Singular Perturbation Expansions for ODEs

Example (overdamped pendulum)

$$\begin{cases} y'' + by' + \omega_0^2 y = 0 , \quad y' = \dfrac{dy}{d\tau} \\ y(0) = y_0 \\ y'(0) = u_0 \end{cases}$$

In the overdamped limit $(b \gg \omega_0)$ inertia is negligible

$\Rightarrow y''$ term becomes vanishingly small

$\Rightarrow$ expect _singular_ perturbation expansion

Rescale variables: $y \to Ax$, $\tau \to \Omega t$

$$y'' + by' + \omega_0^2 y = \underbrace{\Omega^2 A \ddot{x}}_{\varepsilon} + \underbrace{b\Omega A \dot{x}}_{1} + \underbrace{\omega_0^2 A x}_{1} = 0 , \quad \dot{x} = \dfrac{dx}{dt}$$

Solving obtain: $A = \omega_0^{-2}$, $\Omega = 1/(bA) = \omega_0^2/b$,

$$\varepsilon = \Omega^2 A = \frac{\omega_0^4}{b^2} \frac{1}{\omega_0^2} = \left(\frac{\omega_0}{b}\right)^2 \ll 1.$$

Rescaled equation: $\varepsilon \ddot{x} + \dot{x} + x = 0$

Initial conditions: $x(0) = \frac{1}{A} y(0) = \omega_0^2 y_0 = \hat{x}_0$

$\dot{x}(0) = (\Omega A)^{-1} y'(0) = b u_0 = \hat{v}_0$

Seek solution in the form of a perturbation expansion:

$x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$

$$\varepsilon(\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots) + (\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) =$$

$$= (\dot{x}_0 + x_0) + \varepsilon(\ddot{x}_0 + \dot{x}_1 + x_1) + \varepsilon^2(\ddot{x}_1 + \dot{x}_2 + x_2) + \cdots = 0$$

At ε^0: $\dot{x}_0 + x_0 = 0 \Rightarrow x_0 = Ae^{-t}$

ε^1: $\dot{x}_1 + x_1 = -\ddot{x}_0 = -Ae^{-t} \Rightarrow x_1 = (B - At)e^{-t}$

ε^2: $\dot{x}_2 + x_2 = -\ddot{x}_1 = (A + B - At)e^{-t} \Rightarrow x_2 = (C - (B + 2A)t + \frac{A}{2}t^2)e^{-t}$

Collecting everything:

$$x(t) = \left[A + \varepsilon(B - At) + \varepsilon^2\left(C - (B+2A)t + \frac{A}{2}t^2\right) + \cdots\right]e^{-t}$$

Boundary conditions: $x(0) = A + \varepsilon B + \varepsilon^2 C + \cdots = \hat{x}_0$, $\forall \varepsilon$

$$\Rightarrow A = \hat{x}_0, \quad B = C = 0$$

$$\dot{x}(0) = \underbrace{-A - \varepsilon(A+B) - \varepsilon^2(2A+B+C) + \cdots}_{-\hat{x}_0(1 + \varepsilon - 2\varepsilon^2 + \cdots) \neq \hat{V}_0 \,!} = \hat{V}_0$$

Problem: Cannot satisfy both initial conditions at once.

(Of course, the unperturbed problem ($\varepsilon = 0$) is 1st order ODE!)

Same as in algebraic equations: small coeff. at highest order.

Non-uniform convergence: $\lim\limits_{\varepsilon \to 0} \dot{x}(t,\varepsilon) = \dot{x}(t,0)$, $t > 0$

