Georgia Tech PHYS 6124

Mathematical Methods of Physics I

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Homework Set #2

due Sept 6 2011, in class

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems from Stone and Goldbart, but renumbered for this course]

Problem 1.4 Scalar wave equation

The functional $\mathcal{S}$ has, as its argument, a single function u of the two independent variables x and t:

$$\mathcal{S}[u] = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left\{ \bar{\rho} u_t(x,t)^2 - \bar{\kappa} u_x(x,t)^2 \right\},$$

where $\bar{\rho}$ and $\bar{\kappa}$ are constants.

- a) Find the condition on u that makes $\mathcal{S}$ stationary with respect to variations that vanish at all times at the boundary points x_1 and x_2, and at all points at the initial and final times t_1 and t_2.

  [Note: $u_t(x,t) \equiv \partial u/\partial t$ and $u_x(x,t) \equiv \partial u/\partial x$.]

  You have just implemented Hamilton's principle to obtain the equation of motion for transverse displacements of a stretched string of line density $\bar{\rho}$ and tension $\bar{\kappa}$, *i.e.*, the scalar one-dimensional wave equation. $\mathcal{L} = \frac{1}{2}\left\{ \bar{\rho} u_t(x,t)^2 - \bar{\kappa} u_x(x,t)^2 \right\}$ is known as the Lagrange density; $L = \int_{x_1}^{x_2} dx\, \mathcal{L}$ is known as the Lagrangian; and $\mathcal{S} = \int_{t_1}^{t_2} dt\, L$ is known as the action.

- b) Repeat part (a), but now supposing that the line density varies with position, *i.e.*, $\bar{\rho} \to \rho(x)$, and that the Lagrange density has also acquired the additional term $g\rho(x)\, u(x,t)$. State a possible physical origin for such a term.

- c) Show that the vector wave equation follows from the stationarity of the functional

$$\mathcal{W}[\mathbf{u}] = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left\{ \bar{\rho} |\mathbf{u}_t|(x,t)^2 - \bar{\kappa} |\mathbf{u}_x|(x,t)^2 \right\},$$

where $\mathbf{u}(x,t)$ is a vector function of x and t.

Problem 2.1 Higher derivatives

Construct the Euler equation for the functional

$$J[y] = \frac{1}{2}\int_{x_1}^{x_2} dx\, \left\{ \left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 + y^2 \right\}.$$

Of what order is the resulting ordinary differential equation? You may assume that the variation of y and its first derivative vanish at the end-points.

Problem 2.2 Dynamics of fields

A real field $\Phi(\mathbf{x},t)$ obeys the variational principle

$$\delta \int d^3x\, dt\, \mathcal{L}(\mathbf{x},t) = 0.$$

Find the partial differential equation of motion obeyed by $\Phi(\mathbf{x},t)$ for the following cases:

- a) $\mathcal{L}(\mathbf{x},t) = \frac{1}{2}\left\{(\partial_t\Phi)^2 - c^2|\nabla\Phi|^2 - \mu^2\Phi^2\right\}$
- b) $\mathcal{L}(\mathbf{x},t) = \frac{1}{2}\left\{(\partial_t\Phi)^2 - c^2|\nabla\Phi|^2 + 2\mu^2\cos\Phi\right\}$
- c) A real scalar field $\phi(\mathbf{x},t)$ and a real vector field $\mathbf{A}(\mathbf{x},t)$ also obey the variational principle stated above, but with

$$\mathcal{L}(\mathbf{x},t) = \frac{1}{8\pi}\left\{\left|-\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right|^2 - |\nabla\times\mathbf{A}|^2 - \left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2\right\} + c^{-1}\mathbf{j}\cdot\mathbf{A} - \rho\phi,$$

  where $\mathbf{j}(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$ are the current and charge densities, and c is the speed of light *in vacuo*. Find the partial differential equations of motion obeyed by $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$.

