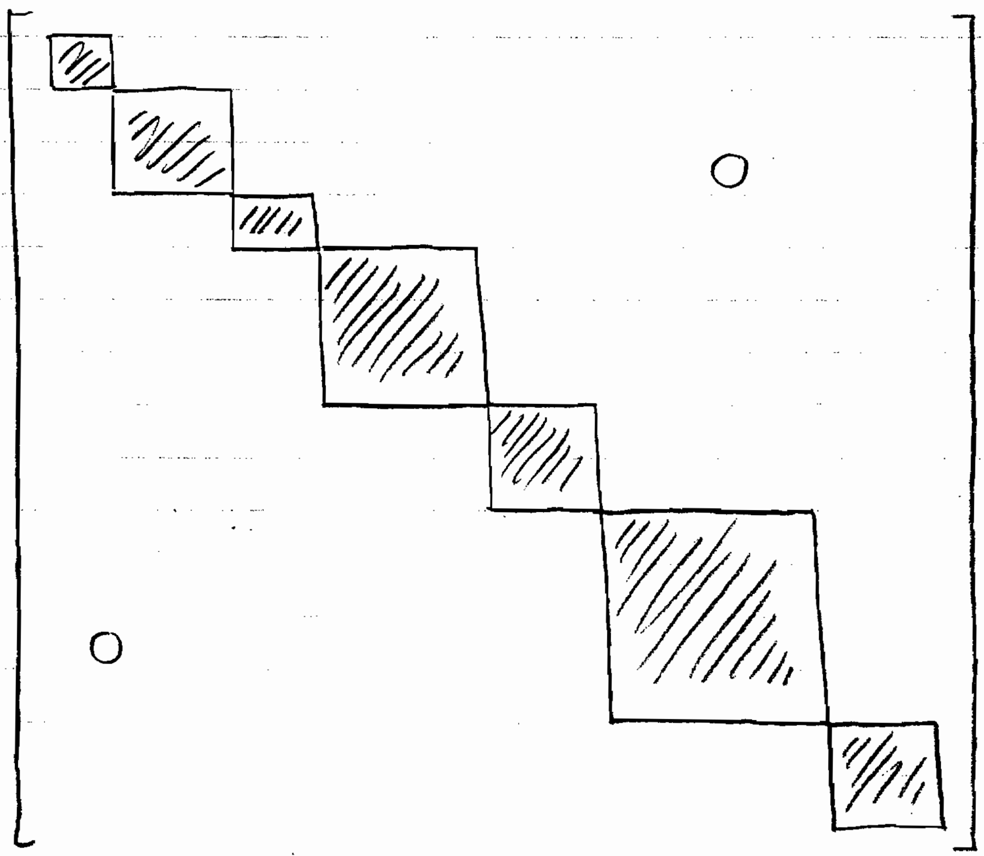
$$D(g) = A(g) \oplus B(g).$$

By further changes of basis (now <u>within</u> each invariant subspace) or equivalently by further similarity transformations that are themselves block-diagonal,

$$S = \left[\begin{array}{c|c} S_A & 0 \\ \hline 0 & S_B \end{array}\right]$$

we may be able to effect further decomposition, now of the reps A and/or B.

By continuing to make further block-diagonal similarity transforms we can, in principle, reach a stage where no further decomposition is possible

$D(g) =$ 

Each block (and its colleagues at other values of $g$) constitute the elements of an <u>irreducible</u> representation (aka irrep) of the representation $D$, and we write

$$D(g) = D^{(\nu_1)} \oplus D^{(\nu_2)} \oplus D^{(\nu_3)} \oplus \cdots$$

where the collection $\{D^{(\nu)}(g)\}_{g\in G}$ constitute the $\nu^{th}$ irrep. of the group $G$.

Notice that a given irrep. may feature several times in a rep.

There is <u>no limit</u> to the number of reps we can form – one can form the direct sum of an arbitrary collection of irreps, repeating any irrep. as often as we want. We can also hide the (de)composition of the rep via a similarity transformation.

But the irreps can be classified and enumerated, and they often have deep physical significance; so it is upon the irreps that we shall be focusing.