mathematical methods - week 10

Finite groups

Georgia Tech PHYS-6124

Homework HW #10 due Monday, October 28, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 10.1 *1-dimensional representation of anything* 1 point
Exercise 10.2 *2-dimensional representation of* S_3 4 points
Exercise 10.3 D_3*: symmetries of an equilateral triangle* 5 points

Bonus points
Exercise 10.4 (a), (b) and (c) *Permutation of three objects* 2 points
Exercise 10.5 *3-dimensional representations of* D_3 3 points

Total of 10 points = 100 % score.

edited October 23, 2019

Week 10 syllabus

Monday, October 21, 2019

> I have given up Twitter in exchange for Tacitus & Thucydides, for Newton & Euclid; & I find myself much the happier.
>
> — Thomas Jefferson to John Adams, 21 January 1812

Mon Groups, permutations, $D_3 \cong C_{3v} \cong S_3$ symmetries of equilateral triangle, rearrangement theorem, subgroups, cosets.

- Chapter 1 *Basic Mathematical Background: Introduction* Dresselhaus *et al.* [3] (click here)
- by Socratica: a delightful introduction to group multiplication (or Cayley) tables.
- ChaosBook.org Chapter 10. Flips, slides and turns
- For deeper insights, read Roger Penrose [7] (click here).

Wed Irreps, unitary reps and Schur's Lemma.

- Chapter 2 *Representation Theory and Basic Theorems* Dresselhaus *et al.* [3], up to and including Sect. 2.4 *The Unitarity of Representations* (click here)

Fri "Wonderful Orthogonality Theorem."

- Dresselhaus *et al.* [3] Sects. 2.5 and 2.6 *Schur's Lemma*. a first go at sect. 2.7

Optional reading

- There is no need to learn all these "Greek" words.
- Bedside crocheting.

10.1 Group presentations

Group multiplication (or Cayley) tables, such as Table 10.1, *define* each distinct discrete group, but they can be hard to digest. A Cayley graph, with links labeled by generators, and the vertices corresponding to the group elements, has the same information as the group multiplication table, but is often a more insightful presentation of the group.

For example, the Cayley graph figure 10.1 is a clear presentation of the dihedral group D_4 of order 8,

$$D_4 = (e, a, a^2, a^3, b, ba, ba^2, ba^3)\,, \quad \text{generators } a^4 = e\,,\ b^2 = e\,. \tag{10.1}$$

Quaternion group is also of order 8, but with a distinct multiplication table / Cayley graph, see figure 10.2. For more of such, see, for example, mathoverflow Cayley graph discussion.

D_3	e	C	C^2	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$
e	e	C	C^2	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$
C	C	C^2	e	$\sigma^{(3)}$	$\sigma^{(1)}$	$\sigma^{(2)}$
C^2	C^2	e	C	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(1)}$
$\sigma^{(1)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$	e	C	C^2
$\sigma^{(2)}$	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(1)}$	C^2	e	C
$\sigma^{(3)}$	$\sigma^{(3)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	C	C^2	e

Table 10.1: The D_3 group multiplication table.

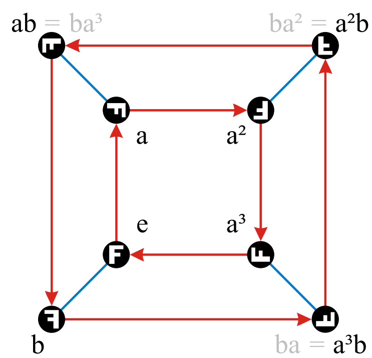
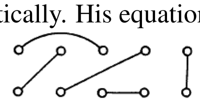


Figure 10.1: A Cayley graph presentation of the dihedral group D_4. The 'root vertex' of the graph, marked e, is here indicated by the letter $\mathbb{F}$, the links are multiplications by two generators: a cyclic rotation by left-multiplication by element a (directed red link), and the flip by b (undirected blue link). The vertices are the 8 possible orientations of the transformed letter $\mathbb{F}$.

10.1.1 Permutations in birdtracks

In 1937 R. Brauer [1] introduced diagrammatic notation for the Kronecker δ_{ij} operation, in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. His equation (39)



(send index 1 to 2, 2 to 4, contract ingoing (3·4), outgoing (1·3)) is the earliest published diagrammatic notation I know about. While in kindergarten (disclosure: we were too poor to afford kindergarten) I sat out to revolutionize modern group theory [2]. But I suffered a terrible setback; in early 1970's Roger Penrose pre-invented my "birdtracks," or diagrammatic notation, for symmetrization operators [6], Levi-Civita tensors [8], and "strand networks" [5]. Here is a little flavor of how one birdtracks:

We can represent the operation of permuting indices (d "billiard ball labels," tensors with d indices) by a matrix with indices bunched together:

$$\sigma_\alpha^\beta = \sigma_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}{}_{,c_q \ldots c_2 c_1}^{d_p \ldots d_1} . \tag{10.2}$$

