Georgia Tech PHYS 6124
Mathematical Methods of Physics I

Instructor: Predrag Cvitanović
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Homework Set #6

due October 9, 2012

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem 1) Linear differential operators

a) Consider the differential operator $L = id/dx$.

   i) Find a weight function w so that the operator is self-adjoint, and find the corresponding surface term Q.

   ii) Consider separately the following boundary conditions for the functions on which L acts:
   $[\alpha]\ u(0) = u(2\pi) = 0, \quad [\beta]\ u(0) = u(2\pi), \quad [\gamma]\ u(0) = u'(2\pi) = 0.$

   In each case, construct the adjoint boundary conditions and hence determine whether the given boundary conditions confer hermiticity on L. For which set(s) of boundary conditions would you anticipate that L possesses a complete set of orthogonal eigenfunctions? In such cases, find that complete set and exhibit the orthogonality explicitly.

b) Show that $-d/dx$ is the formal adjoint of the operator d/dx with respect to the weight function $w = 1$. What is the form of the surface term Q?

c) Consider the Sturm-Liouville operator L defined by $Lu = (pu')' - qu$. Show that, provided p and q are real functions, the Sturm-Liouville operator is self-adjoint. What is the form of the surface term Q?

Problem 2) Sturm-Liouville forms

By constructing appropriate weight functions convert the following common operators into Sturm-Liouville form:

a) $L = (1 - x^2)\, d^2/dx^2 + [(\mu - \nu) - (\mu + \nu + 2)x]\, d/dx$

b) $L = (1-x^2)\, d^2/dx^2 - 3x\, d/dx$

c) $L = d^2/dx^2 - 2x(1-x^2)^{-1}\, d/dx - m^2\,(1-x^2)^{-1}$

Problem 4) Properties of hermitian matrices

The complex conjugate of a matrix M is the matrix M^* whose elements are given by $\left(M^*\right)_{jk} = \left(M_{jk}\right)^*$. The transpose M is the matrix M^T whose elements are given by $\left(M^T\right)_{jk} = M_{kj}$. The hermitian conjugate of M is the matrix $M^\dagger$ defined to be $\left(M^T\right)^*$ or equivalently $\left(M^*\right)^T$ whose elements are given by $\left(M^\dagger\right)_{jk} = \left(M_{kj}\right)^*$. M is said to be hermitian if it equals its own hermitian conjugate, *i.e.*, $M^\dagger = M$. Its matrix elements satisfy $\left(M^\dagger\right)_{jk} \equiv \left(M_{kj}\right)^* = M_{jk}$.

Consider the hermitian matrix M.

a) Prove that the eigenvalues of M are real.

b) Prove that the eigenvectors of M having non-degenerate eigenvalues are orthogonal.

c) Prove that if $\mathbf{u}$ and $\mathbf{v}$ are degenerate eigenvectors of M then $\alpha\mathbf{u} + \beta\mathbf{v}$ (in which α and β may be complex) is also a degenerate eigenvector. Hence show that degenerate eigenvectors may be chosen to be orthogonal.

Optional problems

Problem 3) Difference equations

The purpose of this question is to introduce you to some of the properties of certain difference (rather than differential) equations. We shall, for the sake of simplicity, primarily focus here on the linear first-order variety.

a) Solve the difference equation: $a_{n+1} = a_n + q_n$, where $\{q_n\}_{n=1}^{\infty}$ and a_1 are given.

b) Solve the difference equation: $a_{n+1} = p_n a_n$, where $\{p_n\}_{n=1}^{\infty}$ and a_1 are given.

c) By introducing a *summing factor*, the analogue of an integrating factor, solve the general first-order linear inhomogeneous difference equation: $a_{n+1} = p_n a_n + q_n$, where $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and a_1 are given.

d) Solve the difference equation: $a_{n+1} = n\, a_n / (n+1) + n$, in terms of a_1.

e) Solve the nonlinear difference equation: $a_{n+1} = a_n^2$, in terms of a_1.

f) The discrete derivative Da_n of a discrete function a_n is defined to be $Da_n \equiv a_{n+1} - a_n$. Compute the second and third discrete derivatives of a_n, namely $D^2 a_n$ [$\equiv D(Da_n)$] and $D^3 a_n$ [$\equiv D(D(Da_n))$].

g) The discrete antiderivative b_n of a discrete function a_n is defined to be $b_n \equiv \sum_{j=n_0}^{n} a_j$. The integer function that corresponds to the continuous function $f(x) = x^k$ is the discrete function $f_n = n(n+1)\cdots(n+k-1)$, also having k factors. Calculate Df_n.

h) By taking the logarithm, solve the nonlinear difference equation: $a_{n+2} = a_{n+1}^2 / a_n$.

