Georgia Tech PHYS 6124

Mathematical Methods of Physics I

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Homework Set #7

due October 30 2012

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem 1) Motion of a classical particle
Consider a classical particle of unit mass moving along the x-axis. Suppose that the motion is free, except that at time $\tau = t$, with $0 < t < T$, the particle receives an impulse of unit strength.

a) Write Newton's equation describing the motion of the particle.

   Suppose that at time $\tau = 0$ the particle is located at position x_1 and that at time $\tau = T$ it is located at position x_2.

b) Sketch the position, velocity and acceleration of the particle as a function of time for $0 < \tau < T$.

c) Compute the Green function for the motion of the particle, *i.e.*, solve $d^2G(\tau, t)/d\tau^2 = \delta(\tau - t)$.

d) Consider the applied force $f(\tau)$ (with $0 \leq \tau \leq T$) to be a sequence of impulses. Hence establish the trajectory of the particle in terms of an integral over the applied force.

e) Suppose that the force takes the form $f(\tau) = \tau^2/2$. Find the motion of the particle.

f) (**optional**) Rather than solve for the Green function directly, as you did in part (c), construct the Green function using the eigenfunction expansion technique. Show, by Fourier analysis, that the two schemes for computing the Green function give equivalent results.

Problem 4) Green functions
Consider the second order inhomogeneous linear differential equation: $y'' - y = f(x)$.

a) Suppose that the boundary conditions for y are $y(\pm\infty) = 0$. By solving the associated homogeneous equation, construct the appropriate Green function for this equation.

b) Solve the differential equation when the source term is given by $f(x) = 2\,\mathrm{e}^{-|x|}$.

Problem 6) Green functions and their inverses
The purpose of this question is to show that if $\mathcal{G}$ is the Green function associated with some operator $\mathcal{L}$, then $\mathcal{L}$ is the Green function associated with the operator $\mathcal{G}$.

Consider the operator $\mathcal{L} \equiv (-d^2/d\theta^2 + q^2/2\pi)$, in which the real variable q does not vanish. The functions $u(\theta)$ on which $\mathcal{L}$ acts are chosen to be periodic functions on the interval $0 \leq \theta < 2\pi$, *i.e.*, $u(2\pi) = u(0)$ and $u'(2\pi) = u'(0)$.

a) By using the eigenfunction expansion, show that the Green function associated with the operator $\mathcal{L}$ is given by

$$\mathcal{G}(\theta, \theta') = \frac{1}{2\pi q^2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n(\theta - \theta')}{q^2 + n^2}\,.$$

Now show that $\mathcal{L}$ is the Green function for $\mathcal{G}$ in the following sense. Consider the inhomogeneous integral equation

$$\int_0^{2\pi} d\theta'\, \mathcal{G}(\theta, \theta')\, y(\theta') = f(\theta)$$

for the unknown function $y(\theta)$, in which the source term $f(\theta)$ is presumed known.

(b) Show that the solution $y(\theta)$ is given by

$$y(\theta) = (\mathcal{L}f)(\theta) \equiv (-d^2/d\theta^2 + q^2/2\pi) f(\theta)\,.$$

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Optional problems

Problem 2) Variation of parameters
Use the method of variation of parameters to find the most general solution to the second order linear inhomogeneous differential equation: $x^2 y'' - 4xy' + 6y = x^4$.

Problem 3) General solution to an inhomogeneous equation
Suppose $y = x$, $y = x^2$ and $y = x^3$ each satisfy the second-order inhomogeneous equation $Ly = f(x)$. Find the general solution.

Problem 5) More Green functions
Find the Green function for these operators:

i) $Ly \equiv (xy')'$;

ii) $Ly \equiv (xy')' - n^2 y/x$;

iii) $Ly \equiv x^2 y'' + xy' + (k^2 x^2 - 1)y$.

Problem 7) Eigenfunction expansion for the bowed string Green function
In class we studied the Green function for the response at the driving frequency $\omega(\equiv ck)$ of a bowed stretched string of length ℓ, by solving directly the equation

$$\Big(\frac{d^2}{dx^2} + k^2\Big) G(x, x') = \delta(x - x'),$$

subject to the boundary conditions $G(x, x')|_{x=0,\ell} = 0$. Use the eigenfunction expansion to produce an alternative derivation of this Green function, and demonstrate that the version thus obtained is equivalent to the version

$$G(x, x') = \frac{\sin(kx_<) \sin k(x_> - \ell)}{k \sin(k\ell)},$$

obtained in class, in which $x_< \equiv \min(x, x')$ and $x_> \equiv \max(x, x')$.