To draw this, Brauer style, it is convenient to turn his drawing on a side. For 2-index

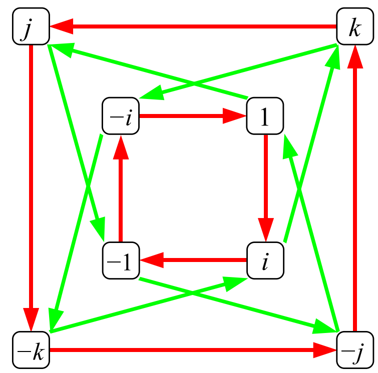
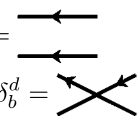


Figure 10.2: A Cayley graph presentation of the quaternion group Q_8. It is also of order 8, but distinct from D_4.

tensors, there are two permutations:

$$\text{identity:} \quad \mathbf{1}_{ab,}{}^{cd} = \delta_a^d \delta_b^c =$$

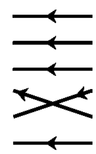
$$\text{flip:} \quad \sigma_{(12)ab,}{}^{cd} = \delta_a^c \delta_b^d = \qquad (10.3)$$



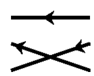
For 3-index tensors, there are six permutations:

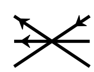
$$\mathbf{1}_{a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} =$$

$$\sigma_{(12)a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} =$$

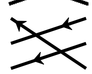


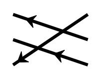
$$\sigma_{(23)} = \quad , \qquad \sigma_{(13)} =$$





$$\sigma_{(123)} = \quad , \qquad \sigma_{(132)} = \quad . \qquad (10.4)$$



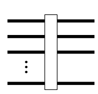


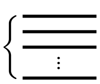
Here group element labels refer to the standard permutation cycles notation. There is really no need to indicate the “time direction" by arrows, so we omit them from now on.

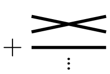
The symmetric sum of all permutations,

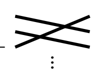
$$S_{a_1 a_2 \ldots a_p,}{}^{b_p \ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$

$$S = \quad = \frac{1}{p!}\left\{ \quad + \quad + \quad + \ldots \right\}, \qquad (10.5)$$





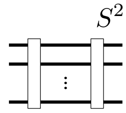


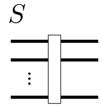


yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines [6] will always denote symmetrization of the lines crossed. A factor of $1/p!$ has been introduced in order for S to satisfy the projection operator normalization

$$S^2 = S$$

$$\quad = \quad . \qquad (10.6)$$

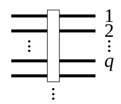




You have already seen such "fully-symmetric representation," in the discussion of discrete Fourier transforms, ChaosBook.org Example A24.3 *'Configuration-momentum' Fourier space duality*, but you are not likely to recognize it. There the average was not over all permutations, but the zero-th Fourier mode $\tilde{\phi}_0$ was the average over only cyclic permutations. Every finite discrete group has such fully-symmetric representation, and in statistical mechanics and quantum mechanics this is often the most important state (the 'ground' state).

A subset of indices $a_1, a_2, \ldots a_q$, $q < p$ can be symmetrized by symmetrization matrix $S_{12\ldots q}$

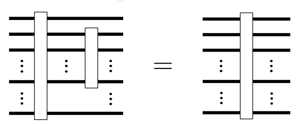
$$
\begin{aligned}
(S_{12\ldots q})_{a_1 a_2 \ldots a_q \ldots a_p},{}^{b_p \ldots b_q \ldots b_2 b_1} &= \\
&\frac{1}{q!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_q}^{b_q} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_q}^{b_q} + \ldots\right\}\delta_{a_q+1}^{b_q+1}\ldots\delta_{a_p}^{b_p} \\
S_{12\ldots q} &=
\end{aligned}
$$



(10.7)

Overall symmetrization also symmetrizes any subset of indices:

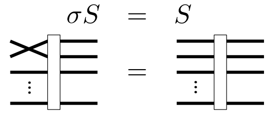
$$SS_{12\ldots q} = S$$



(10.8)

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$



(10.9)

Diagrammatically this means that legs can be crossed and uncrossed at will.

One can construct a projection operator onto the fully antisymmetric space in a similar manner [2]. Other representations are trickier - that's precisely what the theory of finite groups is about.

10.2 Literature

It's a matter of no small pride for a card-carrying dirt physics theorist to claim full and total ignorance of group theory (read sect. A.6 *Gruppenpest* of ref. [4]). The exposition (or the corresponding chapter in Tinkham [9]) that we follow here largely comes from Wigner's classic *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* [10], which is a harder going, but the more group theory you learn the more you'll appreciate it. Eugene Wigner got the 1963 Nobel Prize in Physics, so by mid 60's gruppenpest was accepted in finer social circles.

