# Georgia Tech PHYS 6124 Fall 2012

# Mathematical Methods of Physics I

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## Homework #12 due Tuesday November 29 2012

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== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

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[All problems in this set are from Goldbart]

**Problem 4) Kramers-Krönig relations**

(a) The Debye form of the frequency-dependent generalized response function $\epsilon(\omega)$ is given by

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_0 - \epsilon_\infty}{1 - i\omega T},$$

where $\epsilon_0$, $\epsilon_\infty$ and $T$ are real parameters. Show that this form corresponds to the time-dependent generalized reponse function

$$\alpha(\tau) = \epsilon_\infty \, \delta^{(+)}(\tau) + \left(\epsilon_0 - \epsilon_\infty\right) T^{-1} \, \mathrm{e}^{-\tau/T},$$

where $\delta^{(+)}(\tau)$ is understood to mean $\lim_{\tau_0 \to 0^+} \tau_0^{-1} \exp(-\tau/\tau_0)$ with $\tau_0$ real. Confirm that the Debye form obeys the Kramers-Krönig relations.

(b) The Van Vleck-Weisskopf-Fröhlich form of the time-dependent generalized response function $\alpha(\tau)$ is given by

$$\alpha(\tau) = \epsilon_\infty \, \delta^{(+)}(\tau) + \Delta\epsilon \, T^{-1} \, \mathrm{e}^{-\tau/T} \left(\cos \omega_0 \tau + \omega_0 T \sin \omega_0 \tau\right),$$

where $\Delta\epsilon$ and $\omega_0$ are further real parameters. Determine the corresponding frequency-dependent generalized response function, and confirm that it obeys the Kramers-Krönig relations.

**Problem 5) More applications of Cauchy's theorem**
(**Ablowitz & Fokas**, p. 90-91, p. 231-233)

(a) We wish to evaluate the Fresnel integral $I = \int_0^\infty \exp\left(ix^2\right) dx$. To do this, consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, $R$ and $R\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from $R$ to $R\exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty} \left( \int_0^R \exp\left(ix^2\right) dx - \mathrm{e}^{i\pi/4} \int_0^R \exp\left(-r^2\right) dr \right) = 0$$

and, from the well-known result of real integration $\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2$, deduce that $I = e^{i\pi/4}\sqrt{\pi}/2$.

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# Optional problems

**Problem 1) Winding numbers and topology** (**Needham**, p. 369-372)

(a) Envisage an arbitrarily complicated but nevertheless *simple* contour. By considering the collection of possible values taken by the winding numbers for off-contour points, devise a fast algorithm for establishing whether or not an arbitrary off-contour point lies inside or outside the contour. [Note: You may use this algorithm to impress your friends at dinner parties.]

(b) For each of the following functions $f(z)$, find all the $p$-points lying inside the specified disc and determine their multiplicities.

 (i) $f(z) = \exp 3\pi z$ and $p = i$ for the disc $|z| \leq 4/3$;

 (ii) $f(z) = \cos z$ and $p = 1$ for the disc $|z| \leq 5$;

 (iii) $f(z) = \sin\left(z^4\right)$ and $p = 0$ for the disc $|z| \leq 2$.

 In each case, use a computer to draw the image of the boundary of the circle and, hence, verify the *argument principle*

(c) Use Rouché's theorem to establish the following results:

 (i) If $a$ is greater than 1 then the equation $z^n e^a = e^z$ has $n$ solutions inside the unit circle.

 (ii) If $f(z) = 2z^5$ and $g(z) = 8z - 1$ then all five of the solutions of the equation $f(z) + g(z) = 0$ lie in the disc $|z| < 2$.

 (iii) By reversing the roles of $f$ and $g$, show that there is only one root in the unit disc. Hence, deduce that there are four roots in the annulus $1 < |z| < 2$.

**Problem 2) Cauchy's theorem** (**Needham**, p. 421-423)

(a) Let $K$ be a contour that winds once around $z = 1$, once around $z = 0$, twice around $z = -1$, and not around $z = 1 + i$.

 (i) Evaluate the following integral by factoring the denominator and putting the integrand into partial fractions:

$$\oint_K \frac{z\,dz}{z^2 - iz - 1 - i}.$$

 (ii) Write down the Laurent series (centered at the origin) for $z^{-11}\cos z$. Hence find

$$\oint_K \frac{\cos z}{z^{11}} dz.$$

(b) This exercise illustrates how one type of integral may be evaluated easily using a complex integral. Let $L$ be the straight contour along the real axis from $-R$ to $R$, and let $J$ be the semicircular contour in the upper half-plane back from $R$ to $-R$. The complete contour $L + J$ is thus a closed loop.

(i) By using partial fractions, show that the integral

$$\oint_{L+J} \frac{dz}{z^4+1}$$

vanishes if $R < 1$, and find its value if $R > 1$.

(ii) By using the fact that $z^4 + 1$ is the complex number from $-1$ to $z^4$, write down the minimum of $|z^4+1|$ as $Z$ travels around $J$. Now think of $R$ as large, and use the Darboux inequality to show that the integral of $J$ dies away to zero as $R$ grows to infinity.

(iii) From the previous parts, deduce the value of

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

(iv) Although it can be evaluated easily by ordinary means, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

by the method used in the previous parts of this exercise.

(v) Likewise, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}.$$

(c) (i) Write down the value of $\int_0^{a+ib} dz\, \mathrm{e}^z$.

(ii) By equating your answer to part (i) to the parametric form of the integral taken along the straight contour from $z = 0$ to $z = a + ib$, deduce the values of the integrals $\int_0^1 dx\, \mathrm{e}^{ax} \sin bx$ and $\int_0^1 dx\, \mathrm{e}^{ax} \cos bx$.