Problem 8) Dirac delta function
In this question we shall explore some of the properties of the Dirac delta function (the generalisation to the continuum of the Kronecker delta symbol). The Dirac delta function, $\delta(x)$, has as its argument the real variable, x. It has the following rather striking properties:(i) $\delta(x) = 0$ for $x \neq 0$; and $\int_{-\infty}^{\infty} dx\, \delta(x) = 1$. From these properties you can see that $\delta(x)$ has an infinitely high spike at the origin, is zero elsewhere, and has unit area.

a) Give a heuristic (*i.e.* sloppy) proof that if $f(x)$ is a sufficiently smooth function then $\int_{-\infty}^{\infty} dx\, \delta(x)\, f(x) = f(0)$.

b) Show that $\int_{-\infty}^{\infty} dx\, \delta(x - a)\, f(x) = f(a)$. Often we shall neglect to write the limits on integrals.

c) Using integration by parts, show that $\int_{-\infty}^{\infty} dx\, \delta'(x)\, f(x) = -f'(0)$, where $f'(x) \equiv df(x)/dx$ and $\delta'(x) \equiv d\delta(x)/dx$.

d) Consider the family of gaussian functions,

$$\Delta_\xi(x) \equiv \frac{1}{\sqrt{2\pi\xi^2}} \exp\Big\{ -\frac{x^2}{2\xi^2} \Big\},$$

parametrised by the width ξ. Show that the defining properties of $\delta(x)$ are satisfied by $\lim_{\xi \to 0} \Delta_\xi(x)$ and, hence, that this limit represents $\delta(x)$. Hint: recall that $\int_{-\infty}^{\infty} dy\, \exp(-y^2/2) = \sqrt{2\pi}$.

e) By considering the Fourier transform,

$$\tilde{f}(q) = \int_{-\infty}^{\infty} dx\, f(x)\, \frac{e^{-iqx}}{\sqrt{2\pi}},$$

and its inverse,

$$f(x) = \int_{-\infty}^{\infty} dq\, \tilde{f}(q)\, \frac{e^{iqx}}{\sqrt{2\pi}},$$

establish that

$$f(y) = \int_{-\infty}^{\infty} dx\, f(x) \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iq(y-x)}.$$

Thus, demonstrate that

$$\delta(y) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqy}.$$

This integral representation of the Dirac delta function is known as the Fourier representation of the Dirac delta function.

f) By considering the Fourier representation, show that $\delta(ax) = |a|^{-1}\,\delta(x)$.

g) Prove this result, starting with the gaussian representation of part (d).

We can also have higher dimensional delta functions, by which we mean delta functions with vector arguments. For example, if $\vec{r}$ is a three-dimensional cartesian vector with cartesian components x, y and z, then the delta function, $\delta^{(3)}(\vec{r})$, is defined to be the product $\delta^{(3)}(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$. Often the superscript (3) is omitted.

h) Write down the Fourier representation of $\delta^{(3)}(\vec{r})$ in terms of the vector $\vec{r}$ and an integral over the vector $\vec{q}$.

i) Write down a representation for $\delta^{(3)}(\vec{r})$ in terms of the gaussian function.

j) A real function of a single variable, $f(x)$, has zeros at the set of points $\{x_i\}$. At these zeros, the gradient of f is non-vanishing, and takes the values $\{f_i^{(1)}\}$. Show that

$$\delta\big(f(x)\big) = \sum_i \frac{\delta(x - x_i)}{|f_i^{(1)}|}.$$

k) A real symmetric 3 matrix $\mathcal{A}$ is non-singular (*i.e.* all its eigenvalues are non-zero real numbers). Prove, using either the gaussian or the Fourier representation, that

$$\delta^{(3)}(\mathcal{A}\vec{x}) = \frac{1}{|\det\mathcal{A}|}\,\delta^{(3)}(\vec{x}).$$

A third representation of the delta function is very convenient in the context of complex variables. Consider the real variables ω and ϵ. If Im denotes the imaginary part, use the following strategy to show that

$$\lim_{\epsilon\to 0} \frac{1}{\pi} \operatorname{Im} \frac{1}{\omega - i\epsilon} = \delta(\omega).$$

l.i) Show that

$$\operatorname{Im} \frac{1}{\omega - i\epsilon} = \frac{\epsilon}{\omega^2 + \epsilon^2}.$$

l.ii) Show that

$$\lim_{\omega\to 0} \lim_{\epsilon\to 0} \frac{\epsilon}{\omega^2 + \epsilon^2} = 0, \qquad \lim_{\epsilon\to 0} \lim_{\omega\to 0} \frac{\epsilon}{\omega^2 + \epsilon^2} = \infty.$$

l.iii) Using the substitution $\omega = \epsilon \tan\theta$, or otherwise, show that

$$\int_{-\infty}^{\infty} d\omega \, \frac{\epsilon}{\omega^2 + \epsilon^2} = \pi.$$

l.iv) Put together these pieces to argue that, indeed,

$$\lim_{\epsilon\to 0} \frac{1}{\pi} \operatorname{Im} \frac{1}{\omega - i\epsilon} = \delta(\omega)$$

is a representation of the delta function.