Perturbaticon Theory

Sometimes cannot solve a problem exactly, but can solve it approximately in a limit of small (large) parameter (ε). Often solutoon obtained for extremely small (large) ε works rather well for moderate ε.

Example:

$$x^2 + \varepsilon x - 1 = 0, \quad \varepsilon \ll 1$$

Can solve exactly: $\quad x = \dfrac{-\varepsilon \pm \sqrt{4+\varepsilon^2}}{2} = -\dfrac{\varepsilon}{2} \pm \left(1 + \dfrac{\varepsilon^2}{4}\right)^{1/2} =$

$$= -\frac{1}{2}\varepsilon \pm \left(1 + \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + \dots\right)$$

converge, if $|\varepsilon| < 2$

Truncated series give very good approximaticon of the exact result. For 3% accuracy need

1 term — $|\varepsilon| < 0.05$

2 —"— — —"— 0.5

3 —"— — —"— 1.2

4 —"— — —"— 1.6 ← not small (almost near convergence boundary)!

Algebraic Equations

How do we obtain an approximate answer in a systematic way?

Expansion method

Represent solution as a series in ε:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

Example: $x^2 + \varepsilon x - 1 = 0$ (again).

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) - 1 =$$

$$= (x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2 x_1^2 + 2\varepsilon^2 x_0 x_2 + \cdots) + (\varepsilon x_0 + \varepsilon^2 x_1 + \cdots) - 1 =$$

$$= (x_0^2 - 1) + \varepsilon(2x_0 x_1 + x_0) + \varepsilon^2(x_1^2 + 2x_0 x_2 + x_1) + \cdots = 0, \quad \forall \varepsilon$$

$$\Rightarrow \varepsilon^0: \ x_0^2 - 1 = 0 \quad \Rightarrow x_0 = \pm 1$$

$$\varepsilon^1: \ 2x_0 x_1 + x_0 = 0 \quad \Rightarrow x_1 = -\frac{1}{2}$$

$$\varepsilon^2: \ x_1^2 + 2x_0 x_2 + x_1 = 0 \quad \Rightarrow x_2 = -\frac{1}{2x_0}(x_1 + x_1^2) = \pm\frac{1}{8}$$

$$\cdots$$

$$\Rightarrow x = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \cdots$$ $\leftarrow$ coincides with the expansion of the exact solution up to $O(\varepsilon^3)$

Regular perturbation expansion: as $\varepsilon \to 0$ solution of the perturbed ($\varepsilon \neq 0$) problem approaches the solution of the unperturbed ($\varepsilon = 0$) problem (e.g., $x^2 - 1 = 0$)

Example: $\varepsilon x^2 + x + 1 = 0$

Exact solution: $x = \dfrac{-1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon} = -\dfrac{1}{2\varepsilon} \pm \dfrac{1}{2\varepsilon}(1 - 4\varepsilon)^{\frac{1}{2}} =$

$$= -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}(1 - 2\varepsilon - 2\varepsilon^2 - 4\varepsilon^3 - \cdots) = \left[\begin{array}{l} -1 - \varepsilon - 2\varepsilon^2 + \cdots \\ -\frac{1}{\varepsilon} + 1 + \varepsilon + 2\varepsilon^2 + \cdots \end{array}\right.$$

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

$$\underbrace{(\varepsilon x_0^2 + 2\varepsilon^2 x_0 x_1 + \dots)}_{\varepsilon x^2} + \underbrace{(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)}_{x} + 1 =$$

$$= (x_0 + 1) + \varepsilon(x_0^2 + x_1) + \varepsilon^2(2x_0 x_1 + x_2) + \dots = 0$$

ε^0: $x_0 + 1 = 0 \Rightarrow x_0 = -1$

ε^1: $x_0^2 + x_1 = 0 \Rightarrow x_1 = -x_0^2 = -1$

ε^2: $2x_0 x_1 + x_2 = 0 \Rightarrow x_2 = -2x_0 x_1 = -2$

...

$x = -1 - \varepsilon - 2\varepsilon^2 - \dots \rightarrow$ one solution ???

There is another one that blows up as $\varepsilon \to 0$:

$x(\varepsilon) = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ — singular solution

Substituting into equation gives: $x_{-1} = -1$, $x_0 = 1$, $x_1 = 1$, ...

i.e., $x = -\frac{1}{\varepsilon} + 1 + \varepsilon + \dots$ in agreement with "exact" result.

Singular perturbation expansion: solution of the perturbed problem (e.g., quadratic eqn.) does not approach the solution of the unperturbed problem (e.g., linear eqn: $x + 1 = 0$) as $\varepsilon \to 0$.

- Singular perturbation problems typically arise when small coeff. is in front of the highest order term:

$$\varepsilon x^2 + x + 1 = 0.$$

Note: A similar statement applies to solutions of differential equations with a small prefactor at the highest order derivative.

