Chapter 6

Partial Differential Equations

Most differential equations of physics involve quantities depending on both space and time. Inevitably they involve partial derivatives, and so are partial differential equations (PDE's). Although PDE's are inherently more complicated that ODE's, many of the ideas from the previous chapters — in particular the notion of self adjointness and the resulting completeness of the eigenfunctions — carry over to the partial differential operators that occur in these equations.

6.1 Classification of PDE's

We focus on second-order equations in two variables, such as the wave equation

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = f(x,t), \qquad \text{(Hyperbolic)} \tag{6.1}$$

Laplace or Poisson's equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = f(x,y), \qquad \text{(Elliptic)} \tag{6.2}$$

or Fourier's heat equation

$$\frac{\partial^2\varphi}{\partial x^2} - \kappa\frac{\partial\varphi}{\partial t} = f(x,t). \qquad \text{(Parabolic)} \tag{6.3}$$

What do the names hyperbolic, elliptic and parabolic mean? In high-school co-ordinate geometry we learned that a real quadratic curve

$$ax^2 + 2bxy + cy^2 + fx + gy + h = 0 \tag{6.4}$$

represents a hyperbola, an ellipse or a parabola depending on whether the *discriminant*, $ac - b^2$, is less than zero, greater than zero, or equal to zero, these being the conditions for the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{6.5}$$

to have signature $(+,-)$, $(+,+)$ or $(+,0)$.

By analogy, the equation

$$a(x,y)\frac{\partial^2\varphi}{\partial x^2} + 2b(x,y)\frac{\partial^2\varphi}{\partial x\partial y} + c(x,y)\frac{\partial^2\varphi}{\partial y^2} + (\text{lower orders}) = 0, \tag{6.6}$$

is said to be hyperbolic, elliptic, or parabolic at a point (x,y) if

$$\begin{vmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{vmatrix} = (ac - b^2)|_{(x,y)}, \tag{6.7}$$

is less than, greater than, or equal to zero, respectively. This classification helps us understand what sort of initial or boundary data we need to specify the problem.

There are three broad classes of boundary conditions:

a) **Dirichlet boundary conditions:** The value of the dependent variable is specified on the boundary.
b) **Neumann boundary conditions:** The normal derivative of the dependent variable is specified on the boundary.
c) **Cauchy boundary conditions:** Both the value and the normal derivative of the dependent variable are specified on the boundary.

Less commonly met are *Robin* boundary conditions, where the value of a linear combination of the dependent variable and the normal derivative of the dependent variable is specified on the boundary.

Cauchy boundary conditions are analogous to the initial conditions for a second-order ordinary differential equation. These are given at one end of the interval only. The other two classes of boundary condition are higher-dimensional analogues of the conditions we impose on an ODE at both ends of the interval.

Each class of PDE's requires a different class of boundary conditions in order to have a unique, stable solution.

1) **Elliptic** equations require either Dirichlet or Neumann boundary conditions on a closed boundary surrounding the region of interest. Other

boundary conditions are either insufficient to determine a unique solution, overly restrictive, or lead to instabilities.

2) **Hyperbolic** equations require Cauchy boundary conditions on a open surface. Other boundary conditions are either too restrictive for a solution to exist, or insufficient to determine a unique solution.
3) **Parabolic** equations require Dirichlet or Neumann boundary conditions on a open surface. Other boundary conditions are too restrictive.

6.2 Cauchy data

Given a second-order ordinary differential equation

$$p_0 y'' + p_1 y' + p_2 y = f \tag{6.8}$$

with initial data $y(a)$, $y'(a)$ we can construct the solution incrementally. We take a step $\delta x = \varepsilon$ and use the initial slope to find $y(a+\varepsilon) = y(a) + \varepsilon y'(a)$. Next we find $y''(a)$ from the differential equation

$$y''(a) = -\frac{1}{p_0}(p_1 y'(a) + p_2 y(a) - f(a)), \tag{6.9}$$

and use it to obtain $y'(a+\varepsilon) = y'(a) + \varepsilon y''(a)$. We now have initial data, $y(a+\varepsilon)$, $y'(a+\varepsilon)$, at the point $a+\varepsilon$, and can play the same game to proceed to $a + 2\varepsilon$, and onwards.

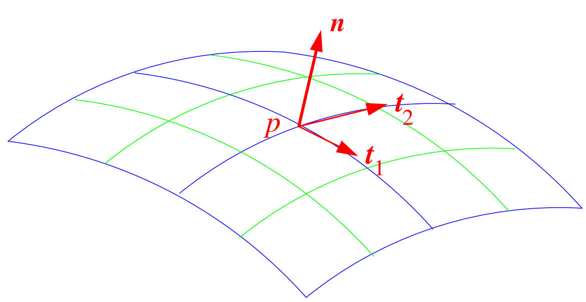

Figure 6.1: *The surface* Γ *on which we are given Cauchy Data.*

Suppose now that we have the analogous situation of a second order partial differential equation

$$a_{\mu\nu}(x)\frac{\partial^2\varphi}{\partial x^\mu \partial x^\nu} + (\text{lower orders}) = 0. \tag{6.10}$$

in $\mathbb{R}^n$. We are also given initial data on a surface, Γ, of co-dimension one in $\mathbb{R}^n$.

At each point p on Γ we erect a basis $\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{n-1}$, consisting of the normal to Γ and $n-1$ tangent vectors. The information we have been given consists of the value of φ at every point p together with

$$\frac{\partial \varphi}{\partial n} \stackrel{\text{def}}{=} n^\mu \frac{\partial \varphi}{\partial x^\mu}, \tag{6.11}$$

the normal derivative of φ at p. We want to know if this *Cauchy data* is sufficient to find the second derivative in the normal direction, and so construct similar Cauchy data on the adjacent surface $\Gamma + \varepsilon \mathbf{n}$. If so, we can repeat the process and systematically propagate the solution forward through $\mathbb{R}^n$.

From the given data, we can construct

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial n \partial t_i} &\stackrel{\text{def}}{=} n^\mu t_i^\nu \frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu}, \\ \frac{\partial^2 \varphi}{\partial t_i \partial t_j} &\stackrel{\text{def}}{=} t_i^\nu t_j^\nu \frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu}, \end{aligned} \tag{6.12}$$

but we do not yet have enough information to determine

$$\frac{\partial^2 \varphi}{\partial n \partial n} \stackrel{\text{def}}{=} n^\mu n^\nu \frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu}. \tag{6.13}$$

Can we fill the data gap by using the differential equation (6.10)? Suppose that

$$\frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu} = \phi_0^{\mu\nu} + n^\mu n^\nu \Phi \tag{6.14}$$

where $\phi_0^{\mu\nu}$ is a guess that is consistent with (6.12), and Φ is as yet unknown, and, because of the factor of $n^\mu n^\nu$, does not affect the derivatives (6.12). We plug into

$$a_{\mu\nu}(x_i) \frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu} + (\text{known lower orders}) = 0. \tag{6.15}$$

and get

$$a_{\mu\nu} n^\mu n^\nu \Phi + (\text{known}) = 0. \tag{6.16}$$

We can therefore find Φ provided that

$$a_{\mu\nu} n^\mu n^\nu \neq 0. \tag{6.17}$$

If this expression *is* zero, we are stuck. It is like having $p_0(x) = 0$ in an ordinary differential equation. On the other hand, knowing Φ tells us the second normal derivative, and we can proceed to the adjacent surface where we play the same game once more.

Definition: A *characteristic surface* is a surface Σ such that $a_{\mu\nu}n^\mu n^\nu = 0$ at all points on Σ. We can therefore propagate our data forward, provided that the initial-data surface Γ is nowhere tangent to a characteristic surface. In two dimensions the characteristic surfaces become one-dimensional curves. An equation in two dimensions is hyperbolic, parabolic, or elliptic at at a point (x, y) if it has two, one or zero characteristic curves through that point, respectively.

Characteristics are both a *curse* and *blessing*. They are a barrier to Cauchy data, but, as we see in the next two subsections, they are also the curves along which information is transmitted.

6.2.1 Characteristics and first-order equations

Suppose we have a linear first-order partial differential equation

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y). \tag{6.18}$$

We can write this in vector notation as $(\mathbf{v} \cdot \nabla)u + cu = f$, where $\mathbf{v}$ is the vector field $\mathbf{v} = (a, b)$. If we define the *flow* of the vector field to be the family of parametrized curves $x(t), y(t)$ satisfying

$$\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y), \tag{6.19}$$

then the partial differential equation (6.18) reduces to an ordinary linear differential equation

$$\frac{du}{dt} + c(t)u(t) = f(t) \tag{6.20}$$

along each flow line. Here,

$$\begin{aligned} u(t) &\equiv u(x(t), y(t)), \\ c(t) &\equiv c(x(t), y(t)), \\ f(t) &\equiv f(x(t), y(t)). \end{aligned} \tag{6.21}$$

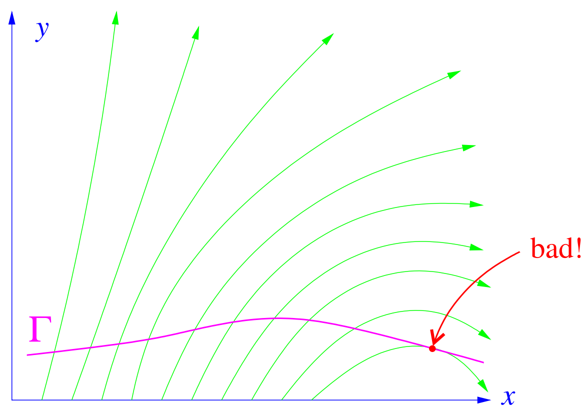

Figure 6.2: *Initial data curve* Γ, *and flow-line characteristics.*

Provided that $a(x, y)$ and $b(x, y)$ are never simultaneously zero, there will be one flow-line curve passing through each point in $\mathbb{R}^2$. If we have been given the initial value of u on a curve Γ that is nowhere tangent to any of these flow lines then we can propagate this data forward along the flow by solving (6.20). On the other hand, if the curve Γ does become tangent to one of the flow lines at some point then the data will generally be inconsistent with (6.18) at that point, and no solution can exist. The flow lines therefore play a role analagous to the characteristics of a second-order partial differential equation, and are therefore also called characteristics. The trick of reducing the partial differential equation to a collection of ordinary differential equations along each of its flow lines is called the *method of characteristics*.

Exercise 6.1: Show that the general solution to the equation

$$\frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial y} - (x - y)\varphi = 0$$

is

$$\varphi(x, y) = e^{-xy} f(x + y),$$

where f is an arbitrary function.

6.2.2 Second-order hyperbolic equations

Consider a second-order equation containing the operator

$$D = a(x, y)\frac{\partial^2}{\partial x^2} + 2b(x, y)\frac{\partial^2}{\partial x \partial y} + c(x, y)\frac{\partial^2}{\partial y^2} \tag{6.22}$$

We can always factorize

$$aX^2 + 2bXY + cY^2 = (\alpha X + \beta Y)(\gamma X + \delta Y), \tag{6.23}$$

and from this obtain

$$\begin{aligned} a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} &= \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)\left(\gamma\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y}\right) + \text{lower}, \\ &= \left(\gamma\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y}\right)\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right) + \text{lower}. \end{aligned} \tag{6.24}$$

Here "lower" refers to terms containing only first order derivatives such as

$$\alpha\left(\frac{\partial \gamma}{\partial x}\right)\frac{\partial}{\partial x}, \quad \beta\left(\frac{\partial \delta}{\partial y}\right)\frac{\partial}{\partial y}, \quad \textit{etc.}$$

A necessary condition, however, for the coefficients $\alpha, \beta, \gamma, \delta$ to be *real* is that

$$\begin{aligned} ac - b^2 &= \alpha\beta\gamma\delta - \frac{1}{4}(\alpha\delta + \beta\gamma)^2 \\ &= -\frac{1}{4}(\alpha\delta - \beta\gamma)^2 \leq 0. \end{aligned} \tag{6.25}$$

A factorization of the leading terms in the second-order operator D as the product of two real first-order differential operators therefore requires that D be *hyperbolic* or *parabolic*. It is easy to see that this is also a *sufficient* condition for such a real factorization. For the rest of this section we assume that the equation is hyperbolic, and so

$$ac - b^2 = -\frac{1}{4}(\alpha\delta - \beta\gamma)^2 < 0. \tag{6.26}$$

With this condition, the two families of flow curves defined by

$$C_1: \quad \frac{dx}{dt} = \alpha(x, y), \quad \frac{dy}{dt} = \beta(x, y), \tag{6.27}$$

and

$$C_2: \quad \frac{dx}{dt} = \gamma(x, y), \quad \frac{dy}{dt} = \delta(x, y), \tag{6.28}$$

are distinct, and are the characteristics of D.

A hyperbolic second-order differential equation $Du = 0$ can therefore be written in either of two ways:

$$\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right) U_1 + F_1 = 0, \tag{6.29}$$

or

$$\left(\gamma\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y}\right) U_2 + F_2 = 0, \tag{6.30}$$

where

$$\begin{aligned} U_1 &= \gamma\frac{\partial u}{\partial x} + \delta\frac{\partial u}{\partial y}, \\ U_2 &= \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y}, \end{aligned} \tag{6.31}$$

and $F_{1,2}$ contain only $\partial u/\partial x$ and $\partial u/\partial y$. Given suitable Cauchy data, we can solve the two first-order partial differential equations by the method of characteristics described in the previous subsection, and so find $U_1(x, y)$ and $U_2(x, y)$. Because the hyperbolicity condition (6.26) guarantees that the determinant

$$\begin{vmatrix} \gamma & \delta \\ \alpha & \beta \end{vmatrix} = \gamma\beta - \alpha\delta$$

is not zero, we can solve (6.31) and so extract from $U_{1,2}$ the individual derivatives $\partial u/\partial x$ and $\partial u/\partial y$. From these derivatives and the initial values of u, we can determine $u(x, y)$.

6.3 Wave equation

The wave equation provides the paradigm for hyperbolic equations that can be solved by the method of characteristics.

6.3.1 d'Alembert's solution

Let $\varphi(x, t)$ obey the wave equation

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \qquad -\infty < x < \infty. \tag{6.32}$$

We use the method of characteristics to propagate Cauchy data $\varphi(x,0) = \varphi_0(x)$ and $\dot\varphi(x,0) = v_0(x)$, given on the curve $\Gamma = \{x \in \mathbb{R}, t = 0\}$, forward in time.

We begin by factoring the wave equation as

$$0 = \left(\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}\right) = \left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial\varphi}{\partial t}\right). \tag{6.33}$$

Thus,

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)(U - V) = 0, \tag{6.34}$$

where

$$U = \varphi' = \frac{\partial\varphi}{\partial x}, \qquad V = \frac{1}{c}\dot\varphi = \frac{1}{c}\frac{\partial\varphi}{\partial t}. \tag{6.35}$$

The quantity $U - V$ is therefore constant along the characteristic curves

$$x - ct = \text{const.} \tag{6.36}$$

Writing the linear factors in the reverse order yields the equation

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)(U + V) = 0. \tag{6.37}$$

This implies that $U + V$ is constant along the characteristics

$$x + ct = \text{const.} \tag{6.38}$$

Putting these two facts together tells us that

$$\begin{aligned} V(x,t') &= \frac{1}{2}[V(x,t') + U(x,t')] + \frac{1}{2}[V(x,t') - U(x,t')] \\ &= \frac{1}{2}[V(x+ct',0) + U(x+ct',0)] + \frac{1}{2}[V(x-ct',0) - U(x-ct',0)]. \end{aligned} \tag{6.39}$$

The value of the variable V at the point (x, t') has therefore been computed in terms of the values of U and V on the initial curve Γ. After changing variables from t' to $\xi = x \pm ct'$ as appropriate, we can integrate up to find that

$$\begin{aligned}\varphi(x,t) &= \varphi(x,0) + c\int_0^t V(x,t')dt' \\ &= \varphi(x,0) + \frac{1}{2}\int_x^{x+ct} \varphi'(\xi,0)\,d\xi + \frac{1}{2}\int_x^{x-ct} \varphi'(\xi,0)\,d\xi + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot\varphi(\xi,0)\,d\xi \\ &= \frac{1}{2}\left\{\varphi(x+ct,0)+\varphi(x-ct,0)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot\varphi(\xi,0)\,d\xi. \qquad (6.40)\end{aligned}$$

This result

$$\varphi(x,t) = \frac{1}{2}\left\{\varphi_0(x+ct)+\varphi_0(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,d\xi \qquad (6.41)$$

is usually known as *d'Alembert's solution* of the wave equation. It was actually obtained first by Euler in 1748.

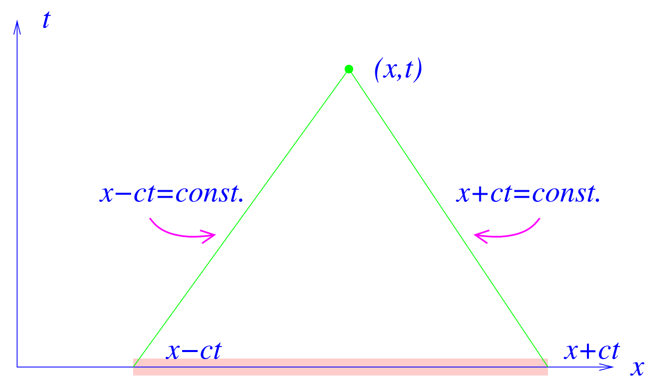

Figure 6.3: *Range of Cauchy data influencing* $\varphi(x,t)$.