The structure of finite groups was understood by late 19th century. A full list of finite groups was another matter. The complete proof of the classification of all finite groups takes about 3 000 pages, a collective 40-years undertaking by over 100 mathematicians, read the wiki. Not all finite groups are as simple or easy to figure out as D_3. For example, the order of the Ree group ${}^2F_4(2)'$ is $212(26+1)(24-1)(23+1)(2-1)/2 = 17\,971\,200$.

From Emory Math Department: A pariah is real! The simple finite groups fit into 18 families, except for the 26 sporadic groups. 20 sporadic groups AKA the Happy Family are parts of the Monster group. The remaining six loners are known as the pariahs.

Question 10.1. Henriette Roux asks

Q What did you do this weekend?

A The same as every other weekend - prepared week's lecture, with my helpers Avi the Little, Edvard the Nordman, and Malbec el Argentino, under Master Roger's watchful eye, see here.

References

[1] R. Brauer, "On algebras which are connected with the semisimple continuous groups", Ann. Math. **38**, 857 (1937).

[2] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[3] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[4] R. Mainieri and P. Cvitanović, "A brief history of chaos", in *Chaos: Classical and Quantum*, edited by P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay (Niels Bohr Inst., Copenhagen, 2017).

[5] R. Penrose, "Angular momentum: An approach to combinatorical space-time", in *Quantum Theory and Beyond*, edited by T. Bastin (Cambridge Univ. Press, Cambridge, 1971).

[6] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial mathematics and its applications*, edited by D. J. J.A. Welsh (Academic, New York, 1971), pp. 221–244.

[7] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

[8] R. Penrose and M. A. H. MacCallum, "Twistor theory: An approach to the quantisation of fields and space-time", Phys. Rep. **6**, 241–315 (1973).

[9] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[10] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

10.1. **1-dimensional representation of anything.** Let $D(g)$ be a representation of a group G. Show that $d(g) = \det D(g)$ is one-dimensional representation of G as well.

(B. Gutkin)

10.2. **2–dimensional representation of S_3.**

(i) Show that the group S_3 of permutations of 3 objects can be generated by two permutations, a transposition and a cyclic permutation:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

(ii) Show that matrices:

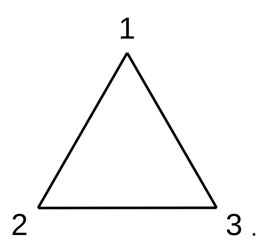
$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \rho(d) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix},$$

with $z = e^{i2\pi/3}$, provide proper (faithful) representation for these elements and find representation for the remaining elements of the group.

(iii) Is this representation irreducible?
One of those tricky questions so simple that one does not necessarily get them. If it were reducible, all group element matrices could be simultaneously diagonalized. A motivational (counter)example: as multiplication tables for D_3 and S_3 are the same, consider D_3. Is the above representation of its C_3 subgroup irreducible?

(B. Gutkin)

10.3. **D_3: symmetries of an equilateral triangle.** Consider group $D_3 \cong C_{3v} \cong S_3$, the symmetry group of an equilateral triangle:



(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D_3.

(c) Find the classes of D_3 and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D_3. (continued as exercise 11.2 and exercise 11.3)

10.4. **Permutation of three objects.** Consider S_3, the group of permutations of 3 objects.

(a) Show that S_3 is a group.

(b) List the equivalence classes of S_3?

(c) Give an interpretation of these classes if the group elements are substitution operations on a set of three objects.

(c) Give a geometrical interpretation in case of group elements being symmetry operations on equilateral triangle.

10.5. **3-dimensional representations of D_3.** The group D_3 is the symmetry group of the equilateral triangle. It has 6 elements

$$\mathrm{D}_3 = \{E, C, C^2, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\},$$

where C is rotation by $2\pi/3$ and $\sigma^{(i)}$ is reflection along one of the 3 symmetry axes.

(i) Prove that this group is isomorphic to S_3

(ii) Show that matrices

$$\mathcal{D}(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{D}(C) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, \mathcal{D}(\sigma^{(1)}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{10.10}$$

generate a 3-dimensional representation $\mathcal{D}$ of D_3. Hint: Calculate products for representations of group elements and compare with the group table (see lecture).

(iii) Show that this is a reducible representation which can be split into one dimensional A and two-dimensional representation Γ. In other words find a matrix R such that

$$\mathbf{R}\mathcal{D}(g)\mathbf{R}^{-1} = \begin{pmatrix} A(g) & 0 \\ 0 & \Gamma(g) \end{pmatrix}$$

for all elements g of D_3. (Might help: D_3 has only one (non-equivalent) 2-dim irreducible representation).

(B. Gutkin)