# Georgia Tech PHYS 6124
# Mathematical Methods of Physics I

Instructor: Predrag Cvitanović
Fall semester 2012

## Homework Set #2

due Sept 4 2012

---

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

---

[All problems from Stone and Goldbart, but renumbered for this course]

**Problem 2.1 Higher derivatives**

Construct the Euler equation for the functional

$$J[y] = \frac{1}{2}\int_{x_1}^{x_2} dx \left\{ \left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 + y^2 \right\}.$$

Of what order is the resulting ordinary differential equation? You may assume that the variation of $y$ and its first derivative vanish at the end-points.

**Problem 2.2 Dynamics of fields**

A real field $\Phi(\mathbf{x}, t)$ obeys the variational principle

$$\delta \int d^3x\, dt\, \mathcal{L}(\mathbf{x}, t) = 0.$$

Find the partial differential equation of motion obeyed by $\Phi(\mathbf{x}, t)$ for the following cases:

- a) $\mathcal{L}(\mathbf{x}, t) = \frac{1}{2}\left\{(\partial_t\Phi)^2 - c^2|\nabla\Phi|^2 - \mu^2\Phi^2\right\}$
- b) $\mathcal{L}(\mathbf{x}, t) = \frac{1}{2}\left\{(\partial_t\Phi)^2 - c^2|\nabla\Phi|^2 + 2\mu^2\cos\Phi\right\}$
- c) A real scalar field $\phi(\mathbf{x}, t)$ and a real vector field $\mathbf{A}(\mathbf{x}, t)$ also obey the variational principle stated above, but with

$$\mathcal{L}(\mathbf{x}, t) = \frac{1}{8\pi}\left\{\left| - \nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right|^2 - |\nabla\times\mathbf{A}|^2 - \left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2\right\} + c^{-1}\mathbf{j}\cdot\mathbf{A} - \rho\phi,$$

where $\mathbf{j}(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ are the current and charge densities, and $c$ is the speed of light *in vacuo*. Find the partial differential equations of motion obeyed by $\mathbf{A}(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t)$.

### Problem 2.4 Lagrange multipliers

- a) Find the stationary values of the function $f(x, y) = 13x^2 + 8xy + 7y^2$ on the circle $x^2 + y^2 = 1$.

- b) Identical particles can be distributed amongst $R$ energy levels, having energies $\{\epsilon_r\}_{r=1}^R$. In a given configuration, $n_r$ is the number of particles occupying level $r$. The total number of particles $\sum_{r=1}^R n_r$ and the total energy of the system $\sum_{r=1}^R n_r \epsilon_r$ are fixed, and take the values $N$ and $E$, respectively. Find the distribution $\{\bar{n}_r\}_{r=1}^R$ that maximises the quantity

$$\Gamma \equiv \frac{N!}{n_1! n_2! \ldots n_R!},$$

  subject to the stated constraints.
  [Hint: You may use Stirling's approximation: $n! \approx \exp\{n \ln(n/e)\}$, valid for large $n$.]

- c) Consider the Hermitian form $Q = \sum_{ij} \psi_i^* H_{ij} \psi_j$, where $\{\psi_k\}$ are the $M$ complex-valued components of a vector $\psi$, and $\{H_{kl}\}$ are the complex-valued elements of an Hermitian $M \times M$ matrix $H$. Show that the condition that $Q$ be stationary with respect to variations in $\psi$, subject to the constraint that $\sum_i \psi_i^* \psi_i = 1$ (*i.e.*, that $\psi$ be normalised to unity) leads to the Hermitian eigenproblem $\sum_j H_{ij} \psi_j = E \psi_i$, where the eigenvalue $E$ is the Lagrange multiplier enforcing the normalisation constraint.

## Optional problems

**Problem 2.3 Scalar wave equation with one free end**

The functional $\mathcal{S}$ has, as its argument, a single function $u$ of the two independent variables $x$ and $t$:

$$\mathcal{S}[u] = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left\{\bar{\rho} u_t(x,t)^2 - \bar{\kappa} u_x(x,t)^2\right\},$$

where $\bar{\rho}$ and $\bar{\kappa}$ are constants. Find the additional condition on $u$, beyond the condition that $u$ satisfies the wave equation, that makes $\mathcal{S}$ stationary with respect to variations that vanish at all points at the initial and final times $t_1$ and $t_2$, and at all times at the boundary point $x_1$, *but with no restriction on the variation at the point* $x_2$.
[Note: $u_t(x,t) \equiv \partial u/\partial t$ and $u_x(x,t) \equiv \partial u/\partial x$.]

**Problem 2.5 Curve of fixed length**

Two fixed points, $A$ and $B$, line in the $xy$-plane at the locations $(x, y) = (0, \pm a)$. They are connected by a curve $\Gamma$ of fixed length $L$ $(> 2a)$ lying in the plane. Assume that the radius vector from the origin to any point on $\Gamma$ cuts $\Gamma$ in at most one point so that $\Gamma$ may be described using plane polar coordinates: $r = r(\theta)$.

