## mathematical methods - week 14

# Probability and statistics

### Georgia Tech PHYS-6124

**Homework HW #14** due Tuesday, November 25, 2014

---

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

---

| | |
|---|---|
| Exercise 14.1 *Who ordered* $\sqrt{\pi}$ *?* | 1 point |
| Exercise 14.2 $d$*-dimensional Gaussian integrals* | 3 points |
| Exercise 14.3 *Errors add up as sums of squares* | 6 points |

**Bonus points**

| | |
|---|---|
| Exercise 14.4 *(b)* $d$*-dimensional Fresnel integral* ((a) done in exercise 5.2) | 4 points |
| Exercise 14.5 *Lyapunov equation* | 12 points |

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

**2014-11-18 Predrag** Lecture 26. Probability and statistics

- Predrag's summary of key concepts for a physicist: ChaosBook Sect. 17.1.3 Moments, cumulants.

**2014-11-20 Predrag** Lecture 27. Probability and statistics

- For binomial theorem, Poisson and Gaussian distributions, see Arfken and Weber [14.1] *Mathematical Methods for Physicists: A Comprehensive Guide*, Chapter 19.
- For a survey of astro-particle experimental methods, study Taboada's lecture notes.
- Integral of a general $d$-dimensional or 'multi-variate' Gaussian, exercise 14.2, was worked out in the class. To see how this integral looks like in quantum theories, work out exercise 14.4. For how independent, uncorrelated errors add up in higher dimensions, work out exercise 14.3.

## 14.1 Literature

**2014-09-11 Predrag** I had been asked to discuss probability in the course, so I have solicited advice from my experimental colleagues. You tell me how to cover this in less than two semesters:)

**2012-09-24 Predrag** Ignacio Taboada writes:

One of my students this year asked me specifically to cover least squares. To me, this is the absolute most basic thing you need to know about data fitting - and usually I use more advanced methods.

For a few things that particle and astroparticle people do often for hypothesis testing, read Li and Ma [14.2], *Analysis methods for results in gamma-ray astronomy*, and Feldman and Cousins [14.3] *Unified approach to the classical statistical analysis of small signals*. Both of these two papers are too complex to cover in this course, but the idea of hypothesis testing can be studied in simpler cases.

**2012-09-24 Predrag** Peter Dimon's thoughts on how to teach math methods needed by experimentlists:

1. Probability theory
   (a) Inference
   (b) random walks
   (c) Conditional probability
   (d) Bayes rule (another look at diffusion)
   (e) Machlup has a classic paper on analysing simple on-off random spectrum. Hand out to students. (no Baysians use of information that you do not have) (Peter takes a dim view)

2. Fourier transforms
3. power spectrum - Wiener-Kitchen for correlation function
   (a) works for stationary system
   (b) useless on drifting system (tail can be due to drift only)
   (c) must check whether the data is stationary
4. measure: power spectrum, work in Fourier space
   (a) do this always in the lab
5. power spectra for processes: Brownian motion,
   (a) Langevin $\rightarrow$ get Lorenzian
   (b) connect to diffusion equation
6. they need to know:
   (a) need to know contour integral to get from Langevin power spectrum, to the correlation function
7. why is power spectrum Lorenzian - look at the tail $1/\omega^2$
   (a) because the cusp at small times that gives the tails
   (b) flat spectrum at origin gives long time lack of correlation
8. position is not stationary
   (a) diffusion
9. Green's function
   (a) $\delta$ fct $\rightarrow$ Gaussian + additivity
10. Nayquist theorem
    (a) sampling up to a Nayquist theorem (easy to prove)
11. Other processes:
    (a) what signal you expect for a given process
12. Fluctuation-dissipation theorem
    (a) connection to response function (lots of them measure that)
    (b) for Brownian motion power spectrum related to imaginary part of response function
13. Use *Numerical Recipes* (stupid on correlation functions)
    (a) zillion filters (murky subject)
    (b) Kalman (?)
14. (last 3 lecturs)
    (a) how to make a measurement
    (b) finite time sampling rates (be intelligent about it)

PS: Did I suggest all that? I thought I mentioned, like, three things.

Did you do the diffusion equation? It's an easy example for PDEs, Green's function, etc. And it has an unphysically infinite speed of information, so you can add a wave term to make it finite. This is called the Telegraph Equation (it was originally used to describe damping in transmission lines).

