Georgia Tech PHYS 6124
Mathematical Methods of Physics I

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Fall semester 2012

Homework Set #6a

due Thursday, October 25, 2012

Notes for lectures 14 and 15: Calculus on smooth manifolds

[based on Deirdre Shoemaker PHYS 479 notes and J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*]

Metric and line element describe spacetime such as the flat spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \text{ and } \eta_{\alpha\beta} = diag(-1, 1, 1) \tag{1}$$

BUT, line elements and metrics are written in terms of a coordinate system, such that different line elements/metrics may describe the same geometry. Examples:

- Transformed line elements under a coordinate transformation are the same as flat spacetime (*i.e.* Cartesian and polar-spherical)
- The static, weak-field gravitational line element cannot be transformed to flat spacetime under any coordinate transformation; it is described by curved spacetime.

The choice of coordinates is arbitrary as long as the uniquely label each point. For example: start with flat spacetime (t, x, y, dz), do the following coordinate transformation, and find a new line element for flat spacetime

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

where

$$\begin{aligned}
dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}\,d\phi \\
&= \sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi \\
dy &= \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}\,d\phi \\
&= \sin\theta\sin\phi\,dr + \cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi \\
dz &= \frac{\partial z}{\partial r}\,dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}\,d\phi = \cos\theta\,dr - r\sin\theta\,d\theta
\end{aligned}$$

Doing the same for dy and dz leads to

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

This line element might not "look" flat but it is since it can be transformed back to the original flat spacetime under a coordinate transformation.

The Good and The Bad

- Good Coordinates uniquely label each point in spacetime, but many coordinates systems fail somewhere. Example:
  Polar coordinates (r, θ, ϕ): the point $\theta = 0$ labels more than one point, different ϕ and r values exist at $\theta = 0$. This is a (mild) coordinate singularity.

- Bad Coordinates exhibit coordinate singularities. Example:

  $$dS^2 = dr^2 + r^2 d\phi^2$$

  is the line element for a 2d plane in polar coordinates. Under a coordinate transform of form $r = a^2/r'$

  $$dS^2 = \frac{a^4}{r'^4}(dr'^2 + r'^2 d\phi^2)$$

  blows up at $r' = 0$ but still describes a flat plane! This is another example of a bad coordinate singularity. There are not many physical singularities, but we one in black holes.

Two coordinate systems in general are labeled by α, β and α', β'. How does dx^α transform?

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^{\alpha'}} dx^{\alpha'}$$

Examples were the dx, dy, and dz that we just did. But what about an arbitrary metric tensor $g_{\alpha\beta}$?

$$g_{\alpha'\beta'} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}}$$

You can see why this is true by looking at the line element

$$ds^2 = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} dx^{\alpha'} \frac{\partial x^\beta}{\partial x^{\beta'}} dx^{\beta'} = g_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} \ .$$

$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ is a general line element and metric where $g_{\alpha\beta} \equiv g_{\alpha\beta}(x)$ is the symmetric matrix

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

called the *metric*. The flat spacetime metric is one example, $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ and $\eta_{\alpha\beta}$ in Cartesian coordinates is

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The metric is a [4×4] matrix with 10 independent components because it is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$.

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{00} dx^0 dx^0 + g_{01} dx^0 dx^1 + g_{02} dx^0 dx^2 + g_{03} dx^0 dx^3 \\ &+ g_{10} dx^1 dx^0 + g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{13} dx^1 dx^3 \\ &+ g_{20} dx^2 dx^0 + g_{21} dx^2 dx^1 + g_{22} dx^2 dx^2 + g_{23} dx^2 dx^3 \\ &+ g_{30} dx^3 dx^0 + g_{31} dx^3 dx^1 + g_{32} dx^3 dx^2 + g_{33} dx^3 dx^3 \end{aligned} \tag{2}$$

but $g_{\alpha\beta}$ is symmetric

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{00} dx^0 dx^0 + 2g_{01} dx^0 dx^1 + 2g_{02} dx^0 dx^2 + 2g_{03} dx^0 dx^3 \\ &+ g_{11} dx^1 dx^1 + 2g_{12} dx^1 dx^2 + 2g_{13} dx^1 dx^3 \\ &+ g_{22} dx^2 dx^2 + 2g_{23} dx^2 dx^3 \\ &+ g_{33} dx^3 dx^3 \end{aligned} \tag{3}$$

If it were also a diagonal metric (*i.e.* like flat spacetime), it would have only four components,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{00} dx^0 dx^0 + g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3 .$$

If you want to find the form of a metric in a new coordinate system, you do not need to go through the line element, we can transform the metric directly.