- a) Express in the form $\delta \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, F(r, \mathrm{d}r/\mathrm{d}\theta, \theta) = 0$ the problem of finding the curve $\Gamma$ for which the area between $\Gamma$ and the chord $AB$ is stationary, and identify an appropriate function $F$.
- b) Construct (but do not attempt to solve) the associated Euler equation.
- c) If one necessarily exists, construct a simpler differential equation satisfied by $r(\theta)$ [*i.e.*, the equation for curve $\Gamma$], and state why it must exist.

[Note: Although you cannot use the following information to address this question, you may wish to know that the sought curve is one of the two arcs of length $L$ of a circle passing through $A$ and $B$.]

**Problem 2.6 Mass distribution for prescribed profile**

You are provided with a light-weight line of length $\pi a/2$ and some lead shot of total mass $M$. Determine how the lead should be distributed along the line if the loaded line is to hang in an arc of a circle of radius $a$ when its ends are attached to two points at the same height.

[Hint: Use plane polar coordinates having their origin at the centre of the circle.]

**Problem 2.7 Flowing river**

A river has parallel straight banks, given by the lines $x = 0$ and $x = b(> 0)$. The velocity $\mathbf{V}$ at which the river water flows is always directed parallel to the banks, but varies with the distance from the banks: $\mathbf{V} = A(x)\,\mathbf{e}_y$, where $A(x)$ is a certain prescribed function. A boat moves at constant speed $C(> |\mathbf{V}|)$ relative to the water, and follows the path $\mathbf{R} = x\,\mathbf{e}_x + Y(x)\,\mathbf{e}_y$. Construct the functional $T[Y]$ that gives the time to cross the river in terms of the path taken [*i.e.*, $Y(x)$ and its derivative(s)], the boat speed $C$, and the river speed $A(x)$. Constructing an Euler equation is *not* required.
[Hint: Observe that the boat velocity relative to the banks has the form $(dx/dt, dy/dt) = (C\sin\alpha, A + C\cos\alpha)$.]

**Problem 2.8 Mechanical equilibrium of a hard ferromagnet**

Let $\mathbf{m}(\mathbf{x})$ be the local value of the magnetisation in a ferromagnet. Suppose that the ferromagnet is *hard*, which means that the *magnitude* of the magnetisation is everywhere equal to unity. Suppose, further, that the free energy of the ferromagnet is given by

$$E = \frac{1}{2}J\int_V d^3x\big(\partial_\mu\, m^a(\mathbf{x})\big)^2 \equiv \frac{1}{2}J\int_V d^3x\sum_{\mu=1}^{3}\sum_{a=1}^{3}\big(\partial_\mu\, m^a(\mathbf{x})\big)^2\,,$$

where $V$ denotes the volume of the sample.

- a)By making a small variation of the magnetisation that vanishes at the boundary of the sample and integrating by parts, and by using a Lagrange multiplier at each position $\mathbf{x}$ to enforce the (non-linear) constraint that $|\mathbf{m}(\mathbf{x})| = 1$, show that the condition for mechanical equilibrium is given by
$$\nabla^2 m^a(\mathbf{x}) - m^a(\mathbf{x})\, m^b(\mathbf{x})\nabla^2 m^b(\mathbf{x}) = 0\,.$$
- b)What do you think is the origin of the non-linearity in this partial differential equation ?
- c)Briefly discuss the number of independent partial differential equations, in the context of the number of dependent variables.

**Problem 2.9 Geodesics in curved spaces**

Suppose that a particle moves along a curve in three dimensions. The time $\delta t$ taken to move from the point $x_i$ $(i = 1, 2, 3)$ to the nearby point $x_i + \delta x_i$

($i = 1, 2, 3$) is given by

$$(\delta t)^2 = \sum_{i,j=1}^{3} g_{ij}(\{x\})\,\delta x_i\,\delta x_j\,.$$

Find the set of coupled ordinary differential equations satisfied by $x_i(s)$ ($i = 1, 2, 3$), the [parametric form of the] path that makes stationary the time taken to move between between two fixed points.

**Problem 2.10 Differential calculus with functionals**

Just as there is a version of Taylor's theorem for functions of several variables, so there is a version for functionals. We can use this version of Taylor's theorem to define *functional derivatives*. Consider the functional $J[y]$. If we make a small shift, $y \to y + \epsilon\eta$, then $J[y] \to J[y + \epsilon\eta]$, where

$$J[y+\epsilon\eta] = J[y] + \epsilon \int dx_1\, J^{(1)}[y; x_1]\,\eta(x_1) + \frac{\epsilon^2}{2!}\int dx_1 dx_2\, J^{(2)}[y; x_1, x_2]\,\eta(x_1)\eta(x_2) + \mathcal{O}(\epsilon^3).$$