Optional problems

Problem 2.3 Scalar wave equation with one free end

The functional $\mathcal{S}$ has, as its argument, a single function u of the two independent variables x and t:

$$\mathcal{S}[u] = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left\{ \bar{\rho} u_t(x,t)^2 - \bar{\kappa} u_x(x,t)^2 \right\},$$

where $\bar{\rho}$ and $\bar{\kappa}$ are constants. Find the additional condition on u, beyond the condition that u satisfies the wave equation, that makes $\mathcal{S}$ stationary with respect to variations that vanish at all points at the initial and final times t_1 and t_2, and at all times at the boundary point x_1, *but with no restriction on the variation at the point* x_2.
[Note: $u_t(x,t) \equiv \partial u/\partial t$ and $u_x(x,t) \equiv \partial u/\partial x$.]

Problem 2.4 Lagrange multipliers

- a) Find the stationary values of the function $f(x,y) = 13x^2 + 8xy + 7y^2$ on the circle $x^2 + y^2 = 1$.
- b) Identical particles can be distributed amongst R energy levels, having energies $\{\epsilon_r\}_{r=1}^{R}$. In a given configuration, n_r is the number of particles occupying level r. The total number of particles $\sum_{r=1}^{R} n_r$ and the total energy of the system $\sum_{r=1}^{R} n_r\epsilon_r$ are fixed, and take the values N and E, respectively. Find the distribution $\{\bar{n}_r\}_{r=1}^{R}$ that maximises the quantity

$$\Gamma \equiv \frac{N!}{n_1!n_2!\ldots n_R!},$$

  subject to the stated constraints.
  [Hint: You may use Stirling's approximation: $n! \approx \exp\{n\ln(n/e)\}$, valid for large n.]
- c) Consider the Hermitian form $Q = \sum_{ij}\psi_i^* H_{ij}\psi_j$, where $\{\psi_k\}$ are the M complex-valued components of a vector ψ, and $\{H_{kl}\}$ are the complex-valued elements of an Hermitian $M \times M$ matrix H. Show that the condition that Q be stationary with respect to variations in ψ, subject to the constraint that $\sum_i \psi_i^*\psi_i = 1$ (*i.e.*, that ψ be normalised to unity) leads to the Hermitian eigenproblem $\sum_j H_{ij}\psi_j = E\psi_i$, where the eigenvalue E is the Lagrange multiplier enforcing the normalisation constraint.

Problem 2.5 Curve of fixed length

Two fixed points, A and B, line in the xy-plane at the locations $(x, y) = (0, \pm a)$. They are connected by a curve Γ of fixed length L $(> 2a)$ lying in the plane. Assume that the radius vector from the origin to any point on Γ cuts Γ in at most one point so that Γ may be described using plane polar coordinates: $r = r(\theta)$.

- a) Express in the form $\delta \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, F(r, \mathrm{d}r/\mathrm{d}\theta, \theta) = 0$ the problem of finding the curve Γ for which the area between Γ and the chord AB is stationary, and identify an appropriate function F.
- b) Construct (but do not attempt to solve) the associated Euler equation.
- c) If one necessarily exists, construct a simpler differential equation satisfied by $r(\theta)$ [*i.e.*, the equation for curve Γ], and state why it must exist.

[Note: Although you cannot use the following information to address this question, you may wish to know that the sought curve is one of the two arcs of length L of a circle passing through A and B.]

Problem 2.6 Mass distribution for prescribed profile

You are provided with a light-weight line of length $\pi a/2$ and some lead shot of total mass M. Determine how the lead should be distributed along the line if the loaded line is to hang in an arc of a circle of radius a when its ends are attached to two points at the same height.
[Hint: Use plane polar coordinates having their origin at the centre of the circle.]