Index rules:

1. Free indices: $g_{\alpha\beta} = g_{\beta\alpha}$ represents 16 equations, there are free indices, not summed over - they can only be summed on the same side of the equation. $g_{\delta\gamma} = g_{\gamma\delta}$ is also fine and represents the same set of 16 equations.

2. Repeated indices: must be superscripts/subscript pairs. For example $g_{\alpha\alpha}$ is incorrect but g^α_α or $g_{\alpha\beta}\, dx^\alpha dx^\beta$ are fine.

3. Location of indices important: $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ the subscripts and superscripts are summed over. A superscript in denominator acts like a subscript when balancing indices in an equation.

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^{\beta'}}\, dx^{\beta'}$$

Problem 1) Balancing indices

Which equations make sense?

$$g_{\alpha\beta}dx^{\alpha}dx^{\beta} = g_{\alpha\beta}dx^{\alpha}dx^{\sigma} \tag{4}$$

$$g_{\alpha\beta}a^{\alpha}b^{\beta} = g_{\alpha\beta}a^{\alpha}c^{\beta} \tag{5}$$

$$\Gamma^{\alpha}_{\alpha\beta} = \Gamma^{\beta}_{\beta\beta} \tag{6}$$

$$\Gamma^{\alpha}_{\alpha\gamma}a^{\gamma} = g_{\alpha\beta}a^{\alpha}b^{\beta} \tag{7}$$

Problem 2) Flat spacetime metric

Given the flat spacetime metric in Cartesian coordinates, show that the metric in coordinates

$$t = t',\ x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

is

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Notes: Covariant derivative

Consider a function $f(x^{\alpha})$ and a curve $x^{\alpha}(\sigma)$ parameterized by curvilinear distance σ. The directional derivative along that curve is

$$\frac{df}{d\sigma} = lim_{\epsilon\to 0}\left[\frac{f(x^{\alpha}(\sigma+\epsilon)) - f(x^{\alpha}(\sigma))}{\epsilon}\right] = \frac{dx^{\alpha}}{d\sigma}\frac{\partial f}{\partial x^{\alpha}}$$

where

$$t^{\alpha} = \frac{dx^{\alpha}}{d\sigma}$$

is the *tangent vector* to the curve. Thus the *directional derivative* is thus

$$\frac{d}{d\sigma} = t^{\alpha}\frac{\partial}{\partial x^{\alpha}}.$$

So we can write any vector as a directional derivative as

$$\mathbf{a} \equiv a^{\alpha}\frac{\partial}{\partial x^{\alpha}}$$

A *dual vector* (covector) ω_α is a linear map from vectors to real numbers,

$$\omega(\mathbf{a}) = \omega_\alpha a^\alpha .$$

Vectors and dual vectors are related by "lowering" or "raising" of indices,

$$a_\alpha = g_{\alpha\beta} a^\beta , \qquad a^\alpha = g^{\alpha\beta} a_\beta .$$

The relationship between the metric and its inverse is encoded in the Kronecker delta

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta = \begin{cases} 1 , & \alpha = \beta \\ 0 , & \alpha \neq \beta . \end{cases} \tag{8}$$

We can express the scalar product between two vectors **a** and **b** as

$$\mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta = a_\alpha b^\beta = a^\alpha b_\beta = g^{\alpha\beta} a_\alpha b_\beta .$$

The gradient of a scalar function $f(x)$ is an example of a dual vector

$$\nabla_\alpha f = \frac{\partial f}{\partial x^\alpha} .$$

For example, the components of the gradient of the square of the Minkowski 'distance' $d(x)^2 = -t^2 + z^2 + y^2 + z^2$ along x^α are:

$$\frac{\partial}{\partial x^\alpha} d^2 = 2\,(-t, x, y, z) = 2\, x_\alpha .$$

Next, differentiate a vector $v^\beta(x)$. We expect a "second-rank" tensor, $\nabla_\alpha v^\beta$.