The value of φ at x, t, is determined by only a finite interval of the initial Cauchy data. In more generality, $\varphi(x,t)$ depends only on what happens in the past *light-cone* of the point, which is bounded by pair of characteristic curves. This is illustrated in figure 6.3

D'Alembert and Euler squabbled over whether φ_0 and v_0 had to be twice differentiable for the solution (6.41) to make sense. Euler wished to apply (6.41) to a plucked string, which has a discontinuous slope at the plucked point, but d'Alembert argued that the wave equation, with its second derivative, could not be applied in this case. This was a dispute that could not be resolved (in Euler's favour) until the advent of the theory of distributions. It highlights an important difference between ordinary and partial differential equations: an ODE with smooth coefficients has smooth solutions; a PDE with with smooth coefficients can admit discontinuous or even distributional solutions.

An alternative route to d'Alembert's solution uses a method that applies most effectively to PDE's with constant coefficients. We first seek a *general solution* to the PDE involving two arbitrary functions. Begin with a change of variables. Let

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct. \end{aligned} \tag{6.42}$$

be *light-cone co-ordinates*. In terms of them, we have

$$\begin{aligned} x &= \frac{1}{2}(\xi + \eta), \\ t &= \frac{1}{2c}(\xi - \eta). \end{aligned} \tag{6.43}$$

Now,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right). \tag{6.44}$$

Similarly

$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right). \tag{6.45}$$

Thus

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) = \left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right) = 4\frac{\partial^2}{\partial \xi \partial \eta}. \tag{6.46}$$

The characteristics of the equation

$$4\frac{\partial^2 \varphi}{\partial \xi \partial \eta} = 0 \tag{6.47}$$

are $\xi = const.$ or $\eta = const.$ There are two characteristics curves through each point, so the equation is still hyperbolic.

With light-cone coordinates it is easy to see that a solution to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 4\frac{\partial^2 \varphi}{\partial \xi \partial \eta} = 0 \tag{6.48}$$

is

$$\varphi(x,t) = f(\xi) + g(\eta) = f(x+ct) + g(x-ct). \tag{6.49}$$

It is this this expression that was obtained by d'Alembert (1746).

Following Euler, we use d'Alembert's general solution to propagate the Cauchy data $\varphi(x,0) \equiv \varphi_0(x)$ and $\dot{\varphi}(x,0) \equiv v_0(x)$ by using this information to determine the functions f and g. We have

$$\begin{aligned} f(x) + g(x) &= \varphi_0(x), \\ c(f'(x) - g'(x)) &= v_0(x). \end{aligned} \tag{6.50}$$

Integration of the second line with respect to x gives

$$f(x) - g(x) = \frac{1}{c}\int_0^x v_0(\xi)\,d\xi + A, \tag{6.51}$$

where A is an unknown (but irrelevant) constant. We can now solve for f and g, and find

$$\begin{aligned} f(x) &= \frac{1}{2}\varphi_0(x) + \frac{1}{2c}\int_0^x v_0(\xi)\,d\xi + \frac{1}{2}A, \\ g(x) &= \frac{1}{2}\varphi_0(x) - \frac{1}{2c}\int_0^x v_0(\xi)\,d\xi - \frac{1}{2}A, \end{aligned} \tag{6.52}$$

and so

$$\varphi(x,t) = \frac{1}{2}\left\{\varphi_0(x+ct) + \varphi_0(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,d\xi. \tag{6.53}$$

The unknown constant A has disappeared in the end result, and again we find "d'Alembert's" solution.

Exercise 6.2: Show that when the operator D in a constant-coefficient second-order PDE $D\varphi = 0$ is *reducible*, meaning that it can be factored into two distinct first-order factors $D = P_1 P_2$, where

$$P_i = \alpha_i \frac{\partial}{\partial x} + \beta_i \frac{\partial}{\partial y} + \gamma_i,$$

then the general solution to $D\varphi = 0$ can be written as $\varphi = \phi_1 + \phi_2$, where $P_1\phi_1 = 0$, $P_2\phi_2 = 0$. Hence, or otherwise, show that the general solution to the equation

$$\frac{\partial^2\varphi}{\partial x \partial y} + 2\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial\varphi}{\partial x} - 2\frac{\partial\varphi}{\partial y} = 0$$

is

$$\varphi(x, y) = f(2x - y) + e^y g(x),$$

where f, g, are arbitrary functions.

Exercise 6.3: Show that when the constant-coefficient operator D is of the form

$$D = P^2 = \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\right)^2,$$

with $\alpha \neq 0$, then the general solution to $D\varphi = 0$ is given by $\varphi = \phi_1 + x\phi_2$, where $P\phi_{1,2} = 0$. (If $\alpha = 0$ and $\beta \neq 0$, then $\varphi = \phi_1 + y\phi_2$.)

6.3.2 Fourier's solution

In 1755 Daniel Bernoulli proposed solving for the motion of a finite length L of transversely vibrating string by setting

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \tag{6.54}$$

but he did not know how to find the coefficients A_n (and perhaps did not care that his cosine time dependence restricted his solution to the intial condition $\dot{y}(x, 0) = 0$). Bernoulli's idea was dismissed out of hand by Euler and d'Alembert as being too restrictive. They simply refused to believe that (almost) any chosen function could be represented by a trigonometric series expansion. It was only fifty years later, in a series of papers starting in 1807, that Joseph Fourier showed how to compute the A_n and insisted that indeed "any" function could be expanded in this way. Mathematicians have expended much effort in investigating the extent to which Fourier's claim is true.

We now try our hand at Bernoulli's game. Because we are solving the wave equation on the infinite line, we seek a solution as a Fourier *integral*. A sufficiently general form is

$$\varphi(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left\{a(k)e^{ikx - i\omega_k t} + a^*(k)e^{-ikx + i\omega_k t}\right\}, \tag{6.55}$$

where $\omega_k \equiv c|k|$ is the *positive* root of $\omega^2 = c^2k^2$. The terms being summed by the integral are each individually of the form $f(x - ct)$ or $f(x + ct)$, and so

$\varphi(x,t)$ is indeed a solution of the wave equation. The positive-root convention means that positive k corresponds to right-going waves, and negative k to left-going waves.

We find the amplitudes $a(k)$ by fitting to the Fourier transforms

$$\begin{aligned} \Phi(k) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \varphi(x,t=0)e^{-ikx}dx, \\ \chi(k) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \dot{\varphi}(x,t=0)e^{-ikx}dx, \end{aligned} \tag{6.56}$$

of the Cauchy data. Comparing

$$\begin{aligned} \varphi(x,t=0) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\Phi(k)e^{ikx}, \\ \dot{\varphi}(x,t=0) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\chi(k)e^{ikx}, \end{aligned} \tag{6.57}$$

with (6.55) shows that

$$\begin{aligned} \Phi(k) &= a(k) + a^*(-k), \\ \chi(k) &= i\omega_k\Big(a^*(-k) - a(k)\Big). \end{aligned} \tag{6.58}$$

Solving, we find

$$\begin{aligned} a(k) &= \frac{1}{2}\left(\Phi(k) + \frac{i}{\omega_k}\chi(k)\right), \\ a^*(k) &= \frac{1}{2}\left(\Phi(-k) - \frac{i}{\omega_k}\chi(-k)\right). \end{aligned} \tag{6.59}$$

The accumulated wisdom of two hundred years of research on Fourier series and Fourier integrals shows that, when appropriately interpreted, this solution is equivalent to d'Alembert's.

6.3.3 Causal Green function

We now add a source term:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = q(x,t). \tag{6.60}$$

We solve this equation by finding a Green function such that

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)G(x,t;\xi,\tau)=\delta(x-\xi)\delta(t-\tau). \tag{6.61}$$

If the only waves in the system are those produced by the source, we should demand that the Green function be *causal*, in that $G(x,t;\xi,\tau)=0$ if $t<\tau$.

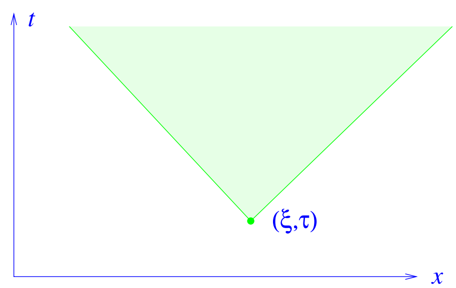

Figure 6.4: *Support of $G(x,t;\xi,\tau)$ for fixed ξ,τ, or the "domain of influence."*

To construct the causal Green function, we integrate the equation over an infinitesimal time interval from $\tau-\varepsilon$ to $\tau+\varepsilon$ and so find Cauchy data

$$\begin{aligned} G(x,\tau+\varepsilon;\xi,\tau) &= 0, \\ \frac{d}{dt}G(x,\tau+\varepsilon;\xi,\tau) &= c^2\delta(x-\xi). \end{aligned} \tag{6.62}$$

We insert this data into d'Alembert's solution to get

$$\begin{aligned} G(x,t;\xi,\tau) &= \theta(t-\tau)\frac{c}{2}\int_{x-c(t-\tau)}^{x+c(t-\tau)}\delta(\zeta-\xi)d\zeta \\ &= \frac{c}{2}\theta(t-\tau)\left\{\theta\Big(x-\xi+c(t-\tau)\Big)-\theta\Big(x-\xi-c(t-\tau)\Big)\right\}. \end{aligned} \tag{6.63}$$

We can now use the Green function to write the solution to the inhomogeneous problem as

$$\varphi(x,t)=\iint G(x,t;\xi,\tau)q(\xi,\tau)\,d\tau d\xi. \tag{6.64}$$

The step-function form of $G(x,t;\xi,\tau)$ allows us to obtain

$$\begin{aligned}\varphi(x,t) &= \iint G(x,t;\xi,\tau)q(\xi,\tau)\,d\tau d\xi,\\ &= \frac{c}{2}\int_{-\infty}^{t} d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\xi,\tau)\,d\xi\\ &= \frac{c}{2}\iint_{\Omega} q(\xi,\tau)\,d\tau d\xi, \end{aligned}\tag{6.65}$$

where the domain of integration Ω is shown in figure 6.5.

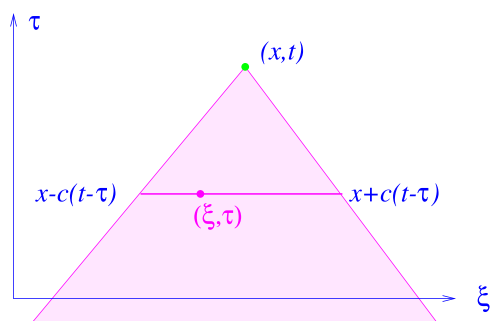

Figure 6.5: *The region Ω, or the "domain of dependence."*

We can write the causal Green function in the form of Fourier's solution of the wave equation. We claim that

$$G(x,t;\xi,\tau) = c^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{\frac{e^{ik(x-\xi)}e^{-i\omega(t-\tau)}}{c^2k^2-(\omega+i\varepsilon)^2}\right\},\tag{6.66}$$

where the $i\varepsilon$ plays the same role in enforcing causality as it does for the harmonic oscillator in one dimension. This is only to be expected. If we decompose a vibrating string into normal modes, then each mode is an independent oscillator with $\omega_k^2 = c^2k^2$, and the Green function for the PDE is simply the sum of the ODE Green functions for each k mode. To confirm our claim, we exploit our previous results for the single-oscillator Green function to evaluate the integral over ω, and we find

$$G(x,t;0,0) = \theta(t)c^2\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\frac{1}{c|k|}\sin(|k|ct).\tag{6.67}$$

Despite the factor of $1/|k|$, there is no singularity at $k = 0$, so no $i\varepsilon$ is needed to make the integral over k well defined. We can do the k integral by recognizing that the integrand is nothing but the Fourier representation, $\frac{2}{k}\sin ak$, of a square-wave pulse. We end up with

$$G(x,t;0,0) = \theta(t)\frac{c}{2}\left\{\theta(x+ct) - \theta(x-ct)\right\}, \tag{6.68}$$

the same expression as from our direct construction. We can also write

$$G(x,t;0,0) = \frac{c}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left(\frac{i}{|k|}\right)\left\{e^{ikx-ic|k|t} - e^{-ikx+ic|k|t}\right\}, \quad t > 0, \tag{6.69}$$

which is in explicit Fourier-solution form with $a(k) = ic/2|k|$.
Illustration: Radiation Damping. Figure 6.6 shows bead of mass M that slides without friction on the y axis. The bead is attached to an infinite string which is initially undisturbed and lying along the x axis. The string has tension T, and a density ρ, so the speed of waves on the string is $c = \sqrt{T/\rho}$. We show that either d'Alembert or Fourier can be used to compute the effect of the string on the motion of the bead.

We first use d'Alembert's general solution to show that wave energy emitted by the moving bead gives rise to an effective viscous damping force on it.

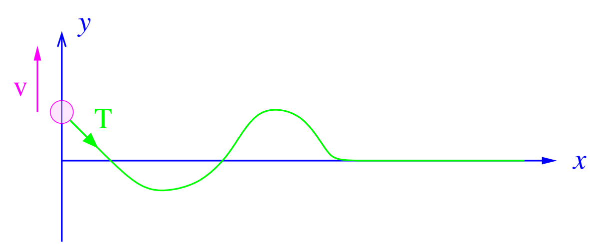

Figure 6.6: *A bead connected to a string.*

The string tension acting on the on the bead leads to the equation of motion $M\dot{v} = Ty'(0,t)$, and from the condition of no incoming waves we know that

$$y(x,t) = y(x-ct). \tag{6.70}$$

Thus $y'(0,t) = -\dot{y}(0,t)/c$. But the bead is attached to the string, so $v(t) = \dot{y}(0,t)$, and therefore

$$M\dot{v} = -\left(\frac{T}{c}\right)v. \tag{6.71}$$

The emitted radiation therefore generates a velocity-dependent drag force with friction coefficient $\eta = T/c$.

We need an infinitely long string for (6.71) to be true for all time. If the string had a finite length L, then, after a period of $2L/c$, energy will be reflected back to the bead and this will complicate matters.

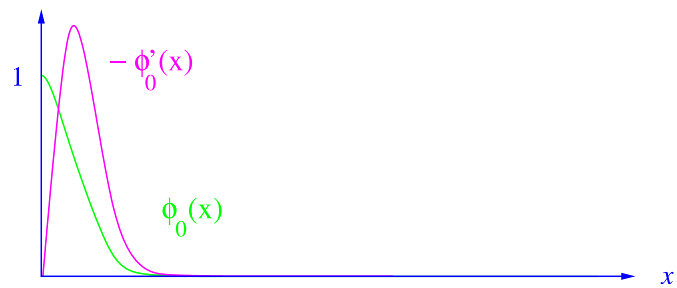

Figure 6.7: *The function* $\phi_0(x)$ *and its derivative.*

We now show that Fourier's mode-decomposition of the string motion, combined with the Caldeira-Leggett analysis of chapter 5, yields the same expression for the radiation damping as the d'Alembert solution. Our bead-string contraption has Lagrangian

$$L = \frac{M}{2}[\dot{y}(0,t)]^2 - V[y(0,t)] + \int_0^L \left\{ \frac{\rho}{2}\dot{y}^2 - \frac{T}{2}y'^2 \right\} dx. \tag{6.72}$$

Here, $V[y]$ is some potential energy for the bead.

To deal with the motion of the bead, we introduce a function $\phi_0(x)$ such that $\phi_0(0) = 1$ and $\phi_0(x)$ decreases rapidly to zero as x increases (see figure 6.7. We therefore have $-\phi_0'(x) \approx \delta(x)$. We expand $y(x,t)$ in terms of $\phi_0(x)$ and the normal modes of a string with fixed ends as

$$y(x,t) = y(0,t)\phi_0(x) + \sum_{n=1}^{\infty} q_n(t)\sqrt{\frac{2}{L\rho}}\sin k_n x. \tag{6.73}$$

Here $k_n L = n\pi$. Because $y(0,t)\phi_0(x)$ describes the motion of only an infinitesimal length of string, $y(0,t)$ makes a negligeable contribution to the string kinetic energy, but it provides a linear coupling of the bead to the string normal modes, $q_n(t)$, through the $Ty'^2/2$ term. Inserting the mode expansion into the Lagrangian, and after about half a page of arithmetic, we end up with

$$L = \frac{M}{2}[\dot{y}(0)]^2 - V[y(0)] + y(0)\sum_{n=1}^{\infty} f_n q_n + \sum_{n=1}^{\infty}\left(\frac{1}{2}\dot{q}_n^2 - \omega_n^2 q_n^2\right) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{f_n^2}{\omega_n^2}\right) y(0)^2, \tag{6.74}$$

where $\omega_n = ck_n$, and

$$f_n = T\sqrt{\frac{2}{L\rho}}k_n. \tag{6.75}$$

This is exactly the Caldeira-Leggett Lagrangian — including their frequency-shift counter-term that reflects that fact that a static displacement of an infinite string results in no additional force on the bead.[1] When L becomes large, the eigenvalue density of states

$$\rho(\omega) = \sum_n \delta(\omega - \omega_n) \tag{6.76}$$

becomes

$$\rho(\omega) = \frac{L}{\pi c}. \tag{6.77}$$

The Caldeira-Leggett spectral function

$$J(\omega) = \frac{\pi}{2}\sum_n \left(\frac{f_n^2}{\omega_n}\right)\delta(\omega - \omega_n), \tag{6.78}$$

is therefore

$$J(\omega) = \frac{\pi}{2}\cdot\frac{2T^2k^2}{L\rho}\cdot\frac{1}{kc}\cdot\frac{L}{\pi c} = \left(\frac{T}{c}\right)\omega, \tag{6.79}$$

where we have used $c = \sqrt{T/\rho}$. Comparing with Caldeira-Leggett's $J(\omega) = \eta\omega$, we see that the effective viscosity is given by $\eta = T/c$, as before. The necessity of having an infinitely long string here translates into the requirement that we must have a *continuum* of oscillator modes. It is only after the sum over discrete modes ω_i is replaced by an integral over the continuum of ω's that no energy is ever returned to the system being damped.

