Georgia Tech PHYS 6124 — Fall 2012

Mathematical Methods of Physics I

Instructor: Predrag Cvitanović

Homework #9 — due Tuesday November 6 2012

(there was no homework #8)

== show all your work for maximum credit,
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem 3) More holomorphic mappings, Needham, pp. 211-213

(a) **(optional)** Use the Cauchy-Riemann conditions to verify that the mapping $z \mapsto \bar{z}$ is not holomorphic.

(b) The mapping $z \mapsto z^3$ acts on an infinitesimal shape and the image is examined. It is found that the shape has been rotated by π, and its linear dimensions expanded by 12. Determine the possibilities for the original location of the shape?

(c) Consider the map $z \mapsto \bar{z}^2/z$. Determine the geometric effect of this mapping. By considering the effect of the mapping on two small arrows emanating from a typical point z, one arrow parallel and one perpendicular to z, show that the map fails to produce an *amplitwist*.

(d) The interior of a simple closed curve $\mathcal{C}$ is mapped by a holomorphic mapping into the exterior of the image of $\mathcal{C}$. If z travels around the curve counterclockwise, which way does the image of z travel around the image of $\mathcal{C}$?

(e) Consider the mapping produced by the function $f(x+iy) = \left(x^2+y^2\right) + i(y/x)$.

   (i) Find and sketch the curves that are mapped by f into horizontal and vertical lines. Notice that f appears to be conformal.

   (ii) Now show that f is *not* in fact a conformal mapping by considering the images of a pair of lines (*e.g.*, one vertical and one horizontal).

   (iii) By using the Cauchy-Riemann conditions confirm that f is not conformal.

   (iv) Show that no choice of $v(x,y)$ makes $f(x+iy) = \left(x^2+y^2\right) + iv(x,y)$ holomorphic.

(f) **(optional)** Show that if f is holomorphic on some connected region then each of the following conditions forces f to reduce to a constant:
(i) $\mathrm{Re}\, f(z) = 0$; (ii) $|f(z)| = \text{const.}$; (iii) $\bar{f}(z)$ is holomorphic too.

(g) **(optional)** Suppose that the holomorphic mapping $z \mapsto f(z)$ is expressed in terms of the modulus R and argument Φ of f, *i.e.*,
$f(z) = R(x,y) \exp i\Phi(x,y)$.

Determine the form of the Cauchy-Riemann conditions in terms of R and Φ.

(h) (i) By sketching the image of an infinitesimal rectangle under a holomorphic mapping, determine the the local magnification factor for the area and compare it with that for a infinitesimal line. Re-derive this result by examining the Jacobian determinant for the transformation.

(ii) Verify that the mapping $z \mapsto \exp z$ satisfies the Cauchy-Riemann conditions, and compute $\big(\exp z\big)'$.

(iii) **(optional)** Let S be the square region given by $A - B \leq \operatorname{Re} z \leq A + B$ and $-B \leq \operatorname{Im} z \leq B$ with A and B positive. Sketch a typical S for which $B < A$ and sketch the image $\tilde{S}$ of S under the mapping $z \mapsto \exp z$.

(iv) **(optional)** Deduce the ratio $(\text{area of } \tilde{S})/(\text{area of } S)$, and compute its limit as $B \to 0^+$.

(v) **(optional)** Compare this limit with the one you would expect from part (i).

Problem 4) Yet more holomorphic mappings, Needham, pp. 258, 264

(a) (i) Show that if $f = u + iv$ is a holomorphic mapping then $(\nabla u) \cdot (\nabla v) = 0$, where ∇ is the two-dimensional gradient operator of ordinary vector calculus. Explain the geometrical content of this result.

(ii) Show that both the real and the imaginary parts of a holomorphic function are *harmonic* (*i.e.*, they both satisfy Laplace's equation).

(iii) **(optional)** Show that each of the following functions is harmonic: $e^x \cos y$; $e^{x^2-y^2} \cos 2xy$; and $\ln |f(z)|$, where $f(z)$ is holomorphic.

(b) Let the position at time t of a particle moving in the complex plane be $z(t) = r(t) \exp i\theta(t)$.

(i) **(optional)** Compute the radial and transverse components of the acceleration of the particle.

(ii) **(optional)** Deduce that if the particle is moving in a central force field centered at the origin then the areal speed $r^2\dot{\theta}/2$ is constant.

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Optional problems

Problem 1) Holomorphic functions, Ahlfors,p. 28

(a) **(optional)** If $g(w)$ and $f(z)$ are holomorphic, show that $h(z) \equiv g\big(f(z)\big)$ is, too.

(b) **(optional)** Verify the Cauchy-Riemann conditions for the holomorphic functions z^2 and z^3.

(c) Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$ (with a, b, c and d real). Determine the conjugate harmonic function by integration of the Cauchy-Riemann conditions.

(d) **(optional)** Show that the functions $\bar{f}(z)$ and $f(\bar{z})$ are simultaneously holomorphic.

(e) **(optional)** Show that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

Problem 2) Partial fractions, Ahlfors, p. 33

(a) If Q is a polynomial with distinct roots $\alpha_1, \ldots, \alpha_N$ and P is a polynomial of degree less than N, show that the rational function $P(z)/Q(z)$ has the partial fraction development

$$\sum_{n=1}^{N} \frac{P(\alpha_n)}{(z - \alpha_n)\, Q'(\alpha_n)}.$$

(b) Develop in partial fractions the functions $z^4 / \left(z^3 - 1\right)$ and $1/z(z+1)^2(z+2)^3$.

(c) **(optional)** What is the general form of a rational function that has modulus unity on the circle $|z| = 1$? How are the zeros and poles related to one another? If a rational function is real on $|z| = 1$, how are the zeros and poles situated?