Trouble: The definition involves differences between vectors at nearby curved manifold points. How do we define the difference between the two vectors $\mathbf{v}(x^\alpha)$ and $\mathbf{v}(x^\alpha + dx^\alpha)$ defined in curved space? The two "nearby" points are separated by $dx^\alpha = t^\alpha \epsilon$, where the vector **t** indicates the direction of displacement. To construct the derivative, $\mathbf{v}(x^\alpha + dx^\alpha)$ is first transported back to x^α, to the vector we denote $\mathbf{v}_{||}(x^\alpha)$. This is called *parallel transport* and is a key notion for defining derivatives of vectors and tensors.

This leads us to the definition of covariant derivative

$$\nabla_t \mathbf{v}(x^\alpha) = lim_{\epsilon \to 0} \frac{[\mathbf{v}(x^\alpha + t^\alpha \epsilon)]_{||\text{trans to } x^\alpha} - \mathbf{v}(x^\alpha)}{\epsilon} .$$

Here the subscript $_t$ indicates that the direction of the derivative is along t. The derivative of a vector necessarily involves the difference between vectors at two different points which means in two different tangent spaces. To define a derivative of a vector, we must transport vectors from one tangent space to another. We will see first how to do so in flat space.

To construct the derivative, the vector $\mathbf{v}(x^\alpha + t^\alpha \epsilon)$ is first transported parallel to itself to the point x^α and now called $\mathbf{v}_{||}(x^\alpha)$. Now it lies in the tangent space of x^α, and $\mathbf{v}(x^\alpha)$ can be subtracted from it. Calculating the covariant derivative in Cartesian coordinates in flat space is straightforward because the components, v^α do not change as they are parallel transported in these coordinates.

$$\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta}\,, \qquad \text{flat space}$$

But even in flat space, if coordinates are curvilinear, this formula does not hold because the angles the vector makes with the basis vectors change. The problem is that we do not know how to implement the parallel transport in general. It should look like

$$v^\alpha_{||}(x) = v^\alpha(x+\epsilon t) + \tilde{\Gamma}^\alpha_{\beta\gamma}(x) v^\gamma(x)(\epsilon t^\beta)\,,$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ is an array of coefficients to be determined. Taking the components of our definition of the covariant derivative

$$\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} v^\gamma\,.$$

The first term comes from the change in the vector field as we go from x^α to $x^\alpha + dx^\alpha$. The second term is the change in the basis vectors. Taken together, the covariant derivative is basis independent. If we always know the local inertial frame, we could find a general coordinate transformation between our coordinates and it, but that is not always so straightforward. In General Relativity we have another tool, the geodesic equation. We know that in a local inertial frame, a geodesic is a straight line whose tangent vector is propagated parallel to itself. If we let $\mathbf{u}$ be the tangent vector, then its covariant derivative is its own direction has to be zero

$$(\nabla_{\mathbf{u}}\mathbf{u})^\alpha = u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} u^\gamma\right) = 0\,.$$

The geodesic equation can be written as

$$u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma\right) = 0\,.$$

where we have expressed the geodesic equation in a coordinate basis. Thus $\tilde{\Gamma}$'s are just the Christoffel symbols defined in a coordinate basis. The covariant derivative in a coordinate basis

$$\nabla_\alpha v^\beta = \frac{\partial v^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} v^\gamma\,. \tag{9}$$

The covariant derivative of a scalar function is just the partial derivative,

$$\nabla_\alpha f \equiv \frac{\partial f}{\partial x^\alpha}$$

We can use Leibnitz' rule to extend the covariant derivative from vectors to other tensors.

$$\nabla_\gamma (v^\alpha w^\beta) = v^\alpha (\nabla_\gamma w^\beta) + (\nabla_\gamma v^\alpha) w^\beta .$$

Combining this with our definition of the covariant derivative we get

$$\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^{\delta\beta} + \Gamma^\alpha_{\gamma\delta} t^{\alpha\delta} .$$

Here's the rule: differentiate the components and add terms with Γ's for each index (subtract them for subindices).