Rescaling

- Rescaling is a useful technique for converting singular perturbation problems into regular perturbation problems.

Example: $\varepsilon x^2 + x - 1 = 0$

Introduce rescaled variable $y = \varepsilon x$ $\left(x = \frac{1}{\varepsilon} y\right)$:

$$\varepsilon\left(\frac{y}{\varepsilon}\right)^2 + \frac{y}{\varepsilon} - 1 = \frac{1}{\varepsilon}\left(y^2 + y - \varepsilon\right) = 0$$

singular root

regular perturbation problem!

How do we find the rescaling without knowing the behavior of singular roots?

a) Introduce scaling factor $\delta(\varepsilon)$: $x = \delta y$,

where $y = O(1)$, i.e., $0 < \left|\lim\limits_{\varepsilon \to 0} y\right| < \infty$

b) Use dominant balance principle to find $\delta(\varepsilon)$ by balancing the orders of magnitude of different terms.

Example: $\varepsilon x^2 + x - 1 = 0$.

$$x = \delta y \Rightarrow \varepsilon \delta^2 y^2 + \delta y - 1 = 0 \quad , \quad y = O(1)$$

a) $\underline{\varepsilon \delta^2 = \delta} \Rightarrow \delta = \frac{1}{\varepsilon} \Rightarrow \underbrace{\underset{\text{large}}{\frac{1}{\varepsilon} y^2} + \underset{\text{large}}{\frac{1}{\varepsilon} y}}_{\text{dominant terms}} - \underset{\text{small}}{1} = 0$

$\varepsilon \delta^2 y^2$ $\quad$ δy

Dominant balance: $\frac{1}{\varepsilon} y^2 + \frac{1}{\varepsilon} y = 0 \Rightarrow y^2 + y = 0 \Rightarrow y = \begin{cases} 0 \\ -1 \end{cases}$

The root $y = 0$ is not $O(1) \Rightarrow y = -1$, $x \approx -\frac{1}{\varepsilon}$ $\leftarrow$ singular root!

b) $\underbrace{\varepsilon\delta^2}_{\varepsilon\delta^2 y^2} = \underbrace{1}_{1}$ => $\delta = \varepsilon^{-1/2}$ => $\underset{\text{small}}{Y^2} + \underset{\text{large}}{\frac{1}{\varepsilon^{1/2}} Y} - \underset{\text{small}}{1} = 0$

No terms balancing $\frac{1}{\varepsilon^{1/2}} Y$!

c) $\underbrace{\delta}_{\delta y} = \underbrace{1}_{1}$ => $\underset{\text{small}}{\varepsilon x^2} + \underset{\text{large}}{x} - \underset{\text{large}}{1} = 0$

Dominant balance: $x - 1 = 0$ => $x = +1$ ← nonsingular root !

Nonintegral Powers

Can we always assume that the expansion $x(\varepsilon)$ is in integral powers of ε?

Example: $(1-\varepsilon)x^2 - 2x + 1 = 0$

For $\varepsilon = 0$: $x^2 - 2x + 1 = (x-1)^2 = 0$ → $x_{1,2} = 1$

Degeneracy – sign of trouble!

$$x(\varepsilon) = \overset{x_0}{1} + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

$$(1-\varepsilon)(1+\varepsilon x_1 + \dots)^2 - 2(1+\varepsilon x_1 + \dots) + 1 =$$

$$= (1-\varepsilon)(1 + 2\varepsilon x_1 + \dots) - 2 - 2\varepsilon x_1 - \dots + 1 =$$

$$= 1 - \varepsilon + 2\varepsilon x_1 + \dots - 2 - 2\varepsilon x_1 - \dots + 1 = 0$$

ε^0: $1 - 2 + 1 = 0$ (chose $x_0 = 1$ correctly)

ε^1: $-1 + 2x_1 - 2x_1 = 0$ – cannot be satisfied for any choice of x_1!

Wrong expansion! Look at the expansion of the exact solution to see what went wrong:

$$x(\varepsilon) = \frac{2 \pm \sqrt{4 - 4(1-\varepsilon)}}{2(1-\varepsilon)} = \frac{1 \pm \varepsilon^{1/2}}{1-\varepsilon} = 1 \pm \varepsilon^{1/2} + \varepsilon \pm \varepsilon^{3/2} + \varepsilon^2 \pm \dots$$

Use rescaling $x = 1+\delta y$: $(1-\varepsilon)(1+\delta y)^2 - 2(1+\delta y)+1 =$

$= \cancel{1}-\varepsilon + 2(\cancel{1}-\varepsilon)\delta y + (1-\varepsilon)\delta^2 y^2 - \cancel{2} - \cancel{2\delta y} + \cancel{1} =$