Problem 2.7 Flowing river

A river has parallel straight banks, given by the lines $x = 0$ and $x = b (> 0)$. The velocity $\mathbf{V}$ at which the river water flows is always directed parallel to the banks, but varies with the distance from the banks: $\mathbf{V} = A(x)\, \mathbf{e}_y$, where $A(x)$ is a certain prescribed function. A boat moves at constant speed $C(> |\mathbf{V}|)$ relative to the water, and follows the path $\mathbf{R} = x\, \mathbf{e}_x + Y(x)\, \mathbf{e}_y$. Construct the functional $T[Y]$ that gives the time to cross the river in terms of the path taken [*i.e.*, $Y(x)$ and its derivative(s)], the boat speed C, and the river speed $A(x)$. Constructing an Euler equation is *not* required.
[Hint: Observe that the boat velocity relative to the banks has the form $(dx/dt, dy/dt) = (C \sin\alpha, A + C\cos\alpha)$.]

Problem 2.8 Mechanical equilibrium of a hard ferromagnet

Let $\mathbf{m}(\mathbf{x})$ be the local value of the magnetisation in a ferromagnet. Suppose that the ferromagnet is *hard*, which means that the *magnitude* of the magnetisation is everywhere equal to unity. Suppose, further, that the free energy of the ferromagnet is given by

$$E = \frac{1}{2} J \int_V d^3x \big(\partial_\mu \, m^a(\mathbf{x})\big)^2 \equiv \frac{1}{2} J \int_V d^3x \sum_{\mu=1}^{3} \sum_{a=1}^{3} \big(\partial_\mu \, m^a(\mathbf{x})\big)^2 \,,$$

where V denotes the volume of the sample.

- a)By making a small variation of the magnetisation that vanishes at the boundary of the sample and integrating by parts, and by using a Lagrange multiplier at each position $\mathbf{x}$ to enforce the (non-linear) constraint that $|\mathbf{m}(\mathbf{x})| = 1$, show that the condition for mechanical equilibrium is given by
$$\nabla^2 m^a(\mathbf{x}) - m^a(\mathbf{x}) \, m^b(\mathbf{x}) \nabla^2 m^b(\mathbf{x}) = 0 \,.$$
- b)What do you think is the origin of the non-linearity in this partial differential equation ?
- c)Briefly discuss the number of independent partial differential equations, in the context of the number of dependent variables.

Problem 2.9 Geodesics in curved spaces

Suppose that a particle moves along a curve in three dimensions. The time δt taken to move from the point x_i $(i = 1, 2, 3)$ to the nearby point $x_i + \delta x_i$ $(i = 1, 2, 3)$ is given by

$$(\delta t)^2 = \sum_{i,j=1}^{3} g_{ij}(\{x\}) \, \delta x_i \, \delta x_j \,.$$

Find the set of coupled ordinary differential equations satisfied by $x_i(s)$ $(i = 1, 2, 3)$, the [parametric form of the] path that makes stationary the time taken to move between between two fixed points.

Problem 2.10 Differential calculus with functionals

Just as there is a version of Taylor's theorem for functions of several variables, so there is a version for functionals. We can use this version of Taylor's theorem to define *functional derivatives*. Consider the functional $J[y]$. If we make a small shift, $y \to y + \epsilon\eta$, then $J[y] \to J[y + \epsilon\eta]$, where

$$J[y+\epsilon\eta] = J[y] + \epsilon \int dx_1 \, J^{(1)}[y; x_1] \, \eta(x_1) + \frac{\epsilon^2}{2!} \int dx_1 dx_2 \, J^{(2)}[y; x_1, x_2] \, \eta(x_1)\eta(x_2) + \mathcal{O}(\epsilon^3).$$

We *define* the coefficient of $\epsilon\eta$, *i.e.*, $J^{(1)}[y; x_1]$, to be the first functional derivative of J with respect to $y(x_1)$, which we denote $\delta J/\delta y(x_1)$. Similarly, we define the coefficient of $\epsilon^2\eta^2/2!$, *i.e.*, $J^{(2)}[y; x_1, x_2]$, to be the second functional derivative of J with respect to $y(x_1)$ and $y(x_2)$, which we denote $\delta^2 J/\delta y(x_1)\delta y(x_2)$, *etc.* Compute the first and second functional derivatives of the following functionals:

