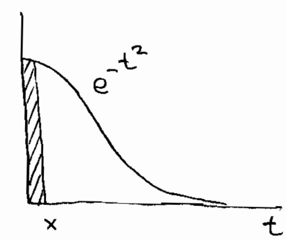
Asymptotic Evaluation of Integrals

Asymptotic expansion: generate a series which appoximates the exact result for small / large parameter

Example: $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dx$$

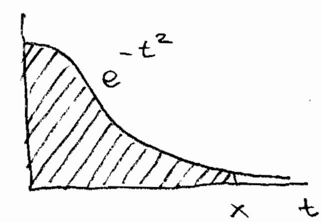


$\boxed{\text{Small } x}$ Use taylor series:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right) dt =$$

$$= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots\right), \quad x \ll 1$$

$\boxed{\text{Large } x}$ Use integration by parts:



$$1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

$$\int_x^\infty e^{-t^2} dt = \int_x^\infty \frac{-1}{2t} d\left(e^{-t^2}\right) = -\frac{e^{-t^2}}{2t}\Bigg|_x^\infty - \int_x^\infty \frac{e^{-t^2}}{2t^2} dt =$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \int_x^\infty \frac{3}{4} \frac{e^{-t^2}}{t^4} dt = \dots$$

$$\mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2} \left(\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + \dots\right), \quad x \gg 1$$

Note: This series does not converge for any x, because the general term eventually diverges:

$$(-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n x^{2n-1}} = -4x\left(-\frac{1}{4x^2}\right)^n \frac{(2n-3)!}{(n-2)!} \to \infty, \quad n \to \infty$$

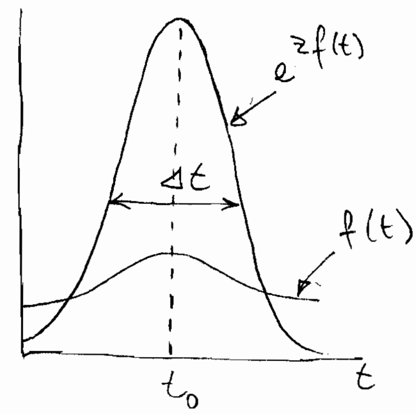
Asymptotic series: $S_n(x)$ is an asymptotic series expansion of $f(x)$ $(S_n(x) \sim f(x))$, if $S_n(x) \to f(x)$, $x \to \infty$ for finite n.

Saddle-Point (Steepest Descent) Method

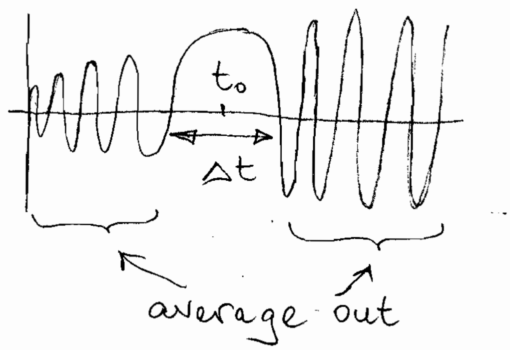
$$I(z) = \int_C g(t)\, e^{z f(t)}\, dt \quad , \quad z - \text{real} , \; z \to \infty$$

<u>$f(t)$ – real</u> $\Rightarrow$ Laplace method

Dominant contribution: $g(t_0)\, e^{z f(t_0)} \Delta t$



<u>$f(t)$ – imaginary</u> $\Rightarrow$ Stationary phase method



Dominant contribution:

$$g(t_0)\, e^{iz\varphi_0} \Delta t,$$

$$\varphi_0 = \operatorname{Im} f(t_0)$$

What happens when $f(t)$ is <u>neither</u> real <u>nor</u> imaginary?

Assume $f(t)$ is <u>analytic</u>:

$\nabla^2 \operatorname{Re} f = 0 \quad \Rightarrow \quad \operatorname{Re} f$ has no maxima (or minima)

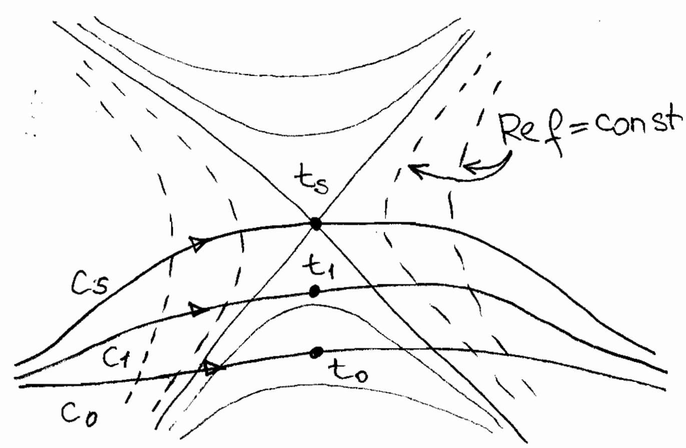
$\frac{df}{dt} = f' = 0$ – saddle for both $\operatorname{Re} f$ and $\operatorname{Im} f$.