$= -\varepsilon - 2\varepsilon\delta y + (1-\varepsilon)\delta^2 y^2 = 0$

can appoximate $1-\varepsilon$ with 1 in doing dominant balance:

$$\delta^2 y^2 - \varepsilon\delta(2y) - \varepsilon = \varepsilon\left(\delta^2/\varepsilon\, y^2 - \delta\cdot 2y - 1\right) = 0$$

a) $\underline{\delta^2/\varepsilon = \delta}$ $\Rightarrow$ $\delta = \varepsilon$ $\Rightarrow$ $\underbrace{\varepsilon\cdot y^2}_{\text{small}} - \underbrace{\varepsilon\cdot 2y}_{\text{small}} - \underbrace{1}_{\text{large}} = 0$ $\boxed{\text{NO BALANCE!}}$

b) $\underline{\delta = 1}$ $\Rightarrow$ $\underbrace{\frac{1}{\varepsilon}y^2}_{\text{large}} - \underbrace{2y}_{\text{small}} - \underbrace{1}_{\text{small}} = 0$ $\Rightarrow$ $y = 0 \neq O(1)$ $\boxed{\text{NO BALANCE!}}$

c) $\underline{\delta^2/\varepsilon = 1}$ $\Rightarrow$ $\delta = \varepsilon^{1/2}$ $\Rightarrow$ $\underbrace{y^2}_{\text{large}} - \underbrace{\varepsilon^{1/2}\cdot 2y}_{\text{small}} - \underbrace{1}_{\text{large}} = 0$

Dominant balance: $y^2 - 1 = 0$ $\Rightarrow$ $y = \pm 1$

$\Rightarrow x = 1 + \varepsilon^{1/2} y = 1 \pm \varepsilon^{1/2} + \dots$

Appropriate expansion: $x(\varepsilon) = 1 + \varepsilon^{1/2}x_1 + \varepsilon x_2 + \varepsilon^{3/2}x_3 + \dots$

Exercise: Find x_1, x_2, x_3 by equating powers of $\varepsilon^{1/2}$

Iterative Method

Sometimes (not always) it is easier and faster to find the right expansion by solving the equation iteratively.

$f(x) = 0$ equivalent to $x = f(x) + x \equiv F(x)$

Iterate $x_{n+1} = F(x_n)$:

Fixed point of iteration $x^* = F(x^*)$ is solution of $f(x^*) = 0$

Convergence of iterations (will see again in stability theory)

Can we find the fixed point by iterating?

In the vicinity of x^*, $x_n = x^* + \delta_n$:

$$x_{n+1} = x^* + \delta_{n+1} = F(x_n) = F(x^* + \delta_n) = \underset{x^*}{\underset{\parallel}{F(x^*)}} + F'(x^*)\delta_n + O(\delta^2)$$

$$\Rightarrow \delta_{n+1} \approx F'(x^*)\delta_n \approx F'(x^*)^2 \delta_{n-1} \approx \cdots \approx F'(x^*)^n \delta_1$$

Iterations converge if $|F'(x^*)| < 1$

<u>Note</u>: Choice of $F(x)$ is <u>not unique</u>, have to choose carefully to ensure convergence

<u>Note</u>: Not every expansion is good, look for expansions that converge <u>quickly</u> $\Rightarrow$ good choice of x_0 is critical!

<u>Example</u>: $(1-\varepsilon)x^2 - 2x + 1 = 0$.

Rewrite as $\varepsilon x^2 = x^2 - 2x - 1 = (x-1)^2$

$\Rightarrow x_{n+1} = 1 \pm \varepsilon^{1/2} x_n$

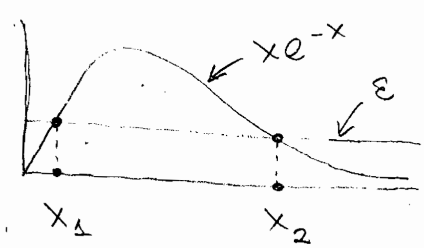
Starting with $x_0 = 1$:

$$x_1 = 1 \pm \varepsilon^{1/2}$$

$$x_2 = 1 \pm \varepsilon^{1/2}(1 \pm \varepsilon^{1/2}) = 1 \pm \varepsilon^{1/2} + \varepsilon$$

$$\ldots$$

<u>Example</u>: $x e^{-x} = \varepsilon$, $\varepsilon > 0$



1) $x_{n+1} = \varepsilon e^{x_n}$: $x_0 = 0$,

$$x_1 = \varepsilon e^{x_0} = \varepsilon$$

$$x_2 = \varepsilon e^{x_1} = \varepsilon e^{\varepsilon} = \varepsilon(1 + \varepsilon + \ldots) = \varepsilon + \varepsilon^2 + \ldots$$

2) $x_{n+1} = \ln(1/\varepsilon) + \ln(x_n)$: $x_0 = \ln(1/\varepsilon)$

extremely slowly converging $\rightarrow$ $x_1 = \ln(1/\varepsilon) + \ln(\ln(1/\varepsilon))$

$$\ldots$$