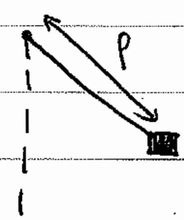
# CONSTRAINED HAMILTONIAN SYSTEMS : DIRAC'S THEORY

We consider a Hamiltonian system given by Hamiltonian $H(z)$, $z = (z_1, \ldots, z_N)$
and Poisson bracket $\{\cdot, \cdot\}$.

We impose $K$ constraints $\Phi_\alpha(z) = 0$ on the dynamics.

example:




$\rho = \sqrt{x^2+y^2+z^2}$ is constant.

We introduce $K$ Lagrange multipliers $\mu_\beta$ $\beta = 1, \ldots, K$.
so that the new Hamiltonian is

$$H_* = H + \mu_\beta \Phi_\beta$$ ; the Poisson bracket is unchanged.

$\mu_\beta$ determined by $\frac{d\Phi_\beta}{dt} = 0 = \{\Phi_\beta, H_*\}$ $(*)$

We define the matrix $C$ of the Poisson brackets between two constraints

$$C_{\alpha\beta} = \{\Phi_\alpha(z), \Phi_\beta(z)\}$$

$(*)$ becomes $C_{\alpha\beta}\mu_\beta = -\{\Phi_\alpha, H\}$

If $C$ is invertible $\mu = -C^{-1}\{\Phi, H\}$
otherwise find additional constraints. (secondary constraints)
$\Rightarrow$ even number of constraints

$$\frac{dF}{dt} = \{F, H_*\} = \{F, H\} + \mu_\alpha\{F, \Phi_\alpha\} + \underbrace{\{F, \mu_\alpha\}\Phi_\alpha}_{=0 \text{ on the surface of constraints.}}$$

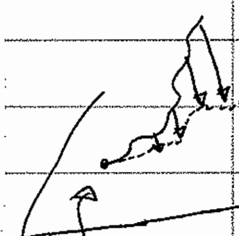
$$\approx \{F, H\} - \{F, \Phi_\alpha\} C^{-1}_{\alpha\beta} \{\Phi_\beta, H\}$$

$\uparrow$ weakly equal (on the surface but not everywhere)

We define the Dirac bracket as

$$\{F,G\}_* = \{F,G\} - \{F,\phi_\alpha\}\, C^{-1}_{\alpha\beta}\, \{\phi_\beta, G\}$$

Is it a Poisson bracket? bilinear, antisymmetric ok
Leibniz : ok
Jacobi : a bit more difficult [Dirac]



$\phi_\alpha(z)=0$

Brief summary:

$$\boxed{\begin{matrix} H \\ \{\cdot,\cdot\} \end{matrix}} \xrightarrow{\text{constrained}} \boxed{\begin{matrix} H_* \\ \{\cdot,\cdot\} \end{matrix}} \xrightarrow{\text{Dirac}} \boxed{\begin{matrix} H \\ \{\cdot,\cdot\}_* \end{matrix}}$$

Additional property: i/ The constraints $\phi_\alpha$ are Casimir invariants

$$\square \quad \{\phi_\gamma, G\}_* = \{\phi_\gamma, G\} - \underbrace{\underbrace{\{\phi_\gamma,\phi_\alpha\}}_{C_{\gamma\alpha}} C^{-1}_{\alpha\beta}}_{\delta_{\gamma\beta}} \{\phi_\beta, G\}$$

$$= \{\phi_\gamma, G\} - \{\phi_\gamma, G\} = 0 \quad \square$$

Casimir invariants are invariants ⇒ preserved by the dynamics.

ii/ Casimir invariants of $\{\cdot,\cdot\}$ are also Casimir invariants of $\{\cdot,\cdot\}_*$ (contrary not true in general)

$$\square \quad \{F,C\}_* = \underset{0}{\underset{\shortparallel}{\{F,C\}}} - \{F,\phi_\alpha\}\, C^{-1}_{\alpha\beta}\, \underset{0}{\underset{\shortparallel}{\{\overset{\phi_\alpha}{\phi_\beta}, C\}}} = 0 \quad \square$$

iii/ There exist conserved quantities for $(H,\{\cdot,\cdot\})$ which are not conserved by $(H,\{\cdot,\cdot\}_*)$ → counterexample: $H = \frac{p_1^2+p_2^2}{2} + V(x_1 - x_2)$