[1] For a *finite* length of string that is fixed at the far end, the string tension *does* add $\frac{1}{2}Ty(0)^2/L$ to the static potential. In the mode expansion, this additional restoring force arises from the first term of $-\phi_0'(x) \approx 1/L + (2/L)\sum_{n=1}^{\infty}\cos k_n x$ in $\frac{1}{2}Ty(0)^2\int(\phi_0')^2\,dx$. The subsequent terms provide the Caldeira-Leggett counter-term. The first-term contribution has been omitted in (6.74) as being unimportant for large L.

For our bead and string, the mode-expansion approach is more complicated than d'Alembert's. In the important problem of the drag forces induced by the emission of radiation from an accelerated charged particle, however, the mode-expansion method leads to an informative resolution[2] of the pathologies of the Abraham-Lorentz equation,

$$M(\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}}) = \mathbf{F}_{\text{ext}}, \quad \tau = \frac{2}{3}\frac{e^2}{Mc^3}\frac{1}{4\pi\varepsilon_0} \tag{6.80}$$

which is plagued by runaway, or apparently acausal, solutions.

6.3.4 Odd *vs.* even dimensions

Consider the wave equation for sound in the three dimensions. We have a velocity potential ϕ which obeys the wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{6.81}$$

and from which the velocity, density, and pressure fluctuations can be extracted as

$$\begin{aligned} v_1 &= \nabla\phi, \\ \rho_1 &= -\frac{\rho_0}{c^2}\dot{\phi}, \\ P_1 &= c^2\rho_1. \end{aligned} \tag{6.82}$$

In three dimensions, and considering only spherically symmetric waves, the wave equation becomes

$$\frac{\partial^2(r\phi)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2(r\phi)}{\partial t^2} = 0, \tag{6.83}$$

with solution

$$\phi(r,t) = \frac{1}{r}f\left(t - \frac{r}{c}\right) + \frac{1}{r}g\left(t + \frac{r}{c}\right). \tag{6.84}$$

Consider what happens if we put a point volume source at the origin (the sudden conversion of a negligeable volume of solid explosive to a large volume

[2]G. W. Ford, R. F. O'Connell, *Phys. Lett.* A **157** (1991) 217.

of hot gas, for example). Let the rate at which volume is being intruded be $\dot{q}$. The gas velocity very close to the origin will be

$$v(r,t) = \frac{\dot{q}(t)}{4\pi r^2}. \tag{6.85}$$

Matching this to an outgoing wave gives

$$\frac{\dot{q}(t)}{4\pi r^2} = v_1(r,t) = \frac{\partial\phi}{\partial r} = -\frac{1}{r^2} f\left(t - \frac{r}{c}\right) - \frac{1}{rc} f'\left(t - \frac{r}{c}\right). \tag{6.86}$$

Close to the origin, in the *near field*, the term $\propto f/r^2$ will dominate, and so

$$-\frac{1}{4\pi}\dot{q}(t) = f(t). \tag{6.87}$$

Further away, in the *far field* or *radiation field*, only the second term will survive, and so

$$v_1 = \frac{\partial\phi}{\partial r} \approx -\frac{1}{rc} f'\left(t - \frac{r}{c}\right). \tag{6.88}$$

The far-field velocity-pulse profile v_1 is therefore the derivative of the near-field v_1 pulse profile.

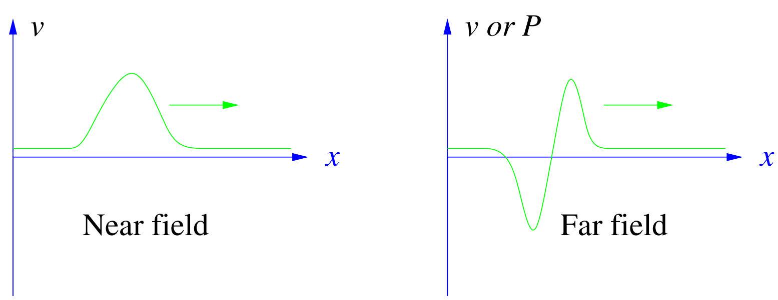

Figure 6.8: *Three-dimensional blast wave.*

The pressure pulse

$$P_1 = -\rho_0\dot{\phi} = \frac{\rho_0}{4\pi r}\ddot{q}\left(t - \frac{r}{c}\right) \tag{6.89}$$

is also of this form. Thus, a sudden localized expansion of gas produces an outgoing pressure pulse which is first positive and then negative.

This phenomenon can be seen in (old, we hope) news footage of bomb blasts in tropical regions. A spherical vapour condensation wave can been seen spreading out from the explosion. The condensation cloud is caused by the air cooling below the dew-point in the low-pressure region which tails the over-pressure blast.

Now consider what happens if we have a sheet of explosive, the simultaneous detonation of every part of which gives us a one-dimensional plane-wave pulse. We can obtain the plane wave by adding up the individual spherical waves from each point on the sheet.

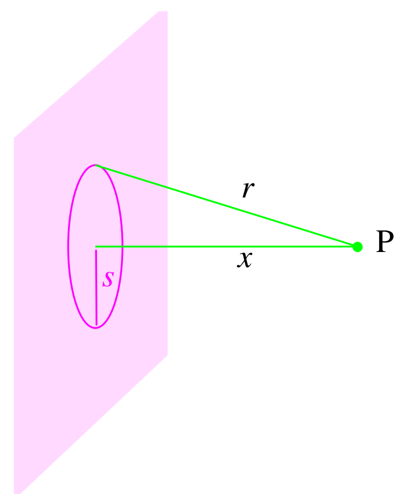

Figure 6.9: *Sheet-source geometry.*

Using the notation defined in figure 6.9, we have

$$\phi(x,t) = 2\pi \int_0^\infty \frac{1}{\sqrt{x^2+s^2}} f\left(t - \frac{\sqrt{x^2+s^2}}{c}\right) s ds \tag{6.90}$$

with $f(t) = -\dot{q}(t)/4\pi$, where now $\dot{q}$ is the rate at which volume is being intruded per unit area of the sheet. We can write this as

$$\begin{aligned} & 2\pi \int_0^\infty f\left(t - \frac{\sqrt{x^2+s^2}}{c}\right) d\sqrt{x^2+s^2}, \\ = \quad & 2\pi c \int_{-\infty}^{t-x/c} f(\tau)\, d\tau, \end{aligned}$$

$$= \quad -\frac{c}{2}\int_{-\infty}^{t-x/c} \dot{q}(\tau)\, d\tau. \tag{6.91}$$

In the second line we have defined $\tau = t - \sqrt{x^2+s^2}/c$, which, *inter alia*, interchanged the role of the upper and lower limits on the integral.

Thus, $v_1 = \phi'(x,t) = \frac{1}{2}\dot{q}(t - x/c)$. Since the near field motion produced by the intruding gas is $v_1(r) = \frac{1}{2}\dot{q}(t)$, the far-field displacement exactly reproduces the initial motion, suitably delayed of course. (The factor 1/2 is because half the intruded volume goes towards producing a pulse in the negative direction.)

In three dimensions, the far-field motion is the first derivative of the near-field motion. In one dimension, the far-field motion is exactly the same as the near-field motion. In two dimensions the far-field motion should therefore be the half-derivative of the near-field motion — but how do you half-differentiate a function? An answer is suggested by the theory of Laplace transformations as

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}} F(t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} \frac{\dot{F}(\tau)}{\sqrt{t-\tau}} d\tau. \tag{6.92}$$

Let us now repeat the explosive sheet calculation for an exploding wire.

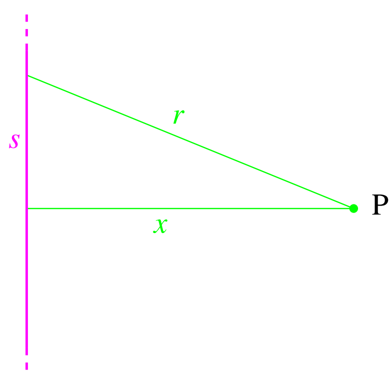

Figure 6.10: *Line-source geometry.*

Using the geometry shown in figure 6.10, we have

$$ds = d\left(\sqrt{r^2 - x^2}\right) = \frac{r\, dr}{\sqrt{r^2-x^2}}, \tag{6.93}$$

and combining the contributions of the two parts of the wire that are the same distance from p, we can write

$$\begin{aligned}\phi(x,t) &= \int_x^\infty \frac{1}{r} f\left(t-\frac{r}{c}\right)\frac{2r\,dr}{\sqrt{r^2-x^2}}\\ &= 2\int_x^\infty f\left(t-\frac{r}{c}\right)\frac{dr}{\sqrt{r^2-x^2}},\end{aligned}\tag{6.94}$$

with $f(t) = -\dot{q}(t)/4\pi$, where now $\dot{q}$ is the volume intruded per unit length. We may approximate $r^2 - x^2 \approx 2x(r-x)$ for the near parts of the wire where $r \approx x$, since these make the dominant contribution to the integral. We also set $\tau = t - r/c$, and then have

$$\begin{aligned}\phi(x,t) &= \frac{2c}{\sqrt{2x}}\int_{-\infty}^{(t-x/c)} f(\tau)\frac{dr}{\sqrt{(ct-x)-c\tau}},\\ &= -\frac{1}{2\pi}\sqrt{\frac{2c}{x}}\int_{-\infty}^{(t-x/c)}\dot{q}(\tau)\frac{d\tau}{\sqrt{(t-x/c)-\tau}}.\end{aligned}\tag{6.95}$$

The far-field velocity is the x gradient of this,

$$v_1(r,t) = \frac{1}{2\pi c}\sqrt{\frac{2c}{x}}\int_{-\infty}^{(t-x/c)}\ddot{q}(\tau)\frac{d\tau}{\sqrt{(t-x/c)-\tau}},\tag{6.96}$$

and is therefore proportional to the 1/2-derivative of $\dot{q}(t - r/c)$.

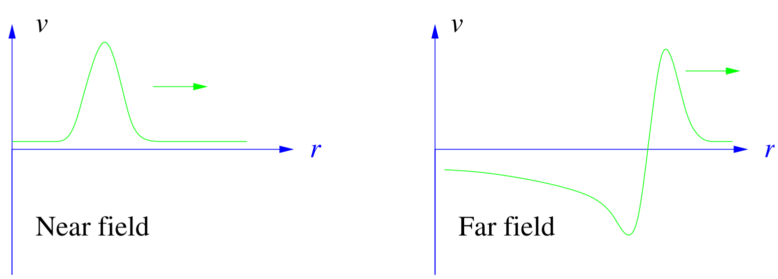

Figure 6.11: *In two dimensions the far-field pulse has a long tail.*

A plot of near field and far field motions in figure 6.11 shows how the far-field pulse never completely dies away to zero. This long tail means that one cannot use digital signalling in two dimensions.

Moral Tale: One of our colleagues was performing numerical work on earthquake propagation. The source of his waves was a long deep linear fault, so he used the two-dimensional wave equation. Not wanting to be troubled by the actual creation of the wave pulse, he took as initial data an outgoing finite-width pulse. After a short propagation time his numerical solution appeared to misbehave. New pulses were being emitted from the fault long after the initial one. He wasted several months in vain attempt to improve the stability of his code before he realized that what he was seeing was real. The lack of a long tail on his pulse meant that it could not have been created by a briefly-active line source. The new "unphysical" waves were a consequence of the source striving to suppress the long tail of the initial pulse. *Moral*: Always check that a solution of the form you seek actually exists before you waste your time trying to compute it.

Exercise 6.4: Use the calculus of improper integrals to show that, provided $F(-\infty)=0$, we have

$$\frac{d}{dt}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\dot{F}(\tau)}{\sqrt{t-\tau}}d\tau\right)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\ddot{F}(\tau)}{\sqrt{t-\tau}}d\tau. \tag{6.97}$$

This means that

$$\frac{d}{dt}\left(\frac{d}{dt}\right)^{\frac{1}{2}}F(t)=\left(\frac{d}{dt}\right)^{\frac{1}{2}}\frac{d}{dt}F(t). \tag{6.98}$$

6.4 Heat equation

Fourier's heat equation

$$\frac{\partial\phi}{\partial t}=\kappa\frac{\partial^2\phi}{\partial x^2} \tag{6.99}$$

is the archetypal parabolic equation. It often comes with initial data $\phi(x,t=0)$, but this is not Cauchy data, as the curve $t=const.$ is a characteristic.

The heat equation is also known as the *diffusion equation*.

6.4.1 Heat kernel

If we Fourier transform the initial data

$$\phi(x,t=0)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\tilde{\phi}(k)e^{ikx}, \tag{6.100}$$

and write

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\phi}(k,t) e^{ikx}, \tag{6.101}$$

we can plug this into the heat equation and find that

$$\frac{\partial \tilde{\phi}}{\partial t} = -\kappa k^2 \tilde{\phi}. \tag{6.102}$$

Hence,

$$\begin{aligned} \phi(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\phi}(k,t) e^{ikx} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\phi}(k,0) e^{ikx-\kappa k^2 t}. \end{aligned} \tag{6.103}$$

We may now express $\tilde{\phi}(k,0)$ in terms of $\phi(x,0)$ and rearrange the order of integration to get

$$\begin{aligned} \phi(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\int_{-\infty}^{\infty} \phi(\xi,0) e^{ik\xi}\, d\xi \right) e^{ikx-\kappa k^2 t} \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-\xi)-\kappa k^2 t} \right) \phi(\xi,0) d\xi \\ &= \int_{-\infty}^{\infty} G(x,\xi,t) \phi(\xi,0)\, d\xi, \end{aligned} \tag{6.104}$$

where

$$G(x,\xi,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-\xi)-\kappa k^2 t} = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left\{ -\frac{1}{4\kappa t}(x-\xi)^2 \right\}. \tag{6.105}$$

Here, $G(x,\xi,t)$ is the *heat kernel*. It represents the spreading of a unit blob of heat.

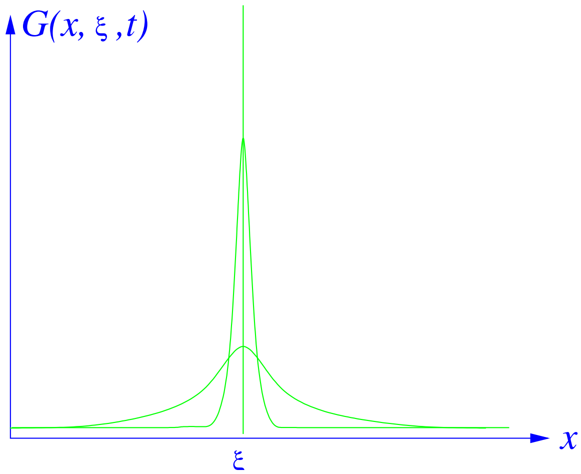

Figure 6.12: *The heat kernel at three successive times.*

As the heat spreads, the total amount of heat, represented by the area under the curve in figure 6.12, remains constant:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}} \exp\left\{-\frac{1}{4\kappa t}(x-\xi)^2\right\} dx = 1. \tag{6.106}$$

The heat kernel possesses a *semigroup property*

$$G(x,\xi,t_1+t_2) = \int_{-\infty}^{\infty} G(x,\eta,t_2)G(\eta,\xi,t_1)d\eta. \tag{6.107}$$

Exercise: Prove this.