- a) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1,z_2)\, y(z_1)\, y(z_2)$
- b) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1,z_2)\, y(z_1)^2\, y(z_2)^2$
- c) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1,z_2)\, y'(z_1)\, y'(z_2)$, where $y'(z)$ denotes dy/dz
- d) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1,z_2)\, y'(z_1)^2\, y'(z_2)^2$

Problem 2.11 Navier-Stokes equation

Consider a fluid with density $\rho(\mathbf{x},t)$ flowing at a velocity $\mathbf{v}(\mathbf{x},t)$. Then, at least for so-called *simple fluids*, the rate of change of the momentum density is given by the forces acting on a small volume element of fluid, *i.e.*, $\partial(\rho v_i)/\partial t = -\partial_k \Pi_{ik}$, where Π_{ik} is the momentum flux density tensor, given by

$$\Pi_{ik} = \rho\, v_i\, v_k + p\,\delta_{ik} - \eta\{\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik}\,\partial_j v_j\} - \zeta\delta_{ik}\,\partial_j v_j\,,$$

p is the pressure, η and ζ are two (positive) coefficients of viscosity, and summation over repeated indices is implied. Show that this form of Π_{ik}, together with the continuity equation $\partial\rho/\partial t = -\nabla\cdot(\rho\mathbf{v})$, produces the Navier-Stokes equation of motion for simple fluids:

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \eta\nabla^2\mathbf{v} + (\zeta + (\eta/3))\nabla(\nabla\cdot\mathbf{v})\,.$$

Problem 2.12 Foucault's pendulum in disguise

A particle of mass μ moving in three dimensions is bound to the origin $\mathcal{O}$ by a harmonic spring of spring constant κ. Let $\mathbf{R}(t)$ denote the position of the particle at time t.

- a) Write down the lagrangian that controls the motion of the system.

  Now suppose that the motion is confined to a certain moving plane that passes through $\mathcal{O}$ and has unit normal vector $\mathbf{N}(t)$.

- b) Write down the appropriate equation of constraint, and use it to construct the appropriate new lagrangian, which involves a lagrangian undetermined multiplier λ.
- c) Construct the classical equation of motion in terms of λ.

  Now restrict your attention to the situation in which the orientation of the plane varies slowly, compared with the natural frequency ω ($\equiv \sqrt{\kappa/\mu}$) of the oscillating particle, *i.e.*, $|\dot{\mathbf{N}}(t)| \ll \omega$. In addition, consider only "linearly polarised" motions, *i.e.*, those that pass through $\mathcal{O}$.

- d) By assuming that the only consequence of the motion of the constraint-plane is the slow variation of the direction of oscillation $\mathbf{A}(t)$, ie, that $\mathbf{R}(t) = \mathbf{A}(t)\ \sin(\omega t + \varphi)$, show that the oscillation-direction $\mathbf{A}(t)$ obeys

$$\dot{\mathbf{A}}(t) \approx \Big(\mathbf{N}(t) \times \dot{\mathbf{N}}(t)\Big) \times \mathbf{A}(t).$$

- e) Show that the magnitude of $\mathbf{A}(t)$ does not vary with time.

- f) Show that $\mathbf{A}(t)$ is not integrable, *i.e.*, that $\mathbf{A}(t)$ cannot be written as $\mathbf{A}(t) = \mathbf{f}\big(\mathbf{N}(t)\big)$.

- g) Suppose that $\mathbf{N}(t)$ is slowly driven around a closed path over a time T, *i.e.*, $\mathbf{N}(T) = \mathbf{N}(0)$. Find a relationship between $\mathbf{A}(T) \cdot \mathbf{A}(0) / |\mathbf{A}(T)|\, |\mathbf{A}(0)|$ and the area of the unit sphere enclosed by the path $\mathbf{N}(t)$.

- h) By using the equation given in part (d) and your answer to parts (g), explain why your answer to part (g) can be described as *geometric*.

[Hint: See the article entitled *The Quantum Phase, Five Years After*, by M. V. Berry, in *Geometric Phases in Physics*, A. Shapere and F. Wilczek (World Scientific, Singapore, 1989), especially p. 8 *et seq*.]