What about Navier-Stokes? There is a really cool exact solution (stationary) in two-dimensions called Jeffery-Hamel flow that involves elliptic functions and has a symmetry-breaking. (It's outlined in Landau and Lifshitz, *Fluid Dynamics*).

**2012-09-24 Predrag** from Mike Schatz:

1. 1D bare minimum:
   (a) temporal signal, time series analysis
   (b) discrete Fourier transform, FFT in 1 and 2D - exercises
   (c) make finite set periodic
2. Image processing:
   (a) Fourier transforms, correlations,
   (b) convolution, particle tracking
3. PDEs in 2D (Matlab): will give it to Predrag

# References

[14.1] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists: A Comprehensive Guide*, 6 ed. (Academic Press, New York, 2005).

[14.2] T.-P. Li and Y.-Q. Ma, Analysis methods for results in gamma-ray astronomy, Astrophys. J. **272**, 317 (1983).

[14.3] G. J. Feldman and R. D. Cousins, Unified approach to the classical statistical analysis of small signals, Phys. Rev. D **57**, 3873 (1998).

[14.4] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, New York, 1986).

[14.5] H. Rome, A direct solution to the linear variance equation of a time-invariant linear system, IEEE Transactions on Automatic Control **14**, 592 (1969).

[14.6] Z. Gajić and M. Qureshi, *Lyapunov Matrix Equation in System Stability and Control* (Academic Press, New York, 1995).

# Exercises

14.1. **Who ordered $\sqrt{\pi}$ ?** Derive the Gaussian integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2a}} = \sqrt{a}\,, \qquad a > 0\,.$$

assuming only that you know to integrate the exponential function $e^{-x}$. Hint, hint: $x^2$ is a radius-squared of something. $\pi$ is related to the area or circumference of something.

14.2. **$d$-dimensional Gaussian integrals.** Show that the Gaussian integral in $d$-dimensions is given by

$$\frac{1}{(2\pi)^{d/2}}\int d^d x e^{-\frac{1}{2}x^\top\cdot M^{-1}\cdot x + x\cdot J} \quad = \quad |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2}J^\top\cdot M\cdot J}\,,$$

where $M$ is a real positive definite $[d\times d]$ matrix, i.e., a matrix with strictly positive eigenvalues, $x$ and $J$ are $d$-dimensional vectors, and $x^\top$ is the transpose of $x$.

This integral you will see over and over in stat mech and quantum field theory: it's called 'free field theory', 'Gaussian model', 'Wick expansion' etc.. This is the starting, 'propagator' term in any perturbation expansion.

Here we require that the real symmetric matrix $M$ in the exponent is strictly positive definite, otherwise the integral is infinite. Negative eigenvalues can be accommodated by taking a contour in the complex plane [14.4], see exercise 14.4 *Fresnel integral*. Zero eigenvalues require stationary phase approximations that go beyond the Gaussian saddle point approximation, typically to the Airy-function type stationary points, see exercise 6.2 *Airy function for large arguments*.

14.3. **Errors add up as sums of squares.** Show that the convolution

$$[f*g](x) = \int d^d y\, f(x-y)g(y)$$

of (normalized) Gaussians

$$\begin{aligned} f(x) &= \frac{1}{\det(2\pi\,\Delta_1)^{1/2}}\, e^{-\frac{1}{2}x^\top\cdot\frac{1}{\Delta_1}\cdot x} \\ g(x) &= \frac{1}{\det(2\pi\,\Delta_2)^{1/2}}\, e^{-\frac{1}{2}x^\top\cdot\frac{1}{\Delta_2}\cdot x} \end{aligned}$$

is given by

$$[f*g](x) \quad = \quad \frac{1}{(2\pi)^{d/2}}\,\frac{1}{|\det(\Delta_1+\Delta_2)|^{1/2}}\, e^{-\frac{1}{2}x^\top\cdot(\Delta_1+\Delta_2)^{-1}\cdot x}\,.$$

In other words, covariances $\Delta_j$ add up. This is the $d$-dimenional statement of the familiar fact that cumulative error squared is the sum of squares of individual errors. When individual errors are small, and you are adding up a sequence of them in time, you get Brownian motion. If the individual errors are small and added independently to a solution of a determinist equation, you get Langevin and Fokker-Planck equations.