6.4.2 Causal Green function

Now we consider the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = q(x,t), \tag{6.108}$$

with initial data $u(x,0) = u_0(x)$. We define a Causal Green function by

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) G(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau) \tag{6.109}$$

and the requirement that $G(x,t;\xi,\tau)=0$ if $t<\tau$. Integrating the equation from $t=\tau-\varepsilon$ to $t=\tau+\varepsilon$ tells us that

$$G(x,\tau+\varepsilon;\xi,\tau)=\delta(x-\xi). \tag{6.110}$$

Taking this delta function as initial data $\phi(x,t=\tau)$ and inserting into (6.104) we read off

$$G(x,t;\xi,\tau)=\theta(t-\tau)\frac{1}{\sqrt{4\pi(t-\tau)}}\exp\left\{-\frac{1}{4(t-\tau)}(x-\xi)^2\right\}. \tag{6.111}$$

We apply this Green function to the solution of a problem involving both a heat source and initial data given at $t=0$ on the entire real line. We exploit a variant of the Lagrange-identity method we used for solving one-dimensional ODE's with inhomogeneous boundary conditions. Let

$$D_{x,t}\equiv\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}, \tag{6.112}$$

and observe that its formal adjoint,

$$D^\dagger_{x,t}\equiv-\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}. \tag{6.113}$$

is a "backward" heat-equation operator. The corresponding "backward" Green function

$$G^\dagger(x,t;\xi,\tau)=\theta(\tau-t)\frac{1}{\sqrt{4\pi(\tau-t)}}\exp\left\{-\frac{1}{4(\tau-t)}(x-\xi)^2\right\} \tag{6.114}$$

obeys

$$D^\dagger_{x,t}G^\dagger(x,t;\xi,\tau)=\delta(x-\xi)\delta(t-\tau), \tag{6.115}$$

with adjoint boundary conditions. These make $G^\dagger$ *anti-causal*, in that $G^\dagger(t-\tau)$ vanishes when $t>\tau$. Now we make use of the two-dimensional Lagrange identity

$$\begin{aligned}\int_{-\infty}^{\infty}dx\int_0^T dt\,\Big\{u(x,t)D^\dagger_{x,t}G^\dagger(x,t;\xi,\tau)-\Big(D_{x,t}u(x,t)\Big)G^\dagger(x,t;\xi,\tau)\Big\}\\ =\int_{-\infty}^{\infty}dx\,\big\{u(x,0)G^\dagger(x,0;\xi,\tau)\big\}-\int_{-\infty}^{\infty}dx\,\big\{u(x,T)G^\dagger(x,T;\xi,\tau)\big\}.\end{aligned} \tag{6.116}$$

Assume that (ξ, τ) lies within the region of integration. Then the left hand side is equal to

$$u(\xi, \tau) - \int_{-\infty}^{\infty} dx \int_{0}^{T} dt \left\{ q(x,t) G^{\dagger}(x,t;\xi,\tau) \right\}. \tag{6.117}$$

On the right hand side, the second integral vanishes because $G^{\dagger}$ is zero on $t = T$. Thus,

$$u(\xi,\tau) = \int_{-\infty}^{\infty} dx \int_{0}^{T} dt \Big\{ q(x,t) G^{\dagger}(x,t;\xi,\tau) \Big\} + \int_{-\infty}^{\infty} \left\{ u(x,0) G^{\dagger}(x,0;\xi,\tau) \right\} dx \tag{6.118}$$

Rewriting this by using

$$G^{\dagger}(x,t;\xi,\tau) = G(\xi,\tau;x,t), \tag{6.119}$$

and relabeling $x \leftrightarrow \xi$ and $t \leftrightarrow \tau$, we have

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t;\xi,0) u_0(\xi)\, d\xi + \int_{-\infty}^{\infty} \int_{0}^{t} G(x,t;\xi,\tau) q(\xi,\tau) d\xi d\tau. \tag{6.120}$$

Note how the effects of any heat source $q(x,t)$ active prior to the initial-data epoch at $t = 0$ have been subsumed into the evolution of the initial data.

6.4.3 Duhamel's principle

Often, the temperature of the spatial boundary of a region is specified in addition to the initial data. Dealing with this type of problem leads us to a new strategy.

Suppose we are required to solve

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{6.121}$$

for the semi-infinite rod shown in figure 6.13. We are given a specified temperature, $u(0,t) = h(t)$, at the end $x = 0$, and for all other points $x > 0$ we are given an initial condition $u(x, 0) = 0$.

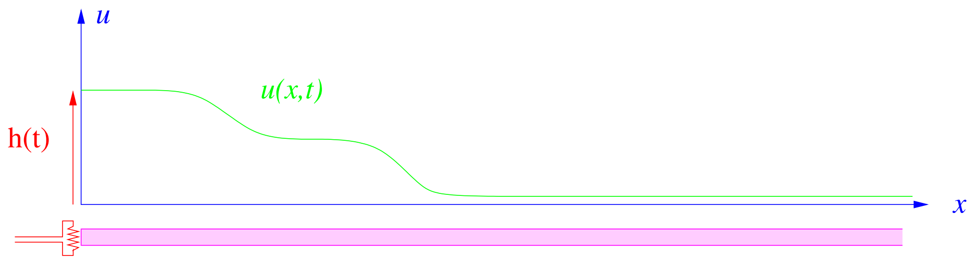

Figure 6.13: *Semi-infinite rod heated at one end.*

We begin by finding a solution $w(x,t)$ that satisfies the heat equation with $w(0,t) = 1$ and initial data $w(x,0) = 0$, $x > 0$. This solution is constructed in problem 6.14, and is

$$w = \theta(t)\left\{1 - \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right\}. \tag{6.122}$$

Here $\mathrm{erf}(x)$ is the *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz. \tag{6.123}$$

which has the properties that $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(x) \to 1$ as $x \to \infty$. See figure 6.14.

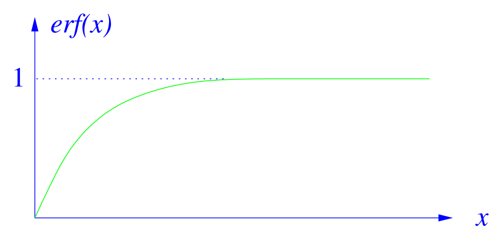

Figure 6.14: *Error function.*

If we were given

$$h(t) = h_0\theta(t - t_0), \tag{6.124}$$

then the desired solution would be

$$u(x,t) = h_0 w(x, t - t_0). \tag{6.125}$$

For a sum

$$h(t) = \sum_n h_n \theta(t - t_n), \tag{6.126}$$

the principle of superposition (*i.e.* the linearity of the problem) tell us that the solution is the corresponding sum

$$u(x,t) = \sum_n h_n w(x, t - t_n). \tag{6.127}$$

We therefore decompose $h(t)$ into a sum of step functions

$$\begin{aligned} h(t) &= h(0) + \int_0^t \dot{h}(\tau)\, d\tau \\ &= h(0) + \int_0^\infty \theta(t - \tau)\dot{h}(\tau)\, d\tau. \end{aligned} \tag{6.128}$$

It is should now be clear that

$$\begin{aligned} u(x,t) &= \int_0^t w(x, t-\tau)\dot{h}(\tau)\, d\tau + h(0)w(x,t) \\ &= -\int_0^t \left(\frac{\partial}{\partial \tau} w(x, t-\tau)\right) h(\tau)\, d\tau \\ &= \int_0^t \left(\frac{\partial}{\partial t} w(x, t-\tau)\right) h(\tau)\, d\tau. \end{aligned} \tag{6.129}$$

This is called *Duhamel's solution*, and the trick of expressing the data as a sum of Heaviside step functions is called Duhamel's principle.

We do not need to be as clever as Duhamel. We could have obtained this result by using the method of images to find a suitable causal Green function for the half line, and then using the same Lagrange-identity method as before.

6.5 Potential theory

The study of boundary-value problems involving the Laplacian is usually known as "'Potential Theory." We seek solutions to these problems in some region Ω, whose boundary we denote by the symbol $\partial\Omega$.

Poisson's equation, $-\nabla^2\chi(\mathbf{r}) = f(\mathbf{r})$, $\mathbf{r} \in \Omega$, and the Laplace equation to which it reduces when $f(\mathbf{r}) \equiv 0$, come along with various boundary conditions, of which the commonest are

$$\begin{aligned} \chi = g(\mathbf{r}) \quad &\text{on} \quad \partial\Omega, \qquad \text{(Dirichlet)} \\ (\mathbf{n}\cdot\nabla)\chi = g(\mathbf{r}) \quad &\text{on} \quad \partial\Omega. \qquad \text{(Neumann)} \end{aligned} \tag{6.130}$$

A function for which $\nabla^2\chi = 0$ in some region Ω is said to be *harmonic* there.

6.5.1 Uniqueness and existence of solutions

We begin by observing that we need to be a little more precise about what it means for a solution to "take" a given value on a boundary. If we ask for a solution to the problem $\nabla^2\varphi = 0$ within $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $\varphi = 1$ on $\partial\Omega$, someone might claim that the function defined by setting $\varphi(x, y) = 0$ for $x^2 + y^2 < 1$ and $\varphi(x, y) = 1$ for $x^2 + y^2 = 1$ does the job—but such a discontinuous "solution" is hardly what we had in mind when we stated the problem. We must interpret the phrase "takes a given value on the boundary" as meaning that the boundary data is the limit, as we approach the boundary, of the solution within Ω.

With this understanding, we assert that a function harmonic in a bounded subset Ω of $\mathbb{R}^n$ is uniquely determined by the values it takes on the boundary of Ω. To see that this is so, suppose that φ_1 and φ_2 both satisfy $\nabla^2\varphi = 0$ in Ω, and coincide on the boundary. Then $\chi = \varphi_1 - \varphi_2$ obeys $\nabla^2\chi = 0$ in Ω, and is zero on the boundary. Integrating by parts we find that

$$\int_\Omega |\nabla\chi|^2 d^n r = \int_{\partial\Omega} \chi(\mathbf{n}\cdot\nabla)\chi \, dS = 0. \tag{6.131}$$

Here dS is the element of area on the boundary and $\mathbf{n}$ the outward-directed normal. Now, because the second derivatives exist, the partial derivatives entering into $\nabla\chi$ must be continuous, and so the vanishing of integral of $|\nabla\chi|^2$ tells us that $\nabla\chi$ is zero everywhere within Ω. This means that χ is constant — and because it is zero on the boundary it is zero everywhere.

An almost identical argument shows that if Ω is a bounded *connected* region, and if φ_1 and φ_2 both satisfy $\nabla^2\varphi = 0$ within Ω and take the same values of $(\mathbf{n}\cdot\nabla)\varphi$ on the boundary, then $\varphi_1 = \varphi_2 + \mathit{const}$. We have therefore shown that, if it exists, the solutions of the Dirichlet boundary value problem is unique, and the solution of the Neumann problem is unique up to the addition of an arbitrary constant.

In the Neumann case, with boundary condition $(\mathbf{n}\cdot\nabla)\varphi = g(\mathbf{r})$, and integration by parts gives

$$\int_\Omega \nabla^2\varphi\, d^n r = \int_{\partial\Omega} (\mathbf{n}\cdot\nabla)\varphi\, dS = \int_{\partial\Omega} g\, dS, \tag{6.132}$$

and so the boundary data $g(\mathbf{r})$ must satisfy $\int_{\partial\Omega} g\, dS = 0$ if a solution to $\nabla^2\varphi = 0$ is to exist. This is an example of the Fredhom alternative that relates the existence of a non-trivial null space to constraints on the source terms. For the inhomogeneous equation $-\nabla^2\varphi = f$, the Fredholm constraint becomes

$$\int_{\partial\Omega} g\, dS + \int_\Omega f\, d^n r = 0. \tag{6.133}$$

Given that we have satisfied any Fredholm constraint, do solutions to the Dirichlet and Neumann problem always exist? That solutions *should* exist is suggested by physics: the Dirichlet problem corresponds to an electrostatic problem with specified boundary potentials and the Neumann problem corresponds to finding the electric potential within a resistive material with prescribed current sources on the boundary. The Fredholm constraint says that if we drive current into the material, we must must let it out somewhere. Surely solutions always exist to these physics problems? In the Dirichlet case we can even make a mathematically plausible argument for existence: We observe that the boundary-value problem

$$\begin{aligned} \nabla^2\varphi &= 0, \quad \mathbf{r}\in\Omega \\ \varphi &= f, \quad \mathbf{r}\in\partial\Omega \end{aligned} \tag{6.134}$$

is solved by taking φ to be the χ that minimizes the functional

$$J[\chi] = \int_\Omega |\nabla\chi|^2 d^n r \tag{6.135}$$

over the set of continuously differentiable functions taking the given boundary values. Since $J[\chi]$ is positive, and hence bounded below, it seems intuitively obvious that there must be some function χ for which $J[\chi]$ is a minimum. The appeal of this *Dirichlet principle* argument led even Riemann astray. The fallacy was exposed by Weierstrass who provided counterexamples.

Consider, for example, the problem of finding a function $\varphi(x, y)$ obeying $\nabla^2\varphi = 0$ within the punctured disc $D' = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$ with boundary data $\varphi(x, y) = 1$ on the outer boundary at $x^2 + y^2 = 1$ and $\varphi(0, 0) = 0$ on the inner boundary at the origin. We substitute the trial functions

$$\chi_\alpha(x, y) = (x^2 + y^2)^\alpha, \quad \alpha > 0, \tag{6.136}$$

all of which satisfy the boundary data, into the positive functional

$$J[\chi] = \int_{D'} |\nabla\chi|^2\, dxdy \tag{6.137}$$

to find $J[\chi_\alpha] = 2\pi\alpha$. This number can be made as small as we like, and so the infimum of the functional $J[\chi]$ is zero. But if there is a minimizing φ, then $J[\varphi] = 0$ implies that φ is a constant, and a constant cannot satisfy the boundary conditions.

An analogous problem reveals itself in three dimensions when the boundary of Ω has a sharp re-entrant spike that is held at a different potential from the rest of the boundary. In this case we can again find a sequence of trial functions $\chi(\mathbf{r})$ for which $J[\chi]$ becomes arbitrarily small, but the sequence of χ's has no limit satisfying the boundary conditions. The physics argument also fails: if we tried to create a physical realization of this situation, the electric field would become infinite near the spike, and the charge would leak off and and thwart our attempts to establish the potential difference. For reasonably smooth boundaries, however, a minimizing function *does* exist.

The Dirichlet-Poisson problem

$$\begin{aligned} -\nabla^2\varphi(\mathbf{r}) &= f(\mathbf{r}), \quad \mathbf{r} \in \Omega, \\ \varphi(\mathbf{r}) &= g(\mathbf{r}), \quad \mathbf{r} \in \partial\Omega, \end{aligned} \tag{6.138}$$

and the Neumann-Poisson problem

$$\begin{aligned} -\nabla^2\varphi(\mathbf{r}) &= f(\mathbf{r}), \quad x \in \Omega, \\ (\mathbf{n}\cdot\nabla)\varphi(\mathbf{r}) &= g(\mathbf{r}), \quad x \in \partial\Omega \end{aligned}$$

supplemented with the Fredholm constraint

$$\int_\Omega f\, d^n r + \int_{\partial\Omega} g\, dS = 0 \tag{6.139}$$

also have solutions when $\partial\Omega$ is reasonably smooth. For the Neumann-Poisson problem, with the Fredholm constraint as stated, the region Ω must be connected, but its boundary need not be. For example, Ω can be the region between two nested spherical shells.

Exercise 6.5: Why did we insist that the region Ω be connected in our discussion of the Neumann problem? (Hint: how must we modify the Fredholm constraint when Ω consists of two or more disconnected regions?)

Exercise 6.6: *Neumann variational principles.* Let Ω be a bounded and connected three-dimensional region with a smooth boundary. Given a function f defined on Ω and such that $\int_\Omega f\, d^3r = 0$, define the functional

$$J[\chi] = \int_\Omega \left\{ \frac{1}{2}|\nabla\chi|^2 - \chi f \right\} d^3r.$$

Suppose that φ is a solution of the Neumann problem

$$\begin{aligned} -\nabla^2\varphi(\mathbf{r}) &= f(\mathbf{r}), \quad \mathbf{r} \in \Omega, \\ (\mathbf{n}\cdot\nabla)\varphi(\mathbf{r}) &= 0, \quad \mathbf{r} \in \partial\Omega. \end{aligned}$$

Show that

$$J[\chi] = J[\varphi] + \int_\Omega \frac{1}{2}|\nabla(\chi-\varphi)|^2\, d^3r \geq J[\varphi] = -\int_\Omega \frac{1}{2}|\nabla\varphi|^2\, d^3r = -\frac{1}{2}\int_\Omega \varphi f\, d^3r.$$

Deduce that φ is determined, up to the addition of a constant, as the function that minimizes $J[\chi]$ over the space of all continuously differentiable χ (and not just over functions satisfying the Neumann boundary condition.)

Similarly, for g a function defined on the boundary $\partial\Omega$ and such that $\int_{\partial\Omega} g\, dS = 0$, set

$$K[\chi] = \int_\Omega \frac{1}{2}|\nabla\chi|^2\, d^3r - \int_{\partial\Omega} \chi g\, dS.$$

Now suppose that ϕ is a solution of the Neumann problem

$$\begin{aligned} -\nabla^2\phi(\mathbf{r}) &= 0, \quad \mathbf{r} \in \Omega, \\ (\mathbf{n}\cdot\nabla)\phi(\mathbf{r}) &= g(\mathbf{r}), \quad \mathbf{r} \in \partial\Omega. \end{aligned}$$

Show that

$$K[\chi] = K[\phi] + \int_\Omega \frac{1}{2}|\nabla(\chi-\phi)|^2\, d^3r \geq K[\phi] = -\int_\Omega \frac{1}{2}|\nabla\phi|^2\, d^3r = -\frac{1}{2}\int_{\partial\Omega} \phi g\, dS.$$

Deduce that ϕ is determined up to a constant as the function that minimizes $K[\chi]$ over the space of all continuously differentiable χ (and, again, not just over functions satisfying the Neumann boundary condition.)