Dominant contribution:

$$I_i \approx g(t_i)\, e^{z \operatorname{Re} f(t_i)} \int_{C_i} e^{iz \operatorname{Im} f(t)}\, dt$$

$e^{z \operatorname{Re} f(t_i)}$: largest for C_0, smallest for C_s

$\int_{C_i} e^{iz \operatorname{Im} f(t)}\, dt$: oscillates fast for C_0, almost constant for C_s



In order to remove dependence on oscillations choose path C with $\mathrm{Im}\, f(t) = \mathrm{Im}\, f(t_s) = \mathrm{const}$ (constant phase)

$C \parallel \mathrm{Im}\, f = \mathrm{const} \Rightarrow$ ← Cauchy-Riemann condition

$C \perp \mathrm{Re}\, f = \mathrm{const} \Rightarrow$

$C \parallel \vec{\nabla} \mathrm{Re}\, f \Rightarrow$

steepest descent (ascent)

Near t_s:

$$zf(t) = zf(t_s) + \tfrac{1}{2} z (t-t_s)^2 f''(t_s) + \tfrac{1}{3!} z (t-t_s)^3 f'''(t_s) + \dots$$

$$g(t) = g(t_s) + (t-t_s) g'(t_s) + \dots$$

Change integration variable ($t \to \tau$; $\Delta\tau = O(1)$):

$$(t-t_s) \equiv z^{-1/2} \tau$$

$$\Rightarrow \begin{cases} zf(t) = zf(t_s) + \tfrac{1}{2}\tau^2 f''(t_s) + O(z^{-1/2}) \\ g(t) = g(t_s) + O(z^{-1/2}) \end{cases}$$

$$I = \int_{\text{saddle } t_s} g(t) e^{zf(t)} dt = \int_{'-\infty'}^{'\infty'} e^{zf(t_s)} e^{\frac{1}{2}\tau^2 \overbrace{f''(t_s)}^{<0}} g(t_s)(1 + O(z^{-1/2})) \frac{d\tau}{z^{1/2}} =$$

highest saddle!

$$= \left[\frac{2\pi}{-zf''(t_s)}\right]^{1/2} e^{zf(t_s)} g(t_s) (1 + O(z^{-1/2}))$$

Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad , x > 0 \qquad (\text{Laplace transform of } t^{x-1})$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

$$\Gamma(x) = \int_0^\infty e^{-t} \frac{1}{x} d(t^x) = \frac{1}{x} t^x e^{-t} \Big|_0^\infty + \frac{1}{x}\int_0^\infty t^x e^{-t} dt = \frac{1}{x}\Gamma(x+1)$$

$$\Rightarrow \Gamma(x+1) = x\Gamma(x)$$

For $x = n$ - integer: $\Gamma(n+1) = n\Gamma(n) = \dots = n \cdot (n-1) \dots 1 = n!$

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty x^{-1} e^{-x^2}\, 2x\,dx = 2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$$

Stirling's Formula

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \int_0^\infty e^{x\ln t - t}\,dt$$

$$\begin{cases} f(t) = x\ln t - t \\ g(t) = 1 \end{cases} \Rightarrow \begin{cases} f' = \frac{x}{t} - 1 = 0 \Rightarrow t_s = x \\ f'' = -\frac{x}{t_s^2} = -\frac{1}{x} < 0 \quad \text{for } x>0. \end{cases}$$

$$\Gamma(x+1) \approx \left(\frac{2\pi}{-(-1/x)}\right)^{1/2} e^{x\ln x - x} = \sqrt{2\pi x}\; x^x e^{-x}, \quad x\to\infty$$

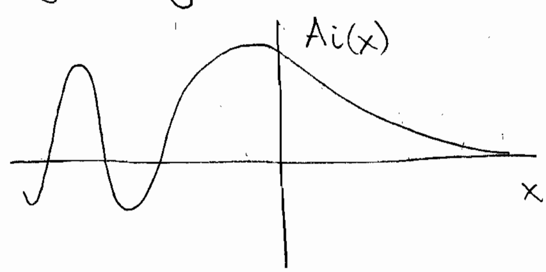
$$n! \approx \sqrt{2\pi n}\; n^n e^{-n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n\to\infty$$