i) By noting that the transcendental functions $\cos x$ and $\cosh x$ satisfy the functional equation $f(2x) = 2f(x)^2 - 1$, solve the nonlinear difference equation: $a_{n+1} = 2a_n^2 - 1$ for the two cases, $|a_1| > 1$ and $|a_1| < 1$.

j) Solve the difference equation with constant coefficients: $a_{n+2} + 3a_{n+1} + 2a_n = 0$.

k) By introducing the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, solve the convolution difference equation: $a_{n+1} = K \sum_{j=0}^{n} a_j a_{n-j}$, with $a_0 = 1$.
[Hint: Find and solve an equation for $f(x)$; expand the solution in a Taylor series; identify a_n from the coefficients.]

Problem 5) One-dimensional motion of the quartic oscillator

Consider a particle of mass m moving in the quartic potential $V(x) = (g/4)x^4$.

a) In terms of the Jacobian elliptic function $\mathrm{cn}(u,k)$, find the trajectory of the particle as a function of time, given the initial position x_0 and the energy E. [Hint: you may use the result that the Jacobian elliptic function $\mathrm{cn}(u,k)$ satisfies the nonlinear ordinary differential equation $\left(d\mathrm{cn}/du\right)^2 = \left(1-\mathrm{cn}^2\right)\left(k'^2 + k^2\mathrm{cn}^2\right)$, in which $k^2 + k'^2 = 1$; see, *e.g.*, Gradshteyn and Ryzhik, *Table of Integrals, Series and Products*, §8.159.2.

b) Sketch $\mathrm{cn}(u,k)$ as a function of u for $k^2 = 1/2$. Note the qualitative similarity with the familiar cosine function. (This similarity becomes an identity when $k = 0$.)
[Hint: See, *e.g.*, Abramowitz and Stegun, *Handbook of Mathematical Functions*, §16, fig. 1; note that this handbook uses the notation $m \equiv k^2$.]

c) Determine the period of the motion in terms of E and the complete elliptic integral of the first kind $K(k)$.

[Hint: See, *e.g.*, Abramowitz and Stegun, §16.1.1 and fig. 1.]

Problem 9) Lyapunov equation

Consider the following system of ordinary differential equations,

$$\frac{d}{dt}Q(t) = A(t)\,Q(t) + Q(t)\,A^{\mathrm{T}}(t) + \Delta(t)\,,$$

in which Q, A and Δ are $[N\times N]$ matrix functions of t, with A and Δ known, and Q sought. The superscript T indicates the transpose of the matrix. Find the solution $Q(t)$, subject to the boundary condition $Q(t_0) = Q_0$, by taking the following steps:

i) Write the solution in the form $Q(t) = J(t)\,Q_0\,J^{\mathrm{T}}(t) + J(t)\,W(t)\,J^{\mathrm{T}}(t)$, with $J(t)$ satisfying

$$\frac{d}{dt}J(t) = A(t)\,J(t)\,,$$

subject to the initial condition $J(t_0) = \mathbf{I}$, in which $\mathbf{I}$ is the $[N\times N]$ identity matrix.

ii) Show that $W(t)$ then satisfies $\frac{d}{dt}W(t) = J^{-1}(t)\,\Delta(t)\,\left(J^{\mathrm{T}}(t)\right)^{-1}$, subject to the initial condition $W(t_0) = \mathbf{O}$.

iii) Integrate the preceding equation to obtain

$$Q(t) = J(t)\, Q_0\, J^{\mathrm{T}}(t) + \int_{t_0}^{t} d\tau\, J(t)\, J^{-1}(\tau)\, \Delta(\tau)\, \left(J^{\mathrm{T}}(\tau)\right)^{-1} J^{\mathrm{T}}(t)\,,$$

in terms of (the as yet unknown) matrix $J(t)$.

iv) $J(t)$ can be determined, usually by numerical integration, as $J(t) = \hat{T} \exp\left\{ \int_{t_0}^{t} d\tau\, A(\tau) \right\}$, where $\hat{T}$ denotes the 'time-ordering' operation.

v) Show that if $A(\tau)$ commutes with itself throughout the interval $t_0 \leq \tau \leq t$ (never happens in real life) then the 'time-ordering' operation is redundant, and we have the explicit solution $J(t, t_0) = \exp\left\{ \int_{t_0}^{t} d\tau\, A(\tau) \right\}$. Show that, in this case, the complete solution reduces to

$$Q(t) = J(t)\ Q_0\ J^{\mathrm{T}}(t) + \int_{t_0}^{t} dt'\, J(t, t')\, \Delta(t')\, J(t, t')^{\mathrm{T}}\,.$$