Show that when f and g fail to satisfy the integral conditions required for the existence of the Neumann solution, the corresponding functionals are not bounded below, and so no minimizing function can exist.

Exercise 6.7: *Helmholtz decomposition* Let Ω be a bounded connected three-dimensional region with smooth boundary $\partial\Omega$.

a) Cite the conditions for the existence of a solution to a suitable Neumann problem to show that if $\mathbf{u}$ is a smooth vector field defined in Ω, then there exist a unique solenoidal (*i.e* having zero divergence) vector field $\mathbf{v}$ with $\mathbf{v}\cdot\mathbf{n} = \mathbf{0}$ on the boundary $\partial\Omega$, and a unique (up to the addition of a constant) scalar field ϕ such that

$$\mathbf{u} = \mathbf{v} + \nabla\phi.$$

Here $\mathbf{n}$ is the outward normal to the (assumed smooth) bounding surface of Ω.

b) In many cases (but not always) we can write a solenoidal vector field $\mathbf{v}$ as $\mathbf{v} = \mathrm{curl}\,\mathbf{w}$. Again by appealing to the conditions for existence and uniqueness of a Neumann problem solution, show that if we *can* write $\mathbf{v} = \mathrm{curl}\,\mathbf{w}$, then $\mathbf{w}$ is not unique, but we can always make it unique by demanding that it obey the conditions $\mathrm{div}\,\mathbf{w} = 0$ and $\mathbf{w}\cdot\mathbf{n} = 0$.

c) Appeal to the Helmholtz decomposition of part a) with $\mathbf{u} \to (\mathbf{v}\cdot\nabla)\mathbf{v}$ to show that in the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P, \quad \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega$$

governing the motion of an incompressible ($\mathrm{div}\,\mathbf{v} = 0$) fluid the instantaneous flow field $\mathbf{v}(x, y, z, t)$ uniquely determines $\partial\mathbf{v}/\partial t$, and hence the time evolution of the flow. (This observation provides the basis of practical algorithms for computing incompressible flows.)

We can always write the solenoidal field as $\mathbf{v} = \mathrm{curl}\,\mathbf{w} + \mathbf{h}$, where $\mathbf{h}$ obeys $\nabla^2\mathbf{h} = 0$ with suitable boundary conditions. See exercise 6.16.

6.5.2 Separation of variables

Cartesian coordinates

When the region of interest is a square or a rectangle, we can solve Laplace boundary problems by separating the Laplace operator in cartesian co-ordinates.

Let

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \tag{6.140}$$

and write

$$\varphi = X(x)Y(y), \tag{6.141}$$

so that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0. \tag{6.142}$$

Since the first term is a function of x only, and the second of y only, both must be constants and the sum of these constants must be zero. Therefore

$$\begin{aligned} \frac{1}{X}\frac{\partial^2 X}{\partial x^2} &= -k^2, \\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} &= k^2, \end{aligned} \tag{6.143}$$

or, equivalently

$$\begin{aligned} \frac{\partial^2 X}{\partial x^2} + k^2 X &= 0, \\ \frac{\partial^2 Y}{\partial y^2} - k^2 Y &= 0. \end{aligned} \tag{6.144}$$

The number that we have, for later convenience, written as k^2 is called a *separation constant*. The solutions are $X = e^{\pm ikx}$ and $Y = e^{\pm ky}$. Thus

$$\varphi = e^{\pm ikx} e^{\pm ky}, \tag{6.145}$$

or a sum of such terms where the allowed k's are determined by the boundary conditions.

How do we know that the separated form $X(x)Y(y)$ captures all possible solutions? We can be confident that we have them all if we can use the separated solutions to solve boundary-value problems with arbitrary boundary data.

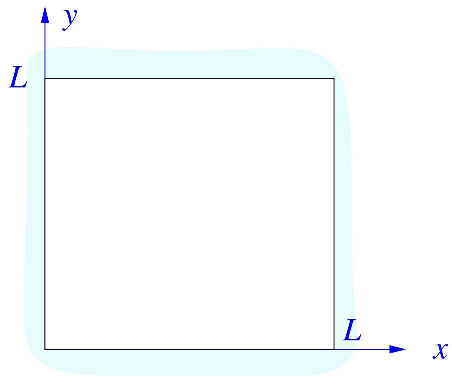

Figure 6.15: *Square region.*

We can use our separated solutions to construct the unique harmonic function taking given values on the sides a square of side L shown in figure 6.15. To see how to do this, consider the four families of functions

$$\begin{aligned}
\varphi_{1,n} &= \sqrt{\frac{2}{L}}\frac{1}{\sinh n\pi}\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L},\\
\varphi_{2,n} &= \sqrt{\frac{2}{L}}\frac{1}{\sinh n\pi}\sinh\frac{n\pi x}{L}\sin\frac{n\pi y}{L},\\
\varphi_{3,n} &= \sqrt{\frac{2}{L}}\frac{1}{\sinh n\pi}\sin\frac{n\pi x}{L}\sinh\frac{n\pi(L-y)}{L},\\
\varphi_{4,n} &= \sqrt{\frac{2}{L}}\frac{1}{\sinh n\pi}\sinh\frac{n\pi(L-x)}{L}\sin\frac{n\pi y}{L}.
\end{aligned} \tag{6.146}$$

Each of these comprises solutions to $\nabla^2\varphi = 0$. The family $\varphi_{1,n}(x,y)$ has been constructed so that every member is zero on three sides of the square, but on the side $y = L$ it becomes $\varphi_{1,n}(x,L) = \sqrt{2/L}\sin(n\pi x/L)$. The $\varphi_{1,n}(x,L)$ therefore constitute an complete orthonormal set in terms of which we can expand the boundary data on the side $y = L$. Similarly, the other other families are non-zero on only one side, and are complete there. Thus, any boundary data can be expanded in terms of these four function sets, and the solution to the boundary value problem is given by a sum

$$\varphi(x,y) = \sum_{m=1}^{4}\sum_{n=1}^{\infty} a_{m,n}\varphi_{m,n}(x,y). \tag{6.147}$$

The solution to $\nabla^2\varphi = 0$ in the unit square with $\varphi = 1$ on the side $y = 1$ and zero on the other sides is, for example,

$$\varphi(x,y) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi}\frac{1}{\sinh(2n+1)\pi}\sin\Big((2n+1)\pi x\Big)\sinh\Big((2n+1)\pi y\Big) \tag{6.148}$$

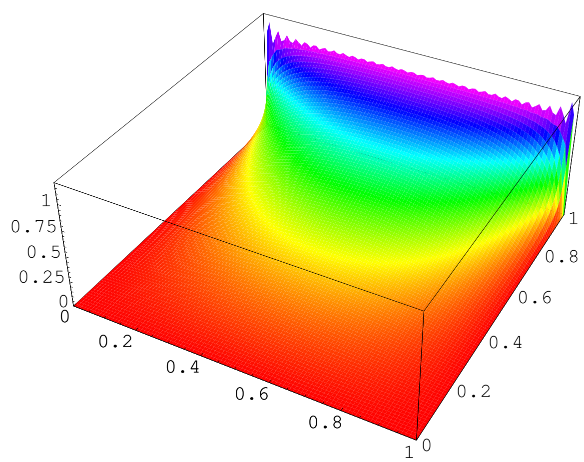

Figure 6.16: *Plot of first thiry terms in equation (6.148).*

For cubes, and higher dimensional hypercubes, we can use similar boundary expansions. For the unit cube in three dimensions we would use

$$\varphi_{1,nm}(x,y,x) = \frac{1}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}\sin(n\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{n^2+m^2}\right),$$

to expand the data on the face $z = 1$, together with five other solution families, one for each of the other five faces of the cube.

If some of the boundaries are at infinity, we may need only need some of these functions.

Example: Figure 6.17 shows three conducting sheets, each infinite in the z direction. The central one has width a, and is held at voltage V_0. The outer two extend to infinity also in the y direction, and are grounded. The resulting potential should tend to zero as $|x|$, $|y| \to \infty$.

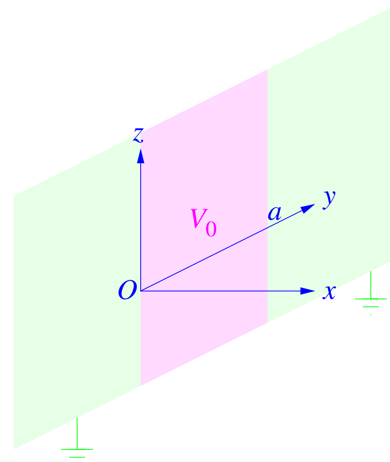

Figure 6.17: *Conducting sheets.*

The voltage in the $x = 0$ plane is

$$\varphi(0, y, z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) e^{-iky}, \tag{6.149}$$

where

$$a(k) = V_0 \int_{-a/2}^{a/2} e^{iky}\, dy = \frac{2V_0}{k} \sin(ka/2). \tag{6.150}$$

Then, taking into account the boundary condition at large x, the solution to $\nabla^2\varphi = 0$ is

$$\varphi(x, y, z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) e^{-iky} e^{-|k||x|}. \tag{6.151}$$

The evaluation of this integral, and finding the charge distribution on the sheets, is left as an exercise.

The Cauchy problem is ill-posed

Although the Laplace equation has no characteristics, the Cauchy data problem is *ill-posed*, meaning that the solution is not a continuous function of the data. To see this, suppose we are given $\nabla^2\varphi = 0$ with Cauchy data on $y = 0$:

$$\begin{aligned} \varphi(x, 0) &= 0, \\ \left.\frac{\partial \varphi}{\partial y}\right|_{y=0} &= \varepsilon \sin kx. \end{aligned} \tag{6.152}$$

Then

$$\varphi(x,y) = \frac{\varepsilon}{k}\sin(kx)\sinh(ky). \tag{6.153}$$

Provided k is large enough — even if ε is tiny — the exponential growth of the hyperbolic sine will make this arbitrarily large. Any infinitesimal uncertainty in the high frequency part of the initial data will be vastly amplified, and the solution, although formally correct, is useless in practice.

Polar coordinates

We can use the separation of variables method in polar coordinates. Here,

$$\nabla^2\chi = \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}. \tag{6.154}$$

Set

$$\chi(r,\theta) = R(r)\Theta(\theta). \tag{6.155}$$

Then $\nabla^2\chi = 0$ implies

$$\begin{aligned} 0 &= \frac{r^2}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \\ &= \qquad m^2 \qquad - \qquad m^2, \end{aligned} \tag{6.156}$$

where in the second line we have written the separation constant as m^2. Therefore,

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0, \tag{6.157}$$

implying that $\Theta = e^{im\theta}$, where m must be an integer if Θ is to be single-valued, and

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - m^2R = 0, \tag{6.158}$$

whose solutions are $R = r^{\pm m}$ when $m \neq 0$, and 1 or $\ln r$ when $m = 0$. The general solution is therefore a sum of these

$$\chi = A_0 + B_0\ln r + \sum_{m\neq 0}(A_m r^{|m|} + B_m r^{-|m|})e^{im\theta}. \tag{6.159}$$

The singular terms, $\ln r$ and $r^{-|m|}$, are not solutions at the origin, and should be omitted when that point is part of the region where $\nabla^2\chi = 0$.

Example: Dirichlet problem in the interior of the unit circle. Solve $\nabla^2\chi = 0$ in $\Omega = \{\mathbf{r} \in \mathbb{R}^2 : |\mathbf{r}| < 1\}$ with $\chi = f(\theta)$ on $\partial\Omega \equiv \{|\mathbf{r}| = 1\}$.

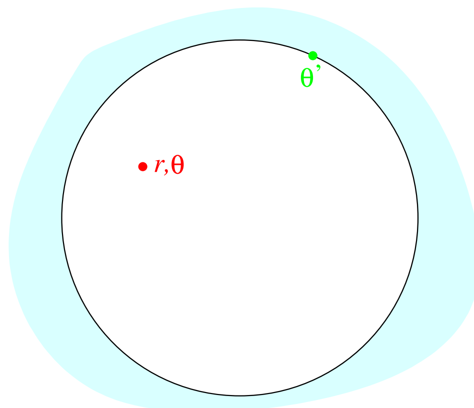

Figure 6.18: *Dirichlet problem in the unit circle.*

We expand

$$\chi(r.\theta) = \sum_{m=-\infty}^{\infty} A_m r^{|m|} e^{im\theta}, \tag{6.160}$$

and read off the coefficients from the boundary data as

$$A_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta'} f(\theta')\, d\theta'. \tag{6.161}$$

Thus,

$$\chi = \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{m=-\infty}^{\infty} r^{|m|} e^{im(\theta-\theta')} \right] f(\theta')\, d\theta'. \tag{6.162}$$

We can sum the geometric series

$$\begin{aligned} \sum_{m=-\infty}^{\infty} r^{|m|} e^{im(\theta-\theta')} &= \left(\frac{1}{1 - re^{i(\theta-\theta')}} + \frac{re^{-i(\theta-\theta')}}{1 - re^{-i(\theta-\theta')}} \right) \\ &= \frac{1 - r^2}{1 - 2r\cos(\theta - \theta') + r^2}. \end{aligned} \tag{6.163}$$

Therefore,

$$\chi(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 - r^2}{1 - 2r\cos(\theta - \theta') + r^2} \right) f(\theta')\, d\theta'. \tag{6.164}$$

This expression is known as the *Poisson kernel formula*. Observe how the integrand sharpens towards a delta function as r approaches unity, and so ensures that the limiting value of $\chi(r,\theta)$ is consistent with the boundary data.

If we set $r = 0$ in the Poisson formula, we find

$$\chi(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\,d\theta'. \tag{6.165}$$

We deduce that if $\nabla^2\chi = 0$ in some domain then the value of χ at a point in the domain is the average of its values on any circle centred on the chosen point and lying wholly in the domain.

This average-value property means that χ can have no local maxima or minima within Ω. The same result holds in $\mathbb{R}^n$, and a formal theorem to this effect can be proved:
Theorem (The mean-value theorem for harmonic functions): If χ *is harmonic* ($\nabla^2\chi = 0$) *within the bounded (open, connected) domain* $\Omega \in \mathbb{R}^n$*, and is continuous on its closure* $\overline{\Omega}$*, and if* $m \le \chi \le M$ *on* $\partial\Omega$*, then* $m < \chi < M$ *within* Ω *— unless, that is,* $m = M$*, when* $\chi = m$ *is constant.*

Pie-shaped regions

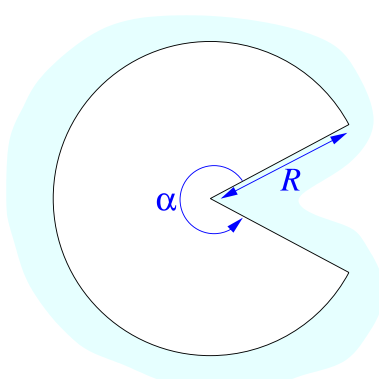

Figure 6.19: *A pie-shaped region of opening angle* α*.*

Electrostatics problems involving regions with corners can often be understood by solving Laplace's equation within a pie-shaped region.

Figure 6.19 shows a pie-shaped region of opening angle α and radius R. If the boundary value of the potential is zero on the wedge and non-zero on the boundary arc, we can seek solutions as a sum of r, θ separated terms

$$\varphi(r,\theta) = \sum_{n=1}^{\infty} a_n r^{n\pi/\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{6.166}$$

Here the trigonometric function is not 2π periodic, but instead has been constructed so as to make φ vanish at $\theta = 0$ and $\theta = \alpha$. These solutions show that close to the edge of a conducting wedge of external opening angle α, the surface charge density σ usually varies as $\sigma(r) \propto r^{\alpha/\pi - 1}$.

If we have non-zero boundary data on the edge of the wedge at $\theta = \alpha$, but have $\varphi = 0$ on the edge at $\theta = 0$ and on the curved arc $r = R$, then the solutions can be expressed as a continuous sum of r, θ separated terms

$$\begin{aligned}\varphi(r,\theta) &= \frac{1}{2i}\int_0^\infty a(\nu)\left(\left(\frac{r}{R}\right)^{i\nu} - \left(\frac{r}{R}\right)^{-i\nu}\right)\frac{\sinh(\nu\theta)}{\sinh(\nu\alpha)}\,d\nu,\\ &= \int_0^\infty a(\nu)\sin[\nu\ln(r/R)]\frac{\sinh(\nu\theta)}{\sinh(\nu\alpha)}\,d\nu.\end{aligned} \tag{6.167}$$

The Mellin sine transformation can be used to computing the coefficient function $a(\nu)$. This transformation lets us write

$$f(r) = \frac{2}{\pi}\int_0^\infty F(\nu)\sin(\nu\ln r)\,d\nu, \quad 0 < r < 1, \tag{6.168}$$

where

$$F(\nu) = \int_0^1 \sin(\nu\ln r) f(r)\,\frac{dr}{r}. \tag{6.169}$$

The Mellin sine transformation is a disguised version of the Fourier sine transform of functions on $[0,\infty)$. We simply map the positive x axis onto the interval $(0,1]$ by the change of variables $x = -\ln r$.