$$\lim\limits_{\varepsilon \to 0} \dot{x}(0,\varepsilon) \neq \dot{x}(0,0)$$

Goal: try to build an asymptotic expansion, such that

1) $\lim\limits_{\varepsilon \to 0} x(t,\varepsilon) = x(t,0)$ 2) $\lim\limits_{\varepsilon \to 0} \dot{x}(t,\varepsilon) = \dot{x}(t,0)$, $\forall t \geq 0$
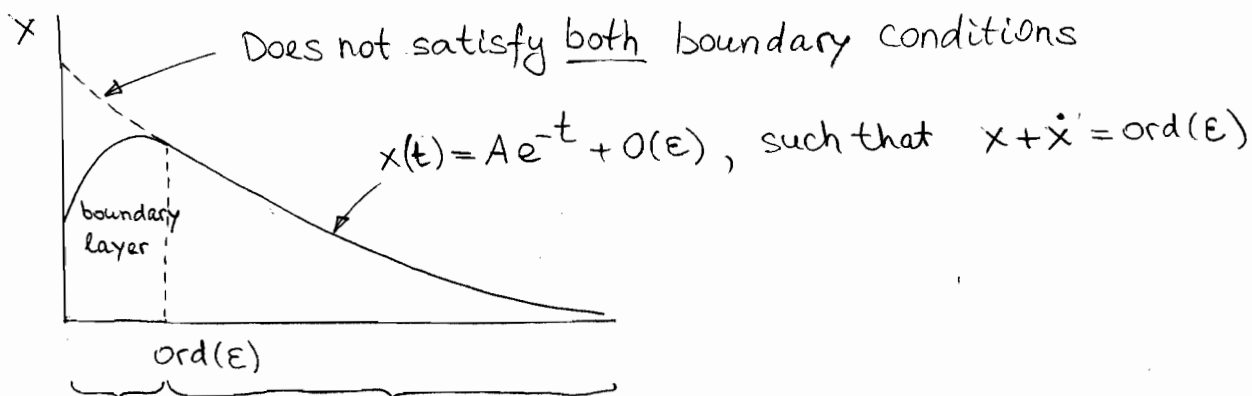
Boundary layers

First, find the characteristic time scales:

$$\varepsilon\ddot{x} + \dot{x} + x = 0 \iff \begin{cases} \dot{x} = V & : \quad x, V = \text{ord}(1) \Rightarrow t = \text{ord}(1), \\ \dot{V} = -\frac{1}{\varepsilon}(x + V) & \Rightarrow t = \text{ord}(\varepsilon) \end{cases}$$

(unless $x + V = \text{ord}(\varepsilon)$)

Generically, $x(0) + V(0) = x(0) + \dot{x}(0) = \hat{x}_0 + \hat{V}_0 = \text{ord}(1)$

Does not satisfy <u>both</u> boundary conditions

$x(t) = Ae^{-t} + O(\epsilon)$, such that $x + \dot{x} = ord(\epsilon)$

boundary layer

$ord(\epsilon)$

"inner solution"
(fast transient)

"outer solution"
(slow relaxation)

<u>outer solution</u>: $t = ord(1)$: $\epsilon \ddot{x} + \dot{x} + x = 0$ — already solved!

Not valid for small $t \Rightarrow$ No boundary conditions

<u>inner solution</u>: $t = ord(\epsilon) \Leftrightarrow t = \epsilon T$, $T = ord(1)$

$\epsilon \ddot{x} + \dot{x} + x = \epsilon \frac{1}{\epsilon^2} \frac{d^2}{dT^2} X + \frac{1}{\epsilon} \frac{d}{dT} X + X = \frac{1}{\epsilon} \left(\frac{d^2 X}{dT^2} + \frac{dX}{dT} + \epsilon X \right) = 0$

$$\begin{cases} \frac{d^2 X}{dT^2} + \frac{dX}{dT} + \epsilon X = 0 \\ X(0) = \hat{X_0} \\ \frac{dX}{dT}(0) = \epsilon \dot{x}(0) = \epsilon \hat{V_0} \end{cases}$$ — regular 2nd order problem!

$X(T, \epsilon) = X_0(T) + \epsilon X_1(T) + \epsilon^2 X_2(T) + \cdots$

a) $\hat{X_0} = X(0) = X_0(0) + \epsilon X_1(0) + \epsilon^2 X_2(0) + \cdots \Rightarrow X_0(0) = \hat{X_0}$,

$X_1(0) = X_2(0) = \cdots = 0$

b) $\epsilon \hat{V_0} = \frac{dX}{dT}(0) = \frac{dX_0}{dT}(0) + \epsilon \frac{dX_1}{dT}(0) + \epsilon^2 \frac{dX_2}{dT}(0) + \cdots \Rightarrow \frac{dX_1}{dT}(0) = \hat{V_0}$

$\frac{dX_0}{dT}(0) = \frac{dX_2}{dT}(0) = \cdots = 0$

Substitute into the equation:

$\frac{d^2 X_0}{dT} + \epsilon \frac{d^2 X_1}{dT} + \epsilon^2 \frac{d^2 X_2}{dT} + \cdots + \frac{dX_0}{dT} + \epsilon \frac{dX_1}{dT} + \epsilon^2 \frac{dX_2}{dT} + \cdots + \epsilon X_0 + \epsilon^2 X_1 + \cdots = 0$