(d) (i) Show that when integrating a product of holomorphic functions we may use the method of integration by parts.

(ii) Let $L$ be a contour between the real numbers $\pm\theta$. Evaluate $\int_L dz\, z\, \mathrm{e}^{iz}$. Verify your result via parametric integration along the line segment between $\pm\theta$.

(e) Let $f(z) = z^{-1}\left(z + z^{-1}\right)^n$, where $n$ is a positive integer.

(i) Use the binomial theorem to find the residue of $f$ at the origin when $n$ is even or odd.

(ii) If $n$ is odd, determine the value of the integral of $f$ around any contour.

(iii) If $n$ is even (and equal to $2m$) and $K$ is a simple contour winding once around the origin, deduce from part (i) that the integral of $f$ around $K$ is given by $2\pi i\,(2m)!/(m!)^2$.

(iv) By taking $K$ to be the unit circle, deduce Wallis' result:

$$\int_0^{2\pi} d\theta\, \cos^{2m}\theta = \frac{2\pi(2m)!}{2^{2m}(m!)^2}.$$

(v) Similarly, by considering functions of the form $z^k f(z)$ with integral $k$, evaluate $\int_0^{2\pi} d\theta\, \cos^n\theta\, \cos k\theta$ and $\int_0^{2\pi} d\theta\, \cos^n\theta\, \sin k\theta$.

(f) Let $E$ be the elliptical orbit $z(t) = a\cos t + ib\sin t$, where $a$ and $b$ are positive and $t$ varies from 0 to $2\pi$. By considering the integral of $1/z$

around $E$, show that

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{2\pi}{ab}.$$

**Problem 3) Cauchy's integral formula** (**Needham**, p. 446)

(a) (i) If $C$ is the unit circle, show that

$$\int_0^{2\pi} \frac{dt}{1 - 2a\cos t + a^2} = \oint_C \frac{i\,dz}{(z-a)(az-1)}.$$

(ii) Use Cauchy's integral formula to deduce that if $0 < a < 1$ then the above integrals are given by $2\pi/(1-a^2)$.

(b) Let $f(z)$ be holomorphic on and inside a circle $K$ defined by $|z-a| = \rho$, and let $M$ be the maximum value of $|f(z)|$ on $K$.

(i) Use Cauchy's integral formula for derivatives to show that $|f^{(n)}(a)| \leq n!\,M/\rho^n$.

(ii) Suppose that $|f(z)| \leq M$ for all $z$, where $M$ is some positive constant. By choosing $n = 1$ in the above inequality, derive Liouville's theorem.

(iii) **(optional)** Suppose that $|f(z)| \leq M|z^n|$ for all $z$, where $n$ is some positive integer. Show that $f^{(n+1)}(z) = 0$, and hence deduce that $f(z)$ must be a polynomial whose degree does not exceed $n$.

(c) **(optional)**

(i) Show that if $C$ is any simple loop around the origin then

$$\frac{1}{2\pi i}\oint_C \frac{(1+z)^n}{z^{r+1}}dz = \binom{n}{r}.$$

(ii) By taking $C$ to be the unit circle, deduce that $\binom{2n}{n} \leq 4^n$.

**Problem 6-2) Evaluation of definite integrals** (**Ablowitz & Fokas**, p. 235-237)

(a) Evaluate the following real integrals via residues (for $a^2, b^2, k > 0$):

(i) $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$ (ii) $\displaystyle\int_{-\infty}^\infty \frac{dx\,\cos kx}{(x^2+a^2)(x^2+b^2)}$ (iii) $\displaystyle\int_0^\infty \frac{dx\,x\,\sin x}{x^2+a^2}$

(iv) $\displaystyle\int_0^\infty \frac{dx\,x^3\,\sin kx}{x^4+a^4}$ (v) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta}$ (vi) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$

(b) **(optional)** Evaluate the following real integrals via residues (for $a^2, b^2, k, m > 0$):

(i) $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2}$ (ii) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ (iii) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

(iv) $\displaystyle\int_{-\infty}^\infty \frac{dx\,x\,\cos kx}{x^2+4x+4}$ (v) $\displaystyle\int_{-\infty}^\infty \frac{dx\,\cos kx\,\cos mx}{x^2+a^2}$ (vi) $\displaystyle\int_0^{\pi/2} d\theta\,\sin^4\theta$

(c) **(optional)** Use an origin-centered sector contour of radius $R$ and angle

$2\pi/5$ to show that (for $a > 0$)

$$\int_0^\infty \frac{dx}{x^5 + a^5} = \frac{\pi}{5a^4 \sin(\pi/5)}.$$

(d) (i) Via a rectangular contour with corners at $b \pm iR$ and $b + 1 \pm iR$, show that

$$\lim_{R\to\infty} \int_{b-iR}^{b+iR} \frac{dz}{2\pi i} \frac{e^{az}}{\sin \pi z} = \frac{1}{\pi} \frac{1}{1 + \exp(-a)} \qquad (0 < b < 1, |\mathrm{Im}\, a| < \pi).$$

(ii) **(optional)** By using a rectangular contour with corners at $\pm R$ and $\pm R + i$, show that

$$\int_0^\infty dx\, \big(\cosh ax / \cosh \pi x\big) = (1/2) \sec(a/2) \qquad (|a| < \pi).$$

(e) **(optional)**

(i) Use a rectangular contour $C_N$ with corners $\big(N + \frac{1}{2}\big)\big(\pm 1 \pm i\big)$ to evaluate

$$\frac{1}{2\pi i} \oint_{C_N} \frac{dz\, \pi \cot \pi z \coth \pi z}{z^3}.$$

(ii) By considering the $N \to \infty$ limit of your answer to part (i) show that

$\sum_{n=1}^\infty n^{-3} \coth n\pi = (7/180)\pi^3$.