Airy Function

$Ai(x)$ – solution of $y'' - xy = 0$

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega z + \omega^3/3)}\,d\omega$$

$$= \frac{1}{2\pi i}\int_C e^{tz - t^3/3}\,dt \qquad t = i\omega$$



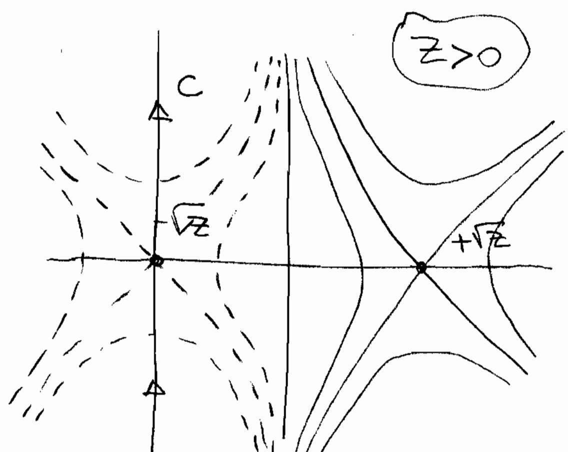
$$f(t) = tz - \frac{t^3}{3} \Rightarrow f'(t) = z - t^2 = 0 \Rightarrow t_s = \pm\sqrt{z}$$

$\underline{z>0}$: $t_s = -z^{1/2}$

$$f(t_s) = -z^{3/2} + \frac{1}{3}z^{3/2} = -\frac{2}{3}z^{3/2}$$

$$f''(t_s) = -2t = 2z^{1/2}$$

$$f'''(t_s) = -2$$



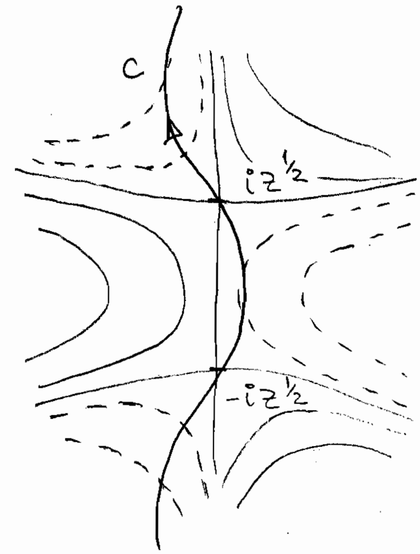
Change variable: $t = z^{-\frac{1}{2}} + iz^{-\frac{1}{4}}\tau$

$$Ai(z) = \frac{1}{2\pi}\int_{'-\infty'}^{'\infty'} \exp\left(-\tfrac{2}{3}z^{3/2} - \tau^2 - \tfrac{i}{3}z^{-3/4}\tau^3\right) z^{-1/4}\,d\tau$$

$$= \frac{\exp(-\frac{2}{3}z^{3/2})}{z^{1/4}\,2\pi}\int_{'-\infty'}^{'\infty'} e^{-\tau^2}\left(1 - \cancel{\tfrac{i}{3}z^{-3/4}\tau^3} - \tfrac{1}{18}z^{-3/2}\tau^6 + \dots\right)d\tau =$$

odd

$$Ai(z) = \frac{1}{z^{1/4}\, 2\pi^{1/2}} e^{-\frac{2}{3}z^{3/2}} \left(1 - \frac{5}{48} z^{-3/2} + \cdots\right), \quad z \to +\infty$$



$\underline{z<0:}$ $\quad t_s = \pm i(-z)^{1/2}$

$$f(t_s) = \pm i \tfrac{2}{3}(-z)^{3/2} \quad (\mathrm{Re}\, f(t_s) = 0)$$

$$f''(t_s) = \mp 2i(-z)^{1/2}$$

$$f'''(t_s) = -2$$

Have to go through <u>both</u> saddles.

For each saddle get $\dfrac{1}{(\pm i(-z)^{1/2})^{1/2}\, 2\pi^{1/2}} e^{\pm \frac{2i}{3}(-z)^{3/2}} + \cdots$

Combining: $\quad Ai(z) = \dfrac{1}{(-z)^{1/4}\pi^{1/2}} \sin\left(\tfrac{2}{3}(-z)^{3/2} + \tfrac{\pi}{4}\right), \quad z \to -\infty$