mathematical methods - week 13

Probability

Georgia Tech PHYS-6124

Homework HW #13 due Monday, November 18, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Bonus points
Exercise 13.1 *Lyapunov equation* 12 points

This week there are no required exercises. Whatever you do, you get bonus points.

edited November 15, 2019

Week 13 syllabus

November 11, 2019

Mon A summary of key concepts

- 20.2 Moments, cumulants

Wed Why Gaussians again?

- 33.2 Brownian diffusion
- 33.3 Noisy trajectories

Fri A glimpse of Orstein-Uhlenbeck, the "harmonic oscillator" of the theory of stochastic processes. And the one "Lyapunov" thing Lyapunov actually did:)

- Noise is your friend
- 33.4 Noisy maps
- 33.5 All nonlinear noise is local

13.1 Literature

Really going into the Ornstein-Uhlenbeck equation might take too much of your time, so this week we skip doing exercises, and if you are curious, and want to try your hand at solving exercise 13.1 *Lyapunov equation*, you probably should first skim through our lectures on the Ornstein-Uhlenbeck spectrum, Sect. 4.1 and Appen. B.1 here. Finally! we get something one expects from a math methods course, an example of why orthogonal polynomials are useful, in this case the Hermite polynomials :) .

The reason why I like this example is that again the standard 'physics' intuition misleads us. Brownian noise spreads with time as $\sqrt{t}$, but the diffusive dynamics of nonlinear flows is fundamentally different - instead of spreading, in the Ornstein-Uhlenbeck example the noise contained and balanced by the nonlinear dynamics.

- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); arXiv:0902.4269
- P. Cvitanović and D. Lippolis, *Knowing when to stop: How noise frees us from determinism*, in M. Robnik and V.G. Romanovski, eds., *Let's Face Chaos through Nonlinear Dynamics* (Am. Inst. of Phys., 2012); arXiv:1206.5506
- J. M. Heninger, D. Lippolis and P. Cvitanović, *Neighborhoods of periodic orbits and the stationary distribution of a noisy chaotic system*; arXiv:1507.00462

Question 13.1. Henriette Roux asks

Q What percentage score on problem sets is a passing grade?

A That might still change, but currently it looks like 60% is good enough to pass the course. 70% for C, 80% for B, 90% for A. Very roughly - will alert you if this changes. Here is the percentage score as of week 10.

Question 13.2. Henriette Roux asks

Q How do I subscribe to the nonlinear and math physics and other seminars mailing lists?

A click here

Exercises

13.1. **Lyapunov equation.** Consider