We *define* the coefficient of $\epsilon\eta$, *i.e.*, $J^{(1)}[y; x_1]$, to be the first functional derivative of $J$ with respect to $y(x_1)$, which we denote $\delta J/\delta y(x_1)$. Similarly, we define the coefficient of $\epsilon^2\eta^2/2!$, *i.e.*, $J^{(2)}[y; x_1, x_2]$, to be the second functional derivative of $J$ with respect to $y(x_1)$ and $y(x_2)$, which we denote $\delta^2 J/\delta y(x_1)\delta y(x_2)$, *etc* Compute the first and second functional derivatives of the following functionals:

- a) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1, z_2)\, y(z_1)\, y(z_2)$
- b) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1, z_2)\, y(z_1)^2\, y(z_2)^2$
- c) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1, z_2)\, y'(z_1)\, y'(z_2)$, where $y'(z)$ denotes $dy/dz$
- d) $\frac{1}{2}\int dz_1\, dz_2\, \alpha(z_1, z_2)\, y'(z_1)^2\, y'(z_2)^2$

**Problem 2.11 Navier-Stokes equation**

Consider a fluid with density $\rho(\mathbf{x}, t)$ flowing at a velocity $\mathbf{v}(\mathbf{x}, t)$. Then, at least for so-called *simple fluids*, the rate of change of the momentum density is given by the forces acting on a small volume element of fluid, *i.e.*, $\partial(\rho v_i)/\partial t = -\partial_k \Pi_{ik}$, where $\Pi_{ik}$ is the momentum flux density tensor, given by

$$\Pi_{ik} = \rho\, v_i\, v_k + p\,\delta_{ik} - \eta\left\{\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik}\,\partial_j v_j\right\} - \zeta\delta_{ik}\,\partial_j v_j\,,$$

$p$ is the pressure, $\eta$ and $\zeta$ are two (positive) coefficients of viscosity, and summation over repeated indices is implied. Show that this form of $\Pi_{ik}$, together with the continuity equation $\partial\rho/\partial t = -\nabla\cdot(\rho\mathbf{v})$, produces the Navier-Stokes equation of motion for simple fluids:

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\big(\mathbf{v}\cdot\nabla\big)\mathbf{v} = -\nabla p + \eta\nabla^2\mathbf{v} + \big(\zeta + (\eta/3)\big)\nabla\big(\nabla\cdot\mathbf{v}\big)\,.$$

**Problem 2.12 Foucault's pendulum in disguise**

A particle of mass $\mu$ moving in three dimensions is bound to the origin $\mathcal{O}$ by a harmonic spring of spring constant $\kappa$. Let $\mathbf{R}(t)$ denote the position of the particle at time $t$.

- a) Write down the lagrangian that controls the motion of the system.

  Now suppose that the motion is confined to a certain moving plane that passes through $\mathcal{O}$ and has unit normal vector $\mathbf{N}(t)$.

- b) Write down the appropriate equation of constraint, and use it to construct the appropriate new lagrangian, which involves a lagrangian undetermined multiplier $\lambda$.

- c) Construct the classical equation of motion in terms of $\lambda$.

  Now restrict your attention to the situation in which the orientation of the plane varies slowly, compared with the natural frequency $\omega$ ($\equiv \sqrt{\kappa/\mu}$) of the oscillating particle, *i.e.*, $|\dot{\mathbf{N}}(t)| \ll \omega$. In addition, consider only "linearly polarised" motions, *i.e.*, those that pass through $\mathcal{O}$.

- d) By assuming that the only consequence of the motion of the constraint-plane is the slow variation of the direction of oscillation $\mathbf{A}(t)$, ie, that $\mathbf{R}(t) = \mathbf{A}(t)\,\sin(\omega t + \varphi)$, show that the oscillation-direction $\mathbf{A}(t)$ obeys

$$\dot{\mathbf{A}}(t) \approx \Big(\mathbf{N}(t) \times \dot{\mathbf{N}}(t)\Big) \times \mathbf{A}(t).$$

- e) Show that the magnitude of $\mathbf{A}(t)$ does not vary with time.

- f) Show that $\mathbf{A}(t)$ is not integrable, *i.e.*, that $\mathbf{A}(t)$ cannot be written as $\mathbf{A}(t) = \mathbf{f}\big(\mathbf{N}(t)\big)$.

- g) Suppose that $\mathbf{N}(t)$ is slowly driven around a closed path over a time $T$, *i.e.*, $\mathbf{N}(T) = \mathbf{N}(0)$. Find a relationship between $\mathbf{A}(T)\cdot\mathbf{A}(0)/|\mathbf{A}(T)|\,|\mathbf{A}(0)|$ and the area of the unit sphere enclosed by the path $\mathbf{N}(t)$.

- h) By using the equation given in part (d) and your answer to parts (g), explain why your answer to part (g) can be described as *geometric*.

[Hint: See the article entitled *The Quantum Phase, Five Years After*, by M. V. Berry, in *Geometric Phases in Physics*, A. Shapere and F. Wilczek (World Scientific, Singapore, 1989), especially p. 8 *et seq*.]