Despite its complexity when expressed in terms of these formulae, the simple solution $\varphi(r,\theta) = a\theta$ is often the physically relevant one when the two sides of the wedge are held at different potentials and the potential is allowed to vary on the curved arc.

Example: Consider a pie-shaped region of opening angle π and radius $R = \infty$. This region can be considered to be the upper half-plane. Suppose that we are told that the positive x axis is held at potential $+1/2$ and the negative

x axis is at potential $-1/2$, and are required to find the potential for positive y. If we separate Laplace's equation in cartesian co-ordinates and match to the boundary data on the x-axes, we end up with

$$\varphi_{xy}(x,y) = \frac{1}{\pi}\int_0^\infty \frac{1}{k}e^{-ky}\sin(kx)\,dk.$$

On the other hand, the function

$$\varphi_{r\theta}(r,\theta) = \frac{1}{\pi}(\pi/2-\theta)$$

satisfies both Laplace's equation and the boundary data. At this point we ought to worry that we do not have enough data to determine the solution uniquely — nothing was said in the statement of the problem about the behavior of φ on the boundary arc at infinity — but a little effort shows that

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \frac{1}{k}e^{-ky}\sin(kx)\,dk &= \frac{1}{\pi}\tan^{-1}\left(\frac{x}{y}\right), \quad y>0,\\ &= \frac{1}{\pi}(\pi/2-\theta),\end{aligned} \tag{6.170}$$

and so the two expressions for $\varphi(x,y)$ are equal.

6.5.3 Eigenfunction expansions

Elliptic operators are the natural analogues of the one-dimensional linear differential operators we studied in earlier chapters.

The operator $L = -\nabla^2$ is formally self-adjoint with respect to the inner product

$$\langle\phi,\chi\rangle = \iint \phi^*\chi\,dxdy. \tag{6.171}$$

This property follows from Green's identity

$$\iint_\Omega \left\{\phi^*(-\nabla^2\chi) - (-\nabla^2\phi)^*\chi\right\}\,dxdy = \int_{\partial\Omega}\{\phi^*(-\nabla\chi)-(-\nabla\phi)^*\chi\}\cdot\mathbf{n}ds \tag{6.172}$$

where $\partial\Omega$ is the boundary of the region Ω and $\mathbf{n}$ is the outward normal on the boundary.

The method of separation of variables also allows us to solve eigenvalue problems involving the Laplace operator. For example, the Dirichlet eigenvalue problem requires us to find the eigenfunctions and eigenvalues of the operator

$$L = -\nabla^2, \qquad \mathcal{D}(L) = \{\phi \in L^2[\Omega] : \phi = 0, \text{ on } \partial\Omega\}. \tag{6.173}$$

Suppose Ω is the rectangle $0 \le x \le L_x$, $0 \le y \le L_y$. The normalized eigenfunctions are

$$\phi_{n,m}(x,y) = \sqrt{\frac{4}{L_x L_y}} \sin\left(\frac{n\pi x}{L_x}\right) \sin\left(\frac{m\pi y}{L_y}\right), \tag{6.174}$$

with eigenvalues

$$\lambda_{n,m} = \left(\frac{n^2\pi^2}{L_x^2}\right) + \left(\frac{m^2\pi^2}{L_y^2}\right). \tag{6.175}$$

The eigenfunctions are orthonormal,

$$\int \phi_{n,m}\phi_{n',m'}\, dxdy = \delta_{nn'}\delta_{mm'}, \tag{6.176}$$

and complete. Thus, any function in $L^2[\Omega]$ can be expanded as

$$f(x,y) = \sum_{m,n=1}^{\infty} A_{nm}\phi_{n,m}(x,y), \tag{6.177}$$

where

$$A_{nm} = \iint \phi_{n,m}(x,y) f(x,y)\, dxdy. \tag{6.178}$$

We can find a complete set of eigenfunctions in product form whenever we can separate the Laplace operator in a system of co-ordinates ξ_i such that the boundary becomes $\xi_i = const$. Completeness in the multidimensional space is then guaranteed by the completeness of the eigenfunctions of each one-dimensional differential operator. For other than rectangular co-ordinates, however, the separated eigenfunctions are not elementary functions.

The Laplacian has a complete set of Dirichlet eigenfunctions in any region, but in general these eigenfunctions cannot be written as separated products of one-dimensional functions.

6.5.4 Green functions

Once we know the eigenfunctions φ_n and eigenvalues λ_n for $-\nabla^2$ in a region Ω, we can write down the Green function as

$$g(\mathbf{r}, \mathbf{r}') = \sum_n \frac{1}{\lambda_n} \varphi_n(\mathbf{r}) \varphi_n^*(\mathbf{r}').$$

For example, the Green function for the Laplacian in the entire $\mathbb{R}^n$ is given by the sum over eigenfunctions

$$g(\mathbf{r}, \mathbf{r}') = \int \frac{d^n k}{(2\pi)^n} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2}. \tag{6.179}$$

Thus

$$-\nabla_{\mathbf{r}}^2 g(\mathbf{r}, \mathbf{r}') = \int \frac{d^n k}{(2\pi)^n} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = \delta^n(\mathbf{r} - \mathbf{r}'). \tag{6.180}$$

We can evaluate the integral for any n by using *Schwinger's trick* to turn the integrand into a Gaussian:

$$\begin{aligned}
g(\mathbf{r}, \mathbf{r}') &= \int_0^\infty ds \int \frac{d^n k}{(2\pi)^n} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-sk^2} \\
&= \int_0^\infty ds \left(\sqrt{\frac{\pi}{s}}\right)^n \frac{1}{(2\pi)^n} e^{-\frac{1}{4s}|\mathbf{r}-\mathbf{r}'|^2} \\
&= \frac{1}{2^n \pi^{n/2}} \int_0^\infty dt\, t^{\frac{n}{2}-2} e^{-t|\mathbf{r}-\mathbf{r}'|^2/4} \\
&= \frac{1}{2^n \pi^{n/2}} \Gamma\left(\frac{n}{2} - 1\right) \left(\frac{|\mathbf{r}-\mathbf{r}'|^2}{4}\right)^{1-n/2} \\
&= \frac{1}{(n-2) S_{n-1}} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)^{n-2}.
\end{aligned} \tag{6.181}$$

Here, $\Gamma(x)$ is Euler's gamma function:

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}, \tag{6.182}$$

and

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{6.183}$$

is the surface area of the n-dimensional unit ball.

For three dimensions we find

$$g(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \qquad n=3. \tag{6.184}$$

In two dimensions the Fourier integral is divergent for small k. We may control this divergence by using *dimensional regularization*. We pretend that n is a continuous variable and use

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1) \tag{6.185}$$

together with

$$a^x = e^{a\ln x} = 1 + a\ln x + \cdots \tag{6.186}$$

to to examine the behaviour of $g(\mathbf{r},\mathbf{r}')$ near $n=2$:

$$\begin{aligned} g(\mathbf{r},\mathbf{r}') &= \frac{1}{4\pi}\frac{\Gamma(n/2)}{(n/2-1)}\left(1-(n/2-1)\ln(\pi|\mathbf{r}-\mathbf{r}'|^2)+O\left[(n-2)^2\right]\right)\\ &= \frac{1}{4\pi}\left(\frac{1}{n/2-1}-2\ln|\mathbf{r}-\mathbf{r}'|-\ln\pi-\gamma+\cdots\right). \end{aligned} \tag{6.187}$$

Here $\gamma = -\Gamma'(1) = .57721\ldots$ is the *Euler-Mascheroni constant.* Although the pole $1/(n-2)$ blows up at $n=2$, it is independent of position. We simply absorb it, and the $-\ln\pi-\gamma$, into an undetermined additive constant. Once we have done this, the limit $n\to 2$ can be taken and we find

$$g(\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'| + \text{const.}, \qquad n=2. \tag{6.188}$$

The constant does not affect the Green-function property, so we can chose any convenient value for it.

Although we have managed to sweep the small-k divergence of the Fourier integral under a rug, the hidden infinity still has the capacity to cause problems. The Green function in $\mathbb{R}^3$ allows us to to solve for $\varphi(\mathbf{r})$ in the equation

$$-\nabla^2\varphi = q(\mathbf{r}),$$

with the boundary condition $\varphi(\mathbf{r})\to 0$ as $|\mathbf{r}|\to\infty$, as

$$\varphi(\mathbf{r}) = \int g(\mathbf{r},\mathbf{r}')q(\mathbf{r}')\,d^3r.$$

In two dimensions, however we try to adjust the arbitrary constant in (6.188), the divergence of the logarithm at infinity means that there can be no solution to the corresponding boundary-value problem unless $\int q(\mathbf{r})\, d^3r = 0$. This is not a Fredholm-alternative constraint because once the constraint is satisfied the solution is unique. The two-dimensional problem is therefore pathological from the viewpoint of Fredholm theory. This pathology is of the same character as the non-existence of solutions to the three-dimensional Dirichlet boundary-value problem with boundary spikes. The Fredholm alternative applies, in general, only to operators a discrete spectrum.

Exercise 6.8: Evaluate our formula for the $\mathbb{R}^n$ Laplace Green function,

$$g(\mathbf{r},\mathbf{r}') = \frac{1}{(n-2)S_{n-1}|\mathbf{r}-\mathbf{r}'|^{n-2}}$$

with $S_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$, for the case $n = 1$. Show that the resulting expression for $g(x, x')$ is not divergent, and obeys

$$-\frac{d^2}{dx^2} g(x, x') = \delta(x - x').$$

Our formula therefore makes sense as a Green function — even though the original integral (6.179) is linearly divergent at $k = 0$! We must defer an explanation of this miracle until we discuss *analytic continuation* in the context of complex analysis.
(Hint: recall that $\Gamma(1/2) = \sqrt{\pi}$)

6.5.5 Boundary-value problems

We now look at how the Green function can be used to solve the interior Dirichlet boundary-value problem in regions where the method of separation of variables is not available. Figure 6.20 shows a bounded region Ω possessing a smooth boundary $\partial\Omega$.

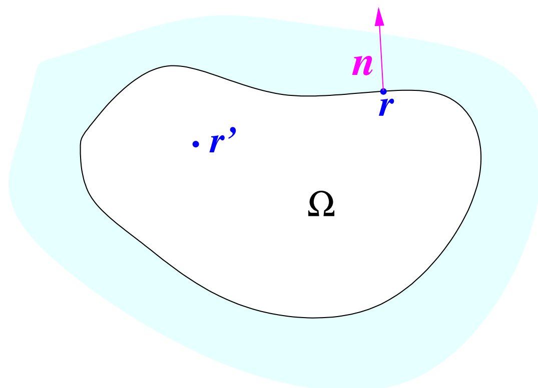

Figure 6.20: *Interior Dirichlet problem.*

We wish to solve $-\nabla^2\varphi = q(\mathbf{r})$ for $\mathbf{r}\in\Omega$ and with $\varphi(\mathbf{r}) = f(\mathbf{r})$ for $\mathbf{r}\in\partial\Omega$. Suppose we have found a Green function that obeys

$$-\nabla^2_{\mathbf{r}} g(\mathbf{r},\mathbf{r}') = \delta^n(\mathbf{r}-\mathbf{r}'), \quad \mathbf{r},\mathbf{r}'\in\Omega, \qquad g(\mathbf{r},\mathbf{r}') = 0, \quad \mathbf{r}\in\partial\Omega. \tag{6.189}$$

We first show that $g(\mathbf{r},\mathbf{r}') = g(\mathbf{r}',\mathbf{r})$ by the same methods we used for one-dimensional self-adjoint operators. Next we follow the strategy that we used for one-dimensional inhomogeneous differential equations: we use Lagrange's identity (in this context called Green's theorem) to write

$$\begin{aligned}\int_\Omega d^n r\,\left\{g(\mathbf{r},\mathbf{r}')\nabla^2_{\mathbf{r}}\varphi(\mathbf{r}) - \varphi(\mathbf{r})\nabla^2_{\mathbf{r}} g(\mathbf{r},\mathbf{r}')\right\}& \\ = \int_{\partial\Omega} d\mathbf{S}_{\mathbf{r}}\cdot\{g(\mathbf{r},\mathbf{r}')\nabla_{\mathbf{r}}\varphi(\mathbf{r}) - \varphi(\mathbf{r})\nabla_{\mathbf{r}} g(\mathbf{r},\mathbf{r}')\},& \end{aligned}\tag{6.190}$$

where $d\mathbf{S}_{\mathbf{r}} = \mathbf{n}\,dS_{\mathbf{r}}$, with $\mathbf{n}$ the outward normal to $\partial\Omega$ at the point $\mathbf{r}$. The left hand side is

$$\begin{aligned}\text{L.H.S.} &= \int_\Omega d^n r\{-g(\mathbf{r},\mathbf{r}')q(\mathbf{r}) + \varphi(\mathbf{r})\delta^n(\mathbf{r}-\mathbf{r}')\},\\ &= -\int_\Omega d^n r\, g(\mathbf{r},\mathbf{r}')\,q(\mathbf{r}) + \varphi(\mathbf{r}'),\\ &= -\int_\Omega d^n r\, g(\mathbf{r}',\mathbf{r})\,q(\mathbf{r}) + \varphi(\mathbf{r}').\end{aligned}\tag{6.191}$$

On the right hand side, the boundary condition on $g(\mathbf{r},\mathbf{r}')$ makes the first term zero, so

$$\text{R.H.S} = -\int_{\partial\Omega} dS_{\mathbf{r}} f(\mathbf{r})(\mathbf{n}\cdot\nabla_{\mathbf{r}})g(\mathbf{r},\mathbf{r}'). \tag{6.192}$$

Therefore,

$$\varphi(\mathbf{r}') = \int_\Omega g(\mathbf{r}',\mathbf{r})\, q(\mathbf{r})\, d^n r - \int_{\partial\Omega} f(\mathbf{r})(\mathbf{n}\cdot\nabla_{\mathbf{r}}) g(\mathbf{r},\mathbf{r}')\, dS_{\mathbf{r}}. \tag{6.193}$$

In the language of chapter 3, the first term is a particular integral and the second (the boundary integral term) is the complementary function.

Exercise 6.9: Assume that the boundary is a smooth surface, Show that the limit of $\varphi(\mathbf{r}')$ as $\mathbf{r}'$ approaches the boundary is indeed consistent with the boundary data $f(\mathbf{r}')$. (Hint: When $\mathbf{r}$, $\mathbf{r}'$ are very close to it, the boundary can be approximated by a straight line segment, and so $g(\mathbf{r},\mathbf{r}')$ can be found by the method of images.)

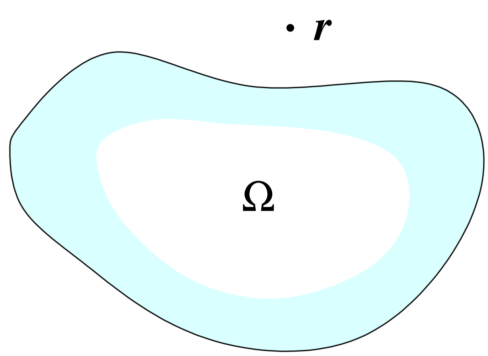

Figure 6.21: *Exterior Dirichlet problem.*

A similar method works for the exterior Dirichlet problem shown in figure 6.21. In this case we seek a Green function obeying

$$-\nabla^2_{\mathbf{r}} g(\mathbf{r},\mathbf{r}') = \delta^n(\mathbf{r}-\mathbf{r}'), \quad \mathbf{r},\mathbf{r}' \in \mathbb{R}^n\setminus\Omega \qquad g(\mathbf{r},\mathbf{r}') = 0, \quad \mathbf{r}\in\partial\Omega. \tag{6.194}$$

(The notation $\mathbb{R}^n\setminus\Omega$ means the region outside Ω.) We also impose a further boundary condition by requiring $g(\mathbf{r},\mathbf{r}')$, and hence $\varphi(\mathbf{r})$, to tend to zero as $|\mathbf{r}|\to\infty$. The final formula for $\varphi(\mathbf{r})$ is the same except for the region of integration and the sign of the boundary term.

The hard part of both the interior and exterior problems is to find the Green function for the given domain.

Exercise 6.10: Suppose that $\varphi(x,y)$ is harmonic in the half-plane $y>0$, tends to zero as $y\to\infty$, and takes the values $f(x)$ on the boundary $y=0$. Show that

$$\varphi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-x')^2+y^2}f(x')\,dx',\quad y>0.$$

Deduce that the "energy" functional

$$S[f]\stackrel{\text{def}}{=}\frac{1}{2}\int_{y>0}|\nabla\varphi|^2\,dxdy\equiv-\frac{1}{2}\int_{-\infty}^{\infty}f(x)\left.\frac{\partial\varphi}{\partial y}\right|_{y=0}dx$$

can be expressed as

$$S[f]=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{f(x)-f(x')}{x-x'}\right\}^2dx'dx.$$

The non-local functional $S[f]$ appears in the quantum version of the Caldeira-Leggett model. See also exercise 2.24.