ϵ^0: $\frac{d^2 X_0}{dT} + \frac{dX_0}{dT} = 0$, $X_0(0) = \hat{X_0}$, $\frac{dX_0}{dT}(0) = 0 \Rightarrow X_0(T) = \hat{X_0}$

ϵ^1: $\frac{d^2 X_1}{dT^2} + \frac{d^2 X_1}{dT^2} = -X_0(T) = -\hat{X_0}$

$$x_1(0) = 0, \quad \frac{dx_1}{dT}(0) = V_0 \implies X_1(T) = (\hat{x}_0 + \hat{v}_0)(1 - e^{-T}) - \hat{x}_0 T$$

$$\varepsilon^2: \quad \frac{d^2x_2}{dT^2} + \frac{dx_2}{dT} = -X_1(T)$$

$$x_2(0) = 0, \quad \frac{dx_2}{dT}(0) = 0 \implies X_2(T) = 3\hat{x}_0 + 2\hat{v}_0 - (2\hat{x}_0 + \hat{v}_0)T + \tfrac{1}{2}\hat{x}_0 T^2$$

$$- \left(3\hat{x}_0 + 2\hat{v}_0 - (\hat{x}_0 + \hat{v}_0)T\right)e^{-T}$$

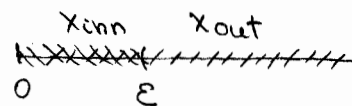Substitute $T = t/\varepsilon$, collect terms of different order in ε:

<u>Inner solution:</u> (valid for $t < O(\varepsilon)$)

$$X_{inn}(t,\varepsilon) = x_0\left(1 - t + \tfrac{1}{2}t^2\right) + \varepsilon\left(\hat{x}_0(1 - 2t) + \hat{v}_0(1 - t) - (\hat{x}_0 + \hat{v}_0)(1 + t)e^{-t/\varepsilon}\right) +$$

$$+ \varepsilon^2\left(3\hat{x}_0 + 2\hat{v}_0\right)\left(1 - e^{-t/\varepsilon}\right) + \cdots$$

<u>Outer solution:</u> (valid for $t > 0$)

$$X_{out}(t,\varepsilon) = Ae^{-t} + \varepsilon(B - At)e^{-t} +$$

$$+ \varepsilon^2\left(C - (B + 2A)t + \tfrac{1}{2}At^2\right)e^{-t} + \cdots$$

Matched Asymptotic Expansions

$$\lim_{\varepsilon \to 0} \left(X_{out}(t,\varepsilon) - X_{inn}(t,\varepsilon)\right) = 0$$

$$\lim_{\varepsilon \to 0} \left(\dot{X}_{out}(t,\varepsilon) - \dot{X}_{inn}(t,\varepsilon)\right) = 0$$

for t such that both expansions are valid

Taylor-expand both X_{out} and X_{inn} for small t ($t < O(\varepsilon)$):

$$X_{inn}(t,\varepsilon) = x_0\left(1 - t + \tfrac{1}{2}t^2\right) + \varepsilon\left(\hat{x}_0 + \hat{v}_0 - (2\hat{x}_0 + \hat{v}_0)t\right) + \varepsilon^2(3\hat{x}_0 + 2\hat{v}_0) + \cdots$$

$$X_{out}(t,\varepsilon) = A\left(1 - t + \tfrac{1}{2}t^2\right) + \varepsilon(B - (A + B)t) + \varepsilon^2 C + \cdots$$

(Discard terms of order $\varepsilon^3, \varepsilon^2 t, \varepsilon t^2, t^3$)

$$\implies A = \hat{x}_0, \quad B = \hat{x}_0 + \hat{v}_0, \quad C = 3\hat{x}_0 + 2\hat{v}_0$$

Coefficients of X_{out} are found through matching, rather than by using boundary conditions:

$$X_{out}(t,\varepsilon) = \left(\hat{x}_0 + \varepsilon(\hat{x}_0 + \hat{v}_0 - \hat{x}_0 t) + \varepsilon^2(3\hat{x}_0 + 2\hat{v}_0 - (3\hat{x}_0 + \hat{v}_0)t + \tfrac{1}{2}\hat{x}_0 t^2) + \cdots\right)e^{-t}$$

Piecewise approximation:

$$x(t, \varepsilon) = \begin{cases} x_{inn}(t, \varepsilon), & t < O(\varepsilon) \\ x_{out}(t, \varepsilon), & t > O(\varepsilon) \end{cases}$$

Can we find a single continuous approximation?