(A hint: Fourier transform of a convolution is the product of Fourier transforms. Or complete a square in the exponent.)

14.4. **Fresnel integral.**

(a) Derive the Fresnel integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2ia}} = \sqrt{ia} = |a|^{1/2} e^{i\frac{\pi}{4}\frac{a}{|a|}}\,.$$

Consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, $R$ and $R\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty}\int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from $R$ to $R\exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty}\left(\int_0^R \exp\left(ix^2\right) dx - \mathrm{e}^{i\pi/4}\int_0^R \exp\left(-r^2\right) dr\right)$$

vanishes, and, from the real integration $\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2$, deduce that

$$\int_0^\infty \exp\left(ix^2\right) dx = \mathrm{e}^{i\pi/4}\sqrt{\pi}/2\,.$$

Now rescale $x$ by real number $a \neq 0$, and complete the derivation of the Fresnel integral.

(b) In exercise 14.2 the exponent in the $d$-dimensional Gaussian integrals was real, so the real symmetric matrix $M$ in the exponent had to be strictly positive definite. However, in quantum physics one often has to evaluate the $d$-dimenional Fresnel integral

$$\frac{1}{(2\pi)^{d/2}}\int d^d\phi\, e^{-\frac{1}{2i}\phi^\top\cdot M^{-1}\cdot\phi + i\,\phi\cdot J}\,,$$

with a Hermitian matrix $M$. Evaluate it. What are conditions on its spectrum in order that the integral be well defined?

14.5. **Lyapunov equation.** Consider the following system of ordinary differential equations,

$$\dot{Q} = A\,Q + Q\,A^\top + \Delta\,, \tag{14.1}$$

in which $\{Q, A, \Delta\} = \{Q(t), A(t), \Delta(t)\}$ are $[d\times d]$ matrix functions of time $t$ through their dependence on a deterministic trajectory, $A(t) = A(x(t))$, etc., with stability matrix $A$ and noise covariance matrix $\Delta$ given, and density covariance matrix $Q$ sought. The superscript $(\;)^\top$ indicates the transpose of the matrix. Find the solution $Q(t)$, by taking the following steps:

(a) Write the solution in the form $Q(t) = J(t)[Q(0) + W(t)]J^\top(t)$, with Jacobian matrix $J(t)$ satisfying

$$\dot{J}(t) = A(t)\,J(t)\,, \qquad J(0) = \mathbf{1}\,, \tag{14.2}$$

with $\mathbf{1}$ the $[d\times d]$ identity matrix. The Jacobian matrix at time $t$

$$J(t) = \hat{T} e^{\int_0^t d\tau\, A(\tau)}\,, \tag{14.3}$$

where $\hat{T}$ denotes the ‘time-ordering’ operation, can be evaluated by integrating (14.2).

(b) Show that $W(t)$ satisfies

$$\dot{W} = \frac{1}{J}\,\Delta\,\frac{1}{J^\top}\,, \qquad W(0) = 0\,. \tag{14.4}$$

(c) Integrate (14.1) to obtain

$$Q(t) = J(t)\left[Q(0) + \int_0^t d\tau\,\frac{1}{J(\tau)}\,\Delta(\tau)\,\frac{1}{J^\top(\tau)}\right]J^\top(t)\,. \tag{14.5}$$

(d) Show that if $A(t)$ commutes with itself throughout the interval $0 \leq \tau \leq t$ then the time-ordering operation is redundant, and we have the explicit solution $J(t) = \exp\left\{\int_0^t d\tau\,A(\tau)\right\}$. Show that in this case the solution reduces to

$$Q(t) = J(t)\;Q(0)\,J(t)^\top + \int_0^t d\tau'\,e^{\int_{\tau'}^t d\tau\,A(t)}\,\Delta(\tau')\,e^{\int_{\tau'}^t d\tau\,A^\top(t)}\,. \tag{14.6}$$

(e) It is hard to imagine a time dependent $A(t) = A(x(t))$ that would be commuting. However, in the neighborhood of an equilibrium point $x^*$ one can approximate the stability matrix with its time-independent linearization, $A = A(x^*)$. Show that in that case (14.3) reduces to

$$J(t) = e^{t\,A}\,,$$

and (14.6) to what?