Problem 3) Covariant derivative

Show that the geodesic equation in terms of the covariant derivative is

$$\nabla_{\mathbf{u}} \mathbf{u} = 0 , \tag{10}$$

that

$$\nabla_\gamma \, g_{\alpha\beta} = 0 . \tag{11}$$

and the dual quantities transform as

$$\nabla_\alpha v_\beta = \frac{\partial v_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma . \tag{12}$$

Notes: Variational principle for free test particle motion

The worldline of a free test particle between two timelike separated points extremizes the proper time between them. In flat space: A particle obeying Newton's or Einstein's law of motion follows a path of extremal action.

- Geodesics: extremal world lines
- Geodesic equation: equation of motion

	Variational Principle	Equation of Motion
flat spacetime	$\delta \int (-\eta_{\alpha\beta} dx^\alpha dx^\beta)^{1/2} = 0$	$\frac{d^2 x^\alpha}{d\tau^2} = 0$
curved spacetime	$\delta \int (-g_{\alpha\beta} dx^\alpha dx^\beta)^{1/2} = 0$	$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$

The procedure for finding the equations for timelike geodesics in spacetime starts with the proper time between two points

$$\tau_{AB} = \int_A^B dt = \int_A^B \left[-g_{\alpha\beta} dx^\alpha dx^\beta \right]^{1/2} \tag{13}$$

The worldline of a timelike geodesic is parameterized by 4-coordinates $x^\alpha(\sigma)$ where σ varies from 0 to 1 at the endpoints. The proper time is then written as

$$\tau_{AB} = \int_0^1 d\sigma \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2} \tag{14}$$

The worldlines that extremize the proper time between A and B are those that satisfy the Lagrange's equation

$$-\frac{d}{d\sigma} \left(\frac{\partial L}{\partial (dx^\alpha / d\sigma)} \right) + \frac{\partial L}{\partial x^\alpha} = 0 \tag{15}$$

for the Lagrangian

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha \right) = \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2} . \tag{16}$$

Problem 4) Equations for geodesic of the plane in polar coordinates

$$dS^2 = dr^2 + r^2 d\phi^2$$

Find the extremum of the action to get the curve in this 2d space (it would be an equation of motion in a 4d space), show that it satisfies

$$\frac{d^2 r}{dS^2} = r \left(\frac{d\phi}{dS} \right)^2 , \qquad \frac{d}{dS} \left(r^2 \frac{d\phi}{dS} \right) = 0 . \tag{17}$$

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Optional problems

Problem 5) Equations for geodesics in wormhole geometry

Line element for the geometry of a wormhole

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin\theta^2 d\phi^2) .$$

Show that the Lagrange's equations give the following equations of motion:

$$\begin{aligned} \frac{d^2 t}{d\tau^2} &= 0 \\ \frac{d^2 r}{d\tau^2} &= r \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2\theta \left(\frac{d\phi}{d\tau} \right)^2 \right] \\ \frac{d}{d\tau} \left[(b^2 + r^2) \frac{d\theta}{d\tau} \right] &= (b^2 + r^2) \sin\theta \cos\theta \left(\frac{d\phi}{d\tau} \right)^2 \\ \frac{d}{d\tau} \left[(b^2 + r^2) \sin^2\theta \frac{d\phi}{d\tau} \right] &= 0 \end{aligned} \tag{18}$$

Notes: Equations for geodesics

This example motivates the form of the equation for geodesics in arbitrary curved spacetimes

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\,. \tag{19}$$

This represents four equations - one for each value of the free index α. The coefficients $\Gamma^\alpha_{\beta\gamma}$ are called Christoffel symbols and are constructed from the metric and its first derivatives. Taken together, these four equations are called the *geodesic equation*. This equation is the basic equation for motion of test particles in curved spacetime.