Notice that $O(1)$ terms in $x_{inn}(t, \varepsilon)$ and $x_{out}(t, \varepsilon)$ are the same:

$$x_0(1 - t + \tfrac{1}{2}t^2) = x_0 e^{-t} + O(t^3)$$

Uniform approximation:

taylor series for $x_{out}(t, \varepsilon)$

$$x_{unif}(t, \varepsilon) = x_{out}(t, \varepsilon) + x_{inn}(t, \varepsilon) - x_{match}(t, \varepsilon) =$$

$$= (\hat{x}_0 + \varepsilon(\hat{x}_0 + \hat{v}_0 - \hat{x}_0 t) + \cdots)e^{-t} + (\varepsilon(\hat{x}_0 + \hat{v}_0)(-1-t) + \cdots)e^{-t/\varepsilon}$$

Exact Solution:

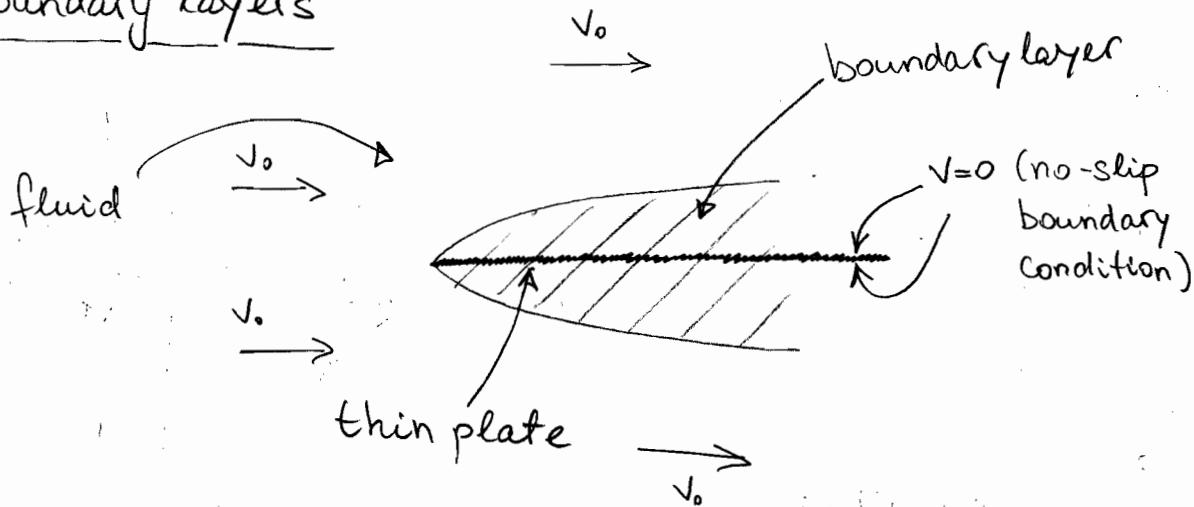$$\varepsilon\ddot{x} + \dot{x} + x = 0 \quad, \quad x = e^{\lambda t} \quad \Rightarrow \quad \varepsilon\lambda^2 + \lambda + 1 = 0$$

We already solved this equation:
$$\lambda_1 = -1 + O(\varepsilon)$$
$$\lambda_2 = -\tfrac{1}{\varepsilon} + O(1)$$

$$\Rightarrow x(t, \varepsilon) = A e^{-t} + B e^{-t/\varepsilon} + \cdots$$

slow relaxation ↗   ↑ fast transient

Fluid boundary layers

fluid $\xrightarrow{V_0}$

V_0

thin plate $\xrightarrow{V_0}$

boundary layer

$V = 0$ (no-slip boundary condition)

x

$x_{out}(t,\varepsilon)$

$x_{inn}(t,\varepsilon)$

$x(0)=1$
$\dot{x}(0)=0$

$\varepsilon = 0.3$

$x(t,\varepsilon)$

1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0

0.5   1   1.5   2   2.5   3

t~

t

x

$x(0) = 1$
$\dot{x}(0) = 1$

$x_{unif}(t, \varepsilon)$

$\varepsilon = 0.1$

1

0.8

0.6

0.4

0.2

0   0.5   1   1.5   2   2.5   3

$t\sim$

$x(t, \varepsilon)$

t