Method of Images

When $\partial\Omega$ is a sphere or a circle we can find the Dirichlet Green functions for the region Ω by using the *method of images*.

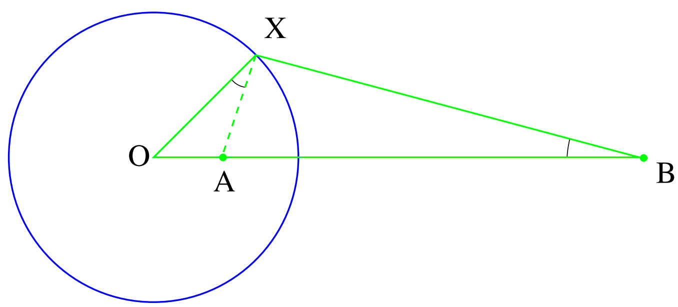

Figure 6.22: *Points inverse with respect to a circle.*

Figure 6.22 shows a circle of radius R. Given a point B outside the circle, and a point X on the circle, we construct A inside and on the line OB, so that $\angle$OBX $=\angle$OXA. We now observe that $\triangle$XOA is similar to $\triangle$BOX, and so

$$\frac{\text{OA}}{\text{OX}}=\frac{\text{OX}}{\text{OB}}.\tag{6.195}$$

Thus, OA × OB = $(\mathrm{OX})^2 \equiv R^2$. The points A and B are therefore *mutually inverse* with respect to the circle. In particular, the point A does not depend on which point X was chosen.

Now let AX= r_i, BX= r_0 and OB= B. Then, using similar triangles again, we have

$$\frac{\mathrm{AX}}{\mathrm{OX}} = \frac{\mathrm{BX}}{\mathrm{OB}}, \tag{6.196}$$

or

$$\frac{R}{r_i} = \frac{B}{r_0}, \tag{6.197}$$

and so

$$\frac{1}{r_i}\left(\frac{R}{B}\right) - \frac{1}{r_0} = 0. \tag{6.198}$$

Interpreting the figure as a slice through the centre of a sphere of radius R, we see that if we put a unit charge at B, then the insertion of an *image charge* of magnitude $q = -R/B$ at A serves to the keep the entire surface of the sphere at zero potential.

Thus, in three dimensions, and with Ω the region exterior to the sphere, the Dirichlet Green function is

$$g_\Omega(\mathbf{r}, \mathbf{r}_\mathrm{B}) = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_\mathrm{B}|} - \left(\frac{R}{|\mathbf{r}_\mathrm{B}|}\right)\frac{1}{|\mathbf{r}-\mathbf{r}_\mathrm{A}|}\right). \tag{6.199}$$

In two dimensions, we find similarly that

$$g_\Omega(\mathbf{r}, \mathbf{r}_\mathrm{B}) = -\frac{1}{2\pi}\Big(\ln|\mathbf{r}-\mathbf{r}_\mathrm{B}| - \ln|\mathbf{r}-\mathbf{r}_\mathrm{A}| - \ln\left(|\mathbf{r}_\mathrm{B}|/R\right)\Big), \tag{6.200}$$

has $g_\Omega(\mathbf{r}, \mathbf{r}_\mathrm{B}) = 0$ for $\mathbf{r}$ on the circle. Thus, this is the Dirichlet Green function for Ω, the region exterior to the circle.

We can use the same method to construct the interior Green functions for the sphere and circle.

6.5.6 Kirchhoff *vs.* Huygens

Even if we do not have a Green function tailored for the specific region in which were are interested, we can still use the whole-space Green function to convert the differential equation into an *integral equation*, and so make progress. An example of this technique is provided by Kirchhoff's partial justification of Huygens' construction.

The Green function $G(\mathbf{r},\mathbf{r}')$ for the elliptic Helmholtz equation

$$(-\nabla^2+\kappa^2)G(\mathbf{r},\mathbf{r}')=\delta^3(\mathbf{r}-\mathbf{r}') \tag{6.201}$$

in $\mathbb{R}^3$ is given by

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2+\kappa^2}=\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-\kappa|\mathbf{r}-\mathbf{r}'|}. \tag{6.202}$$

Exercise 6.11: Perform the k integration and confirm this.

For solutions of the wave equation with $e^{-i\omega t}$ time dependence, we want a Green function such that

$$\left[-\nabla^2-\left(\frac{\omega^2}{c^2}\right)\right]G(\mathbf{r},\mathbf{r}')=\delta^3(\mathbf{r}-\mathbf{r}'), \tag{6.203}$$

and so we have to take κ^2 negative. We therefore have two possible Green functions

$$G_\pm(\mathbf{r},\mathbf{r}')=\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{\pm ik|\mathbf{r}-\mathbf{r}'|}, \tag{6.204}$$

where $k=|\omega|/c$. These correspond to taking the real part of κ^2 negative, but giving it an infinitesimal imaginary part, as we did when discussing resolvent operators in chapter 5. If we want outgoing waves, we must take $G\equiv G_+$.

Now suppose we want to solve

$$(\nabla^2+k^2)\psi=0 \tag{6.205}$$

in an arbitrary region Ω. As before, we use Green's theorem to write

$$\begin{aligned}\int_\Omega \left\{G(\mathbf{r},\mathbf{r}')(\nabla^2_{\mathbf{r}}+k^2)\psi(\mathbf{r})-\psi(\mathbf{r})(\nabla^2_{\mathbf{r}}+k^2)G(\mathbf{r},\mathbf{r}')\right\}\,d^nx\\ =\int_{\partial\Omega}\{G(\mathbf{r},\mathbf{r}')\nabla_{\mathbf{r}}\psi(\mathbf{r})-\psi(\mathbf{r})\nabla_{\mathbf{r}}G(\mathbf{r},\mathbf{r}')\}\cdot d\mathbf{S}_{\mathbf{r}}\end{aligned} \tag{6.206}$$

where $d\mathbf{S}_{\mathbf{r}}=\mathbf{n}\,dS_{\mathbf{r}}$, with $\mathbf{n}$ the outward normal to $\partial\Omega$ at the point $\mathbf{r}$. The left hand side is

$$\int_\Omega \psi(\mathbf{r})\delta^n(\mathbf{r}-\mathbf{r}')\,d^nx=\begin{cases}\psi(\mathbf{r}'), & \mathbf{r}'\in\Omega\\ 0, & \mathbf{r}'\notin\Omega\end{cases} \tag{6.207}$$

and so

$$\psi(\mathbf{r}') = \int_{\partial\Omega} \{G(\mathbf{r},\mathbf{r}')(\mathbf{n}\cdot\nabla_x)\psi(\mathbf{r}) - \psi(\mathbf{r})(\mathbf{n}\cdot\nabla_{\mathbf{r}})G(\mathbf{r},\mathbf{r}')\}\, dS_{\mathbf{r}}, \quad \mathbf{r}' \in \Omega. \tag{6.208}$$

This must *not* be thought of as solution to the wave equation in terms of an integral over the boundary, analogous to the solution (6.193) of the Dirichlet problem that we found in the last section. Here, unlike that earlier case, $G(\mathbf{r},\mathbf{r}')$ knows nothing of the boundary $\partial\Omega$, and so both terms in the surface integral contribute to ψ. We therefore have a formula for $\psi(\mathbf{r})$ in the interior in terms of both Dirichlet *and* Neumann data on the boundary $\partial\Omega$, and giving *both* over-prescribes the problem. If we take arbitrary values for ψ and $(\mathbf{n}\cdot\nabla)\psi$ on the boundary, and plug them into (6.208) so as to compute $\psi(\mathbf{r})$ within Ω then there is no reason for the resulting $\psi(\mathbf{r})$ to reproduce, as $\mathbf{r}$ approaches the boundary, the values ψ and $(\mathbf{n}\cdot\nabla)\psi$ appearing in the integral. If we demand that the output $\psi(\mathbf{r})$ *does* reproduce the input boundary data, then this is equivalent to demanding that the boundary data come from a solution of the differential equation in a region encompassing Ω.

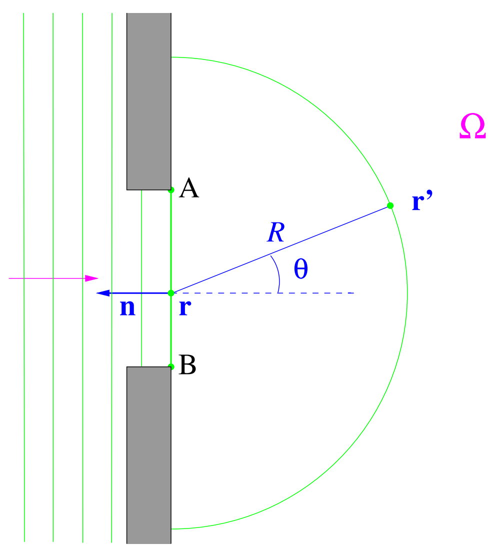

Figure 6.23: *Huygens' construction.*

The mathematical inconsistency of assuming arbitrary boundary data notwithstanding, this is exactly what we do when we follow Kirchhoff and use (6.208) to provide a justification of Huygens' construction as used in optics. Consider the problem of a plane wave, $\psi = e^{ikx}$, incident on a screen from the left and passing though the aperture labelled AB in figure 6.23.

We take as the region Ω everything to the right of the obstacle. The Kirchhoff approximation consists of assuming that the values of ψ and $(\mathbf{n}\cdot\nabla)\psi$ on the surface AB are e^{ikx} and $-ike^{ikx}$, the same as they would be if the obstacle were not there, and that they are identically zero on all other parts of the boundary. In other words, we completely ignore any scattering by the material in which the aperture resides. We can then use our formula to estimate ψ in the region to the right of the aperture. If we further set

$$\nabla_{\mathbf{r}} G(\mathbf{r},\mathbf{r}') \approx ik\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2} e^{ik|\mathbf{r}-\mathbf{r}'|}, \tag{6.209}$$

which is a good approximation provided we are more than a few wavelengths away from the aperture, we find

$$\psi(\mathbf{r}') \approx \frac{k}{4\pi i}\int_{\text{aperture}} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}(1+\cos\theta)dS_{\mathbf{r}}. \tag{6.210}$$

Thus, each part of the wavefront on the surface AB acts as a source for the diffracted wave in Ω.

This result, although still an approximation, provides two substantial improvements to the naïve form of Huygens' construction as presented in elementary courses:

i) There is factor of $(1+\cos\theta)$ which suppresses backward propagating waves. The traditional exposition of Huygens construction takes no notice of which way the wave is going, and so provides no explanation as to why a wavefront does not act a source for a backward wave.

ii) There is a factor of $i^{-1} = e^{-i\pi/2}$ which corrects a 90° error in the phase made by the naïve Huygens construction. For two-dimensional slit geometry we must use the more complicated two-dimensional Green function (it is a Bessel function), and this provides an $e^{-i\pi/4}$ factor which corrects for the 45° phase error that is manifest in the Cornu spiral of Fresnel diffraction.

For this reason the Kirchhoff approximation is widely used.

Problem 6.12: Use the method of images to construct i) the Dirichlet, and ii) the Neumann, Green function for the region Ω, consisting of everything to the right of the screen. Use your Green functions to write the solution to the diffraction problem in this region a) in terms of the values of ψ on the aperture surface AB, b) in terms of the values of $(\mathbf{n}\cdot\nabla)\psi$ on the aperture surface. In each case, assume that the boundary data are identically zero on the dark side of the screen. Your expressions should coincide with the *Rayleigh-Sommerfeld diffraction integrals* of the first and second kind, respectively.[3] Explore the differences between the predictions of these two formulæ and that of Kirchhoff for case of the diffraction of a plane wave incident on the aperture from the left.

6.6 Further exercises and problems

Problem 6.13: *Critical Mass.* An infinite slab of fissile material has thickness L. The neutron density $n(x)$ in the material obeys the equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} + \lambda n + \mu,$$

where $n(x,t)$ is zero at the surface of the slab at $x = 0, L$. Here, D is the neutron diffusion constant, the term λn describes the creation of new neutrons by induced fission, and the constant μ is the rate of production per unit volume of neutrons by spontaneous fission.

a) Expand $n(x,t)$ as a series,

$$n(x,t) = \sum_m a_m(t)\varphi_m(x),$$

where the $\varphi_m(x)$ are a complete set of functions you think suitable for solving the problem.

b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their intial values $a_m(0)$.
c) Determine the critical thickness L_{crit} above which the slab will explode.
d) Assuming that $L < L_{\text{crit}}$, find the equilibrium distribution $n_{\text{eq}}(x)$ of neutrons in the slab. (You may either sum your series expansion to get an explicit closed-form answer, or use another (Green function?) method.)

Problem 6.14: *Semi-infinite Rod.* Consider the heat equation

$$\frac{\partial \theta}{\partial t} = D\nabla^2\theta, \quad 0 < x < \infty,$$

with the temperature $\theta(x,t)$ obeying the initial condition $\theta(x,0) = \theta_0$ for $0 < x < \infty$, and the boundary condition $\theta(0,t) = 0$.

[3] M. Born and E. Wolf *Principles of Optics* 7th (expanded) edition, section 8.11.

a) Show that the boundary condition at $x = 0$ may be satisfied at all times by introducing a suitable mirror image of the initial data in the region $-\infty < x < 0$, and then applying the heat kernel for the entire real line to this extended initial data. Show that the resulting solution of the semi-infinite rod problem can be expressed in terms of the *error function*

$$\mathrm{erf}\,(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi,$$

as

$$\theta(x,t) = \theta_0 \,\mathrm{erf}\left(\frac{x}{\sqrt{4t}}\right).$$

b) Solve the same problem by using a Fourier integral expansion in terms of $\sin kx$ on the half-line $0 < x < \infty$ and obtaining the time evolution of the Fourier coefficients. Invert the transform and show that your answer reduces to that of part a). (Hint: replace the initial condition by $\theta(x,0) = \theta_0 e^{-\epsilon x}$, so that the Fourier transform converges, and then take the limit $\epsilon \to 0$ at the end of your calculation.)

Exercise 6.15: *Seasonal Heat Waves.* Suppose that the measured temperature of the air above the arctic permafrost at time t is expressed as a Fourier series

$$\theta(t) = \theta_0 + \sum_{n=1}^{\infty} \theta_n \cos n\omega t,$$

where the period $T = 2\pi/\omega$ is one year. Solve the heat equation for the soil temperature,

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}, \quad 0 < z < \infty$$

with this boundary condition, and find the temperature $\theta(z,t)$ at a depth z below the surface as a function of time. Observe that the sub-surface temperature fluctuates with the same period as that of the air, but with a phase lag that depends on the depth. Also observe that the longest-period temperature fluctuations penetrate the deepest into the ground. (Hint: for each Fourier component, write θ as $\mathrm{Re}[A_n(z) \exp in\omega t]$, where A_n is a complex function of z.)

The next problem is an illustration of a Dirichlet principle.

Exercise 6.16: *Helmholtz-Hodge decomposition.* Given a three-dimensional region Ω with smooth boundary $\partial\Omega$, introduce the real Hilbert space $L^2_{\text{vec}}(\Omega)$ of finite-norm vector fields, with inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d^3x.$$

Consider the spaces $\mathcal{L} = \{\mathbf{v} : \mathbf{v} = \nabla\phi\}$ and $\mathcal{T} = \{\mathbf{v} : \mathbf{v} = \mathrm{curl}\,\mathbf{w}\}$ consisting of vector fields in $L^2_{\mathrm{vec}}(\Omega)$ that can can be written as gradients and curls, respectively. (Strictly speaking, we should consider the completions of these spaces.)

a) Show that if we demand that either (or both) of ϕ and the tangential component of $\mathbf{w}$ vanish on $\partial\Omega$, then the two spaces $\mathcal{L}$ and $\mathcal{T}$ are mutually orthogonal with respect to the the $L^2_{\mathrm{vec}}(\Omega)$ inner product.