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma \tag{20}$$

is the geodesic equation for *timelike geodesics*. The Christoffel symbols are symmetric in the lower two indices

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$$

They are written out as

$$g_{\alpha\beta}\Gamma^\beta_{\delta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\delta} - \frac{\partial g_{\delta\gamma}}{\partial x^\alpha}\right)\,. \tag{21}$$

Notes: Using the geodesic equation to derive equations of motion

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

- Other way of writing the Christoffel symbols in terms of the *inverse metric*, $g^{\alpha\beta}$.
$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\epsilon}\left(\frac{\partial g_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial g_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\epsilon}\right)$$

- $g^{\alpha\beta}$ is the inverse of the metric - operationally, this is just the inverse of a 4x4 matrix. For a diagonal metric:

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}\,.$$

The inverse, $g^{\alpha\beta}$ is given by

$$g^{\alpha\beta} = \begin{pmatrix} g^{00} & 0 & 0 & 0 \\ 0 & g^{11} & 0 & 0 \\ 0 & 0 & g^{22} & 0 \\ 0 & 0 & 0 & g^{33} \end{pmatrix}.$$

where $g^{00} = \frac{1}{g_{00}}$ and likewise.

- Example 8.1 Again, Modified - Use Geodesic Equations to find the straight lines in polar coordinates

$$dS^2 = dr^2 + r^2 d\phi^2$$

1. Compute Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\epsilon} \left(\frac{\partial g_{\epsilon\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}} \right)$$

where α and β are running over 1 and 2 such that $x^{\alpha} = (r, \phi)$.

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

and

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

In terms of components we have $g_{rr} = g^{rr} = 1$, $g_{\phi\phi} = r^2$, $g^{\phi\phi} = r^{-2}$ and $g_{r\phi} = g^{r\phi} = 0$. So the Christoffel symbols are

$$\begin{aligned}
\Gamma^{r}_{\phi\phi} &= \frac{1}{2} g^{r\alpha} (g_{\alpha\phi,\phi} + g_{\alpha\phi,\phi} - g_{\phi\phi,\alpha}) \\
&= \frac{1}{2} g^{rr} (g_{r\phi.\phi} + g_{r\phi,\phi} - g_{\phi\phi,r}) + \frac{1}{2} g^{r\phi} (g_{\phi\phi.\phi} + g_{\phi\phi,\phi} - g_{\phi\phi,\phi}) \\
&= \frac{1}{2} (0 + 0 - \frac{\partial r^2}{\partial r}) \\
&= -r \\
\Gamma^{\phi}_{r\phi} &= \frac{1}{2} g^{\phi\alpha} (g_{\alpha\phi,r} + g_{\alpha\phi,r} - g_{\phi r,\alpha}) \\
&= \frac{1}{2} g^{\phi\phi} (g_{\phi\phi,r} + g_{r\phi\phi,r} - g_{r\phi,\phi}) \\
&= \frac{1}{2} g_{\phi\phi,r} \\
&= \frac{1}{2} r^{-2} (2r) = 1/r \\
\Gamma^{\phi}_{\phi r} &= \Gamma^{\phi}_{r\phi} = 1/r. \qquad (22)
\end{aligned}$$

All other Christoffel symbols are 0.

2. Now plug the Christoffel symbols into the geodesic equation for spacelike geodesics

$$\frac{d^2x^\alpha}{ds^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} \tag{23}$$

We have two equations, one for r and one for ϕ.

$$\begin{aligned}
\frac{d^2x^r}{ds^2} &= -\Gamma^r_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = -\Gamma^r_{\phi\phi}\left(\frac{d\phi}{ds}\right)^2 \\
\frac{d^2r}{ds^2} &= r\left(\frac{d\phi}{ds}\right)^2 \\
\frac{d^2x^\phi}{ds^2} &= -\Gamma^\phi_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = -2\Gamma^\phi_{r\phi}\frac{dr}{ds}\frac{d\phi}{ds} \\
\frac{d^2\phi}{ds^2} &= -\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds}
\end{aligned} \tag{24}$$

- The geodesic equation is invariant under coordinate transformations.
- Consider the Special Relativity application of the equations, $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $g_{\alpha\beta,\gamma} = 0$ which means that all the Christoffel symbols are 0. This means that the geodesic equation in SR

$$\frac{d^2x^\alpha}{d\tau^2} = 0$$

Uniform motion extremizes proper time in Special Relativity (free-falling motion does so in GR).