Exercise 5.A.1 Diagonalization of propagators

Consider the stepping matrix (...) for a n-site lattice with periodic boundary conditions (assume 1-dimensional, for now):

$$S = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot \\ \cdot & & & \ddots & \\ 1 & \cdot & \cdot & & 0 \end{pmatrix}$$

S satisfies $S^n = 1$, with eigenvalues $\lambda_r = e^{i2\pi\frac{r}{n}}$. It can be diagonalized by

$$S = C \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix} C^\dagger$$

$$C_{jr} = \frac{1}{\sqrt{n}} e^{i2\pi\frac{rj}{n}} \; , \; (C^\dagger)_{rj} = C^*_{jr}$$

Check that

$$S_{ij} = \sum_r C_{ir} \lambda_r C^\dagger_{rj} = \delta_{i+1,j} \; .$$

Now the propagator (5.5) can be diagonalized

$$\Delta_{ij} = \sum_r C_{ir} \frac{s}{1 - h(\lambda_r + \lambda_r^*)} C^\dagger_{rj} = \frac{s}{n} \sum \frac{e^{i2\pi r\frac{i-j}{n}}}{1 - 2h\cos(2\pi\frac{r}{n})}$$

This is the <u>lattice propagator</u>. To take its continuum limit, introduce lattice spacing, and the continuum length and momentum variables:

lattice spacing $a = \frac{L}{n}$ ← arbitrary length scale

momentum $k = \frac{2\pi r}{L}$; coordinates $x = ia, y = ja, \ldots$

For small momenta (i.e. distances much larger than the lattice spacing)

$$\Delta(x,y) = as\int \frac{dk}{2\pi}\ \frac{e^{ik(x-y)}}{1-2h+ha^2k^2+\dots}$$

By the probability conservation $1-2h=s$. We replace the hopping parameter by the mass parameter

$$m^2 = \frac{s}{ha^2}$$

If the particle does not like hopping ($h\to 0$), the mass is infinite, and there is no propagation. If the particle does not like stopping ($s\to 0$), the mass is zero. The argument is the same for arbitrary dimension, and it yields the Euclidean continuum propagator:

$$\Delta(x,y) = m^2a^d\int \frac{d^dk}{(2\pi)^d}\ \frac{e^{ik\cdot(x-y)}}{m^2+k^2}$$

(The factor a^dm^2 is usually absorbed into the definition of $\Delta(x,y)$).

Exercise 5.A.2 Probability conservation. Check that $\Delta(x,y)$ satisfies probability conservation: $\int dy\,\Delta(x,y) = 1$.

Exercise 5.A.3 Massless propagators. Show that for a massless particle the continuum propagator is given by

$$\Delta(x,y) = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}\ \frac{1}{|x-y|^{d-2}}$$

Can you derive this by a Euclidean random walk argument?

Exercise 5.A.4 Fermion propagators. Random walk now has a tough constraint: the walks must be non-self-intersecting. Show that by taking into account vacuum fluctuations you get the same propagator as in the bose case.

Exercise 5.A.5 Spinning particles. Construct propagators for (massive) spin-1 and spin 1/2 particles.

Exercise 5.A.6 Derive spin-statistics theorem; probability conservation requires integer spins to be bose particles, half-integer spins to be fermi particles.

Exercise 5.A.7 Pull out the propagator from a different hat (suggestions: stochastic quantization; microcanonical field theory; heat-kernel formulation).

Exercise 5.A.8 Construct propagator for some non-trivial geometry, such as random walk on a non-abelian Lie group manifold.

add exercise on boundary conditions: hint; S^k counts correctly for $k = L$ (paths same for torus and plane, hence corrections of order h^L. By prob conservation $h \leq \frac{1}{2d}$, this drops like ton of rocks