Let $\mathbf{u} \in L^2_{\mathrm{vec}}(\Omega)$. We will try to express $\mathbf{u}$ as the sum of a gradient and a curl by seeking to make the distance functional

$$\begin{aligned} F_{\mathbf{u}}[\phi, \mathbf{w}] &= \|\mathbf{u} - \nabla\phi - \mathrm{curl}\,\mathbf{w}\|^2 \\ &\stackrel{\mathrm{def}}{=} \int_\Omega |\mathbf{u} - \nabla\phi - \mathrm{curl}\,\mathbf{w}|^2 \, d^3x \end{aligned}$$

equal to zero.

b) Show that if we find a $\mathbf{w}$ and ϕ that minimize $F_{\mathbf{u}}[\phi, \mathbf{w}]$, then the residual vector field

$$\mathbf{h} \stackrel{\mathrm{def}}{=} \mathbf{u} - \nabla\phi - \mathrm{curl}\,\mathbf{w}$$

obeys $\mathrm{curl}\,\mathbf{h} = 0$ and $\mathrm{div}\,\mathbf{h} = 0$, together with boundary conditions determined by the constraints imposed on ϕ and $\mathbf{w}$:

i) If ϕ is unconstrained on $\partial\Omega$, but the tangential boundary component of $\mathbf{w}$ is required to vanish, then the component of $\mathbf{h}$ normal to the boundary must be zero.

ii) If $\phi = 0$ on $\partial\Omega$, but the tangential boundary component of $\mathbf{w}$ is unconstrained, then the tangential boundary component of $\mathbf{h}$ must be zero.

iii) If $\phi = 0$ on $\partial\Omega$ and also the tangential boundary component of $\mathbf{w}$ is required to vanish, then $\mathbf{h}$ need satisfy no boundary condition.

c) Assuming that we can find suitable minimizing ϕ and $\mathbf{w}$, deduce that under each of the three boundary conditions of the previous part, we have a *Helmholtz-Hodge decomposition*

$$\mathbf{u} = \nabla\phi + \mathrm{curl}\,\mathbf{w} + \mathbf{h}$$

into unique parts that are mutually $L^2_{\mathrm{vec}}(\Omega)$ orthogonal. Observe that the residual vector field $\mathbf{h}$ is *harmonic* — *i.e.* it satisfies the equation $\nabla^2\mathbf{h} = 0$, where

$$\nabla^2\mathbf{h} \stackrel{\mathrm{def}}{=} \nabla(\mathrm{div}\,\mathbf{h}) - \mathrm{curl}\,(\mathrm{curl}\,\mathbf{h})$$

is the *vector Laplacian* acting on $\mathbf{h}$.

If $\mathbf{u}$ is sufficiently smooth, there will exist ϕ and $\mathbf{w}$ that minimize the distance $\|\mathbf{u} - \nabla\phi - \operatorname{curl}\mathbf{w}\|$ and satisfy the boundary conditions. Whether or not $\mathbf{h}$ is needed in the decomposition is another matter. It depends both on how we constrain ϕ and $\mathbf{w}$, and on the topology of Ω. At issue is whether or not the boundary conditions imposed on $\mathbf{h}$ are sufficient to force it to be zero. If Ω is the interior of a torus, for example, then $\mathbf{h}$ can be non-zero whenever its tangential component is unconstrained.

The Helmholtz-Hodge decomposition is closely related to the vector-field eigenvalue problems commonly met with in electromagnetism or elasticity. The next few exercises lead up to this connection.

Exercise 6.17: *Self-adjointness and the vector Laplacian.* Consider the vector Laplacian (defined in the previous problem) as a linear operator on the Hilbert space $L^2_{\rm vec}(\Omega)$.

a) Show that

$$\begin{aligned}\int_\Omega d^3x \,\{\mathbf{u}\cdot(\nabla^2\mathbf{v}) - \mathbf{v}\cdot(\nabla^2\mathbf{u})\} \quad &= \quad \int_{\partial\Omega}\{(\mathbf{n}\cdot\mathbf{u})\operatorname{div}\mathbf{v} - (\mathbf{n}\cdot\mathbf{v})\operatorname{div}\mathbf{u} \\ &\qquad -\mathbf{u}\cdot(\mathbf{n}\times\operatorname{curl}\mathbf{v}) + \mathbf{v}\cdot(\mathbf{n}\times\operatorname{curl}\mathbf{u})\}\,dS\end{aligned}$$

b) Deduce from the identity in part a) that the domain of ∇^2 coincides with the domain of $(\nabla^2)^\dagger$, and hence the vector Laplacian defines a truly self-adjoint operator with a complete set of mutually orthogonal eigenfunctions, when we take as boundary conditions one of the following:
   o) Dirichlet-Dirichlet: $\mathbf{n}\cdot\mathbf{u} = 0$ and $\mathbf{n}\times\mathbf{u} = 0$ on $\partial\Omega$,
   i) Dirichlet-Neumann: $\mathbf{n}\cdot\mathbf{u} = 0$ and $\mathbf{n}\times\operatorname{curl}\mathbf{u} = 0$ on $\partial\Omega$,
   ii) Neumann-Dirichlet: $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{n}\times\mathbf{u} = 0$ on $\partial\Omega$,
   iii) Neumann-Neumann: $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{n}\times\operatorname{curl}\mathbf{u} = 0$ on $\partial\Omega$.

c) Show that the more general *Robin* boundary conditions

$$\begin{aligned}\alpha(n\cdot\mathbf{u}) + \beta\operatorname{div}\mathbf{u} &= 0,\\ \lambda(\mathbf{n}\times\mathbf{u}) + \mu(\mathbf{n}\times\operatorname{curl}\mathbf{u}) &= 0,\end{aligned}$$

where α β, μ ν can be position dependent, also give rise to a truly self-adjoint operator.

Problem 6.18: *Cavity electrodynamics and the Hodge-Weyl decomposition.* Each of the self-adjoint boundary conditions in the previous problem gives rise to a complete set of mutually orthogonal vector eigenfunctions obeying

$$-\nabla^2\mathbf{u}_n = k_n^2\mathbf{u}_n.$$

For these eigenfunctions to describe the normal modes of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ (which we identify with $\mathbf{H}$ as we will use units in which $\mu_0 = \epsilon_0 = 1$) within a cavity bounded by a perfect conductor, we need to additionally impose the Maxwell equations $\operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0$ everywhere within Ω, and to satisfy the perfect-conductor boundary conditions $\mathbf{n}\times\mathbf{E} = \mathbf{n}\cdot\mathbf{B} = 0$.

a) For each eigenfunction $\mathbf{u}_n$ corresponding to a non-zero eigenvalue k_n^2, define

$$\mathbf{v}_n = \frac{1}{k_n^2}\operatorname{curl}(\operatorname{curl}\mathbf{u}_n), \quad \mathbf{w}_n = -\frac{1}{k_n^2}\nabla(\operatorname{div}\mathbf{u}_n),$$

so that $\mathbf{u}_n = \mathbf{v}_n + \mathbf{w}_n$. Show that $\mathbf{v}_n$ and $\mathbf{w}_m$ are, if non-zero, each eigenfunctions of $-\nabla^2$ with eigenvalue k_n^2. The vector eigenfunctions that are not in the null-space of ∇^2 can therefore be decomposed into their *transverse* (the $\mathbf{v}_n$, which obey $\operatorname{div}\mathbf{v}_n = 0$) and *longitudinal* (the $\mathbf{w}_n$, which obey $\operatorname{curl}\mathbf{w}_n = 0$) parts. However, it is not immediately clear what boundary conditions the $\mathbf{v}_n$ and $\mathbf{w}_n$ separately obey.

b) The boundary-value problems of relevance to electromagnetism are:

$$\text{i)} \quad \begin{cases} -\nabla^2\mathbf{h}_n = k_n^2\mathbf{h}_n, & \text{within } \Omega, \\ \mathbf{n}\cdot\mathbf{h}_n = 0, \quad \mathbf{n}\times\operatorname{curl}\mathbf{h}_n = 0, & \text{on } \partial\Omega; \end{cases}$$

$$\text{ii)} \quad \begin{cases} -\nabla^2\mathbf{e}_n = k_n^2\mathbf{e}_n, & \text{within } \Omega, \\ \operatorname{div}\mathbf{e}_n = 0, \quad \mathbf{n}\times\mathbf{e}_n = 0, & \text{on } \partial\Omega; \end{cases}$$

$$\text{iii)} \quad \begin{cases} -\nabla^2\mathbf{b}_n = k_n^2\mathbf{b}_n, & \text{within } \Omega, \\ \operatorname{div}\mathbf{b}_n = 0, \quad \mathbf{n}\times\operatorname{curl}\mathbf{b}_n = 0, & \text{on } \partial\Omega, \end{cases}$$

These problems involve, respectively, the Dirichlet-Neumann, Neumann-Dirichlet, and Neumann-Neumann boundary conditions from the previous problem.

Show that the divergence-free transverse eigenfunctions

$$\mathbf{H}_n \stackrel{\text{def}}{=} \frac{1}{k_n^2}\operatorname{curl}(\operatorname{curl}\mathbf{h}_n), \quad \mathbf{E}_n \stackrel{\text{def}}{=} \frac{1}{k_n^2}\operatorname{curl}(\operatorname{curl}\mathbf{e}_n), \quad \mathbf{B}_n \stackrel{\text{def}}{=} \frac{1}{k_n^2}\operatorname{curl}(\operatorname{curl}\mathbf{b}_n),$$

obey $\mathbf{n}\cdot\mathbf{H}_n = \mathbf{n}\times\mathbf{E}_n = \mathbf{n}\times\operatorname{curl}\mathbf{B}_n = 0$ on the boundary, and that from these and the eigenvalue equations we can deduce that $\mathbf{n}\times\operatorname{curl}\mathbf{H}_n = \mathbf{n}\cdot\mathbf{B}_n = \mathbf{n}\cdot\operatorname{curl}\mathbf{E}_n = 0$ on the boundary. The perfect-conductor boundary conditions are therefore satisfied.

Also show that the corresponding longitudinal eigenfunctions

$$\boldsymbol{\eta}_n \stackrel{\text{def}}{=} \frac{1}{k_n^2}\nabla(\operatorname{div}\mathbf{h}_n), \quad \boldsymbol{\epsilon}_n \stackrel{\text{def}}{=} \frac{1}{k_n^2}\nabla(\operatorname{div}\mathbf{e}_n), \quad \boldsymbol{\beta}_n \stackrel{\text{def}}{=} \frac{1}{k_n^2}\nabla(\operatorname{div}\mathbf{b}_n)$$

obey the boundary conditions $\mathbf{n}\cdot\boldsymbol{\eta}_n = \mathbf{n}\times\boldsymbol{\epsilon}_n = \mathbf{n}\times\boldsymbol{\beta}_n = 0$.

c) By considering the counter-example provided by a rectangular box, show that the Dirichlet-Dirichlet boundary condition is not compatible with a longitudinal+transverse decomposition. (A purely transverse wave incident on such a boundary will, on reflection, acquire a longitudinal component.)

d) Show that

$$0=\int_\Omega \boldsymbol{\eta}_n\cdot \mathbf{H}_m\, d^3x=\int_\Omega \boldsymbol{\epsilon}_n\cdot \mathbf{E}_m\, d^3x=\int_\Omega \boldsymbol{\beta}_n\cdot \mathbf{B}_m\, d^3x,$$

but that the $\mathbf{v}_n$ and $\mathbf{w}_n$ obtained from the Dirichlet-Dirichlet boundary condition $\mathbf{u}_n$'s are not in general orthogonal to each other. Use the continuity of the $L^2_{\rm vec}(\Omega)$ inner product

$$\mathbf{x}_n\to\mathbf{x}\quad\Rightarrow\quad \langle \mathbf{x}_n,\mathbf{y}\rangle\to\langle\mathbf{x},\mathbf{y}\rangle$$

to show that this individual-eigenfunction orthogonality is retained by limits of sums of the eigenfunctions. Deduce that, for each of the boundary conditions i)-iii) (but *not* for the Dirichlet-Dirichlet case), we have the *Hodge-Weyl* decomposition of $L^2_{\rm vec}(\Omega)$ as the orthogonal direct sum

$$L^2_{\rm vec}(\Omega)=\mathcal{L}\oplus\mathcal{T}\oplus\mathcal{N},$$

where $\mathcal{L}$, $\mathcal{T}$ are respectively the spaces of functions representable as infinite sums of the longitudinal and transverse eigenfunctions, and $\mathcal{N}$ is the finite-dimensional space of harmonic (nullspace) eigenfunctions.

Complete sets of vector eigenfunctions for the interior of a rectangular box, and for each of the four sets of boundary conditions we have considered, can be found in Morse and Feshbach §13.1.

Problem 6.19: *Hodge-Weyl and Helmholtz-Hodge.* In this exercise we consider the problem of what classes of vector-valued functions can be expanded in terms of the various families of eigenfunctions of the previous problem. It is tempting (but wrong) to think that we are restricted to expanding functions that obey the same boundary conditions as the eigenfunctions themselves. Thus, we might erroniously expect that the $\mathbf{E}_n$ are good only for expanding functions whose divergence vanishes and have vanishing tangential boundary components, or that the $\boldsymbol{\eta}_n$ can expand out only curl-free vector fields with vanishing normal boundary component. That this supposition can be false was exposed in section 2.2.3, where we showed that functions that are zero at the endpoints of an interval can be used to expand out functions that are not zero there. The key point is that each of our four families of $\mathbf{u}_n$ constitute a *complete orthonormal set* in $L^2_{\rm vec}(\Omega)$, and can therefore be used expand *any* vector field. As a consequence, the infinite sum $\sum a_n \mathbf{B}_n \in \mathcal{T}$ can, for example, represent any vector-valued function $\mathbf{u} \in L^2_{\rm vec}(\Omega)$ provided only that $\mathbf{u}$ possesses no component lying either in the subspace $\mathcal{L}$ of the longitudinal eigenfunctions $\boldsymbol{\beta}_n$, or in the nullspace $\mathcal{N}$.

a) Let $\mathcal{T} = < \mathbf{E}_n >$ be space of functions representable as infinite sums of the $\mathbf{E}_n$. Show that

$$< \mathbf{E}_n >^{\perp} = \{\mathbf{u} : {\rm curl}\,\mathbf{u} = 0 \text{ within } \Omega,\, \mathbf{n}\times\mathbf{u} = 0 \text{ on } \partial\Omega\}.$$

Find the corresponding perpendicular spaces for each of the other eight orthogonal decomposition spaces.

b) Exploit your knowledge of $< \mathbf{E}_n >^{\perp}$ acquired in part (a) to show that $< \mathbf{E}_n >$ itself is the Hilbert space

$$< \mathbf{E}_n > = \{\mathbf{u} : {\rm div}\,\mathbf{u} = 0 \text{ within } \Omega, \text{ no condition on } \partial\Omega\}.$$

Similarly show that

$$\begin{aligned}
< \boldsymbol{\epsilon}_n > &= \{\mathbf{u} : {\rm curl}\,\mathbf{u} = 0 \text{ within } \Omega,\ \mathbf{n}\times\mathbf{u} = 0 \text{ on } \partial\Omega\},\\
< \boldsymbol{\eta}_n > &= \{\mathbf{u} : {\rm curl}\,\mathbf{u} = 0 \text{ within } \Omega,\ \text{no condition on } \partial\Omega\},\\
< \mathbf{H}_n > &= \{\mathbf{u} : {\rm div}\,\mathbf{u} = 0 \text{ within } \Omega,\, \mathbf{n}\cdot\mathbf{u} = 0 \text{ on } \partial\Omega\},\\
< \boldsymbol{\beta}_n > &= \{\mathbf{u} : {\rm curl}\,\mathbf{u} = 0 \text{ within } \Omega,\ \mathbf{n}\times\mathbf{u} = 0 \text{ on } \partial\Omega\},\\
< \mathbf{B}_n > &= \{\mathbf{u} : {\rm div}\,\mathbf{u} = 0 \text{ within } \Omega,\, \mathbf{n}\cdot\mathbf{u} = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

c) Conclude from the previous part that any vector vector field $\mathbf{u} \in L^2_{\rm vec}(\Omega)$ can be uniquely decomposed as the $L^2_{\rm vec}(\Omega)$ orthogonal sum

$$\mathbf{u} = \nabla\phi + {\rm curl}\,\mathbf{w} + \mathbf{h},$$

where $\nabla\phi \in \mathcal{L}$, ${\rm curl}\,\mathbf{w} \in \mathcal{T}$, and $\mathbf{h} \in \mathcal{N}$, under each of the following sets of conditions:

i) The scalar ϕ is unrestricted, but $\mathbf{w}$ obeys $\mathbf{n}\times\mathbf{w} = 0$ on $\partial\Omega$, and the harmonic $\mathbf{h}$ obeys $\mathbf{n}\cdot\mathbf{h} = 0$ on $\partial\Omega$. (The condition on $\mathbf{w}$ makes ${\rm curl}\,\mathbf{w}$ have vanishing normal boundary component.)

ii) The scalar ϕ is zero on $\partial\Omega$, while $\mathbf{w}$ is unrestricted on $\partial\Omega$. The harmonic $\mathbf{h}$ obeys $\mathbf{n}\times\mathbf{h} = 0$ on $\partial\Omega$. (The condition on ϕ makes $\nabla\phi$ have zero tangential boundary component.)

iii) The scalar ϕ is zero on $\partial\Omega$, the vector $\mathbf{w}$ obeys $\mathbf{n}\times\mathbf{w} = 0$ on $\partial\Omega$, while the harmonic $\mathbf{h}$ requires no boundary condition. (The conditions on ϕ and $\mathbf{w}$ make $\nabla\phi$ have zero tangential boundary component and ${\rm curl}\,\mathbf{w}$ have vanishing normal boundary component.)

d) As an illustration of the practical distinctions between the decompositions in part (c), take Ω to be the unit cube in $\mathbb{R}^3$, and $\mathbf{u} = (1, 0, 0)$ a constant vector field. Show that with conditions (i) we have $\mathbf{u} \in \mathcal{L}$, but for (ii) we have $\mathbf{u} \in \mathcal{T}$, and for (iii) we have $\mathbf{u} \in \mathcal{N}$.

We see that the Hodge-Weyl decompositions of the eigenspaces correspond one-to-one with the Helmholtz-Hodge decompositions of problem 6.16.