mathematical methods - week 4

Complex variables

Georgia Tech PHYS-6124

Homework HW #4 due Tuesday, September 16, 2014

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 4.1 *Unique solutions – a matrix caricature*	6 points
Exercise 4.2 *More holomorphic mappings*	10 (+6 bonus) points

Bonus points

Exercise 4.3 *Two constrained particles*	6 points

Total of 16 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

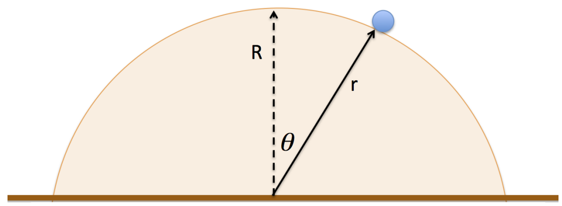

Figure 4.1: Bead rolls off frictionless hemisphere.

2014-09-09 Predrag Lecture 7

As a preparation for Professor Kennedy's classical mechanics lecture on holonomic constraints we will motivate and describe the rank-nullity theorem, Lagrangian constraints, and the Lagrangian multipliers.

Kimberly will work through the classic example of a bead rolling off a frictionless hemisphere, figure 4.1. If one delays inserting the constraint $R = r$ until after the Lagrangian equations of motion are found, the constraining force Q—in this case, Q is equal to the normal force—can be determined. From this, we see that the virtual displacements δq_i (*à la* d'Alembert) are linearly dependent. This result is general:

$$\sum_i \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i \right] \delta q_i = 0\,.$$

In order to get a solvable system, i.e., where the number of unknowns equals the number of [nonredundant] equations, we have to introduce constraint forces, Q_i. Further, if the dimension of configuration space is n (equal to the usual number of degrees of freedom of unconstrained systems), and N is the number of generalized coordinates (which is equal to the number of [nontrivial] Lagrange equations of motion), then it must be the case that

$$N + k = n\,,$$

where k is the number of constraints in the system. This relates to the rank-nullity theorem in the following way: Let the system of equations, including the constraint equations, be represented as an $[m \times n]$ matrix A. $\mathrm{Rank}(A)$ equals the number of generalized coordinates, and $\mathrm{Nul}(A) = k$ is the number of constraints. Then

$$\mathrm{Rank}(A) + \mathrm{Nul}(A) = n\,.$$

The rank-nullity theorem is compactly stated in Stone and P. Goldbart [4.1], *Mathematics for Physics: A Guided Tour for Graduate Students*, Appendix A, and pictorially explained in Thoo [4.2], *A picture is worth a thousand words*, fetch it here. A simple discussion of Lagrange multipliers is taken from Dayan and Abbott [4.3] *Theoretical Neuroscience*, Appendix A.2.

2014-09-11 Predrag Lecture 8

Complex variables: Cauchy-Riemann relations; ...

Arfken and Weber [4.4] Chapter 6 on complex numbers.

References

[4.1] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge, 2009).

[4.2] J. B. Thoo, A picture is worth a thousand words, College Math. J. **29**, 408 (1998).

[4.3] P. Dayan and L. F. Abbott, *Theoretical Neuroscience* (MIT Press, Cambridge, MA, 2001).

[4.4] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists: A Comprehensive Guide*, 6 ed. (Academic Press, New York, 2005).

Exercises

4.1. **Unique solutions – a matrix caricature.** The purpose of this problem is to provide a matrix caricature of the notion in differential equations that to obtain a unique solution one must search for functions from the appropriate collection. (In other words, one must choose appropriate boundary conditions for differential equations.)

Consider the following simultaneous equations for the three unknown vector components x, y and z:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\alpha \\ \beta + \gamma \\ \beta - \gamma \end{pmatrix} ,$$

where we have, without loss of generality, parametrized the arbitrary components of the *source* vector (or *inhomogeneous*) term in terms of the three independent real variables α, β and γ.

a) For arbitrary α, β and γ, can a solution (x, y, z) be found? Explain why?

b) For arbitrary α and β, but $\gamma = 0$, explain if a solution (x, y, z) be found? Is it unique?

c) For arbitrary α and β, but $\gamma = 0$, can a solution (x, y, z) be found from the restricted collection of vectors which are orthogonal to $(0, 1, -1)$? If so, is it unique?