□ où $V(x_1) = +V(-x_1)$

$P = p_1 + p_2$ conserved quantity

$$\{F,G\}_* = \frac{\partial F}{\partial q_2}\frac{\partial G}{\partial p_2} - \frac{\partial F}{\partial p_2}\frac{\partial G}{\partial q_2}$$

$$\dot{P} = \{P,H\}_* = -\frac{\partial H}{\partial q_2} = \frac{\partial V}{\partial x} \neq 0 \quad \square$$

Example 1: rigid pendulum

$$\mathcal{L}(r,\dot r,\theta,\dot\theta) = \frac{1}{2}\left(\dot r^2 + r^2\dot\theta^2\right) + g\, r\cos\theta$$

generalized momenta: $p_r = \dot r \qquad p_\theta = r^2\dot\theta$

$$H(r,p_r,\theta,p_\theta) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - r\cos\theta$$

$(r,p_r)$ $(\theta,p_\theta)$ canonically conjugate pairs of variables.

$$\{F,G\} = \frac{\partial F}{\partial r}\frac{\partial G}{\partial p_r} - \frac{\partial F}{\partial p_r}\frac{\partial G}{\partial r} + \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial p_\theta} - \frac{\partial F}{\partial p_\theta}\frac{\partial G}{\partial \theta}$$

Why this change of variables (w.r.t. Cartesian coordinates)?

because the constraint is easily expressed in this set

$$\phi_1 = r - a \qquad (a \text{ length})$$

$C_{11} = \{\phi_1, \phi_1\} = 0$ (antisymmetry of the Poisson bracket)

$\Rightarrow$ secondary constraint.

$$\phi_2 = p_r$$

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ invertible} \qquad C^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Dirac's contribution: $-\{F,\phi_\alpha\}C^{-1}_{\alpha\beta}\{\phi_\beta,G\}$

$$-\{F,\phi_1\}C^{-1}_{12}\{\phi_2,G\} = \frac{\partial F}{\partial p_r}\frac{\partial G}{\partial r}$$

$$-\{F,\phi_2\}C^{-1}_{21}\{\phi_1,G\} = -\frac{\partial F}{\partial r}\frac{\partial G}{\partial p_r}$$

hence $\{F,G\}_D = \dfrac{\partial F}{\partial \theta}\dfrac{\partial G}{\partial p_\theta} - \dfrac{\partial F}{\partial p_\theta}\dfrac{\partial G}{\partial \theta}$

the degree of freedom $(r,p_r)$ has been frozen.

Example 2: incompressible Euler equation.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\nabla\cdot(\rho u) \\ \dfrac{\partial u}{\partial t} = -(u\cdot\nabla)u - \dfrac{\nabla p}{\rho} \end{cases} + \begin{cases} \rho - \rho_0 = 0 \\ \nabla\cdot u = 0 \end{cases}$$

incompressibility condition.
at each point in $\mathbb{R}^3$
$\Rightarrow \infty$ number of constraints.

$$C_{22}(x,x') = \{\nabla\cdot u(x), \nabla\cdot u(x')\}$$

$$\{F,G\} = -\int d^3x \left[ F_\rho \nabla\cdot G_u - \nabla\cdot F_u G_\rho + \frac{\nabla\times u}{\rho}\cdot F_u \times G_u \right]$$

$\downarrow$ Dirac's procedure

$$\{F,G\}_* = \int d^3x \, \frac{\nabla\times u}{\rho}\cdot \overline{F_u}\times\overline{G_u}$$

where $\overline{F_u}$ is the constrained functional derivative.

$$\overline{F_u} = F_u - \nabla\Delta^{-1}\nabla\cdot F_u$$

It is evident that $\rho$ is a Casimir invariant. (no $F_\rho$)

Exercise: show that $\nabla\cdot u$ is a Casimir invariant of $\{\cdot,\cdot\}_*$.