4.2. **More holomorphic mappings.** **Needham**, pp. 211-213

(a) **(optional)** Use the Cauchy-Riemann conditions to verify that the mapping $z \mapsto \bar{z}$ is not holomorphic.

(b) The mapping $z \mapsto z^3$ acts on an infinitesimal shape and the image is examined. It is found that the shape has been rotated by π, and its linear dimensions expanded by 12. Determine the possibilities for the original location of the shape?

(c) Consider the map $z \mapsto \bar{z}^2/z$. Determine the geometric effect of this mapping. By considering the effect of the mapping on two small arrows emanating from a typical point z, one arrow parallel and one perpendicular to z, show that the map fails to produce an *amplitwist*.

(d) The interior of a simple closed curve $\mathcal{C}$ is mapped by a holomorphic mapping into the exterior of the image of $\mathcal{C}$. If z travels around the curve counterclockwise, which way does the image of z travel around the image of $\mathcal{C}$?

(e) Consider the mapping produced by the function $f(x+iy) = \big(x^2+y^2\big) + i\big(y/x\big)$.

  (i) Find and sketch the curves that are mapped by f into horizontal and vertical lines. Notice that f appears to be conformal.

  (ii) Now show that f is *not* in fact a conformal mapping by considering the images of a pair of lines (e.g. , one vertical and one horizontal).

  (iii) By using the Cauchy-Riemann conditions confirm that f is not conformal.

  (iv) Show that no choice of $v(x, y)$ makes $f(x+iy) = \big(x^2+y^2\big) + iv(x, y)$ holomorphic.

(f) **(optional)** Show that if f is holomorphic on some connected region then each of the following conditions forces f to reduce to a constant:
(i) $\operatorname{Re} f(z) = 0$; (ii) $|f(z)| = \text{const.}$; (iii) $\bar{f}(z)$ is holomorphic too.

(g) **(optional)** Suppose that the holomorphic mapping $z \mapsto f(z)$ is expressed in terms of the modulus R and argument Φ of f, i.e.,
$f(z) = R(x, y) \exp i\Phi(x, y)$.
Determine the form of the Cauchy-Riemann conditions in terms of R and Φ.

(h) (i) By sketching the image of an infinitesimal rectangle under a holomorphic mapping, determine the the local magnification factor for the area and compare it with that for a infinitesimal line. Re-derive this result by examining the Jacobian determinant for the transformation.

  (ii) Verify that the mapping $z \mapsto \exp z$ satisfies the Cauchy-Riemann conditions, and compute $\big(\exp z\big)'$.

  (iii) **(optional)** Let S be the square region given by $A - B \leq \operatorname{Re} z \leq A + B$ and $-B \leq \operatorname{Im} z \leq B$ with A and B positive. Sketch a typical S for which $B < A$ and sketch the image $\tilde{S}$ of S under the mapping $z \mapsto \exp z$.

  (iv) **(optional)** Deduce the ratio $(\text{area of } \tilde{S})/(\text{area of } S)$, and compute its limit as $B \to 0^+$.

  (v) **(optional)** Compare this limit with the one you would expect from part (i).

4.3. **Two constrained particles.** Consider two particles of masses m_1 and m_2. Let m_1 be confined to move on a circle of radius a in the $z = 0$ plane, centered at $x = y = 0$. Let m_2 be confined to move on a circle of radius b in the $z = c$ plane centered at $x = y = 0$. A light (massless) spring of spring constant k is attached between the particles.

(a) Find the Lagrangian of the system

(b) Solve the problem using the Lagrange multipliers and give physical interpretation for each multiplier.

Note: Here you are supposed to use coordinates x_1, y_1 for the first particle and x_2, y_2 for the second one. These coordinates are not independent, thus Lagrange multipliers are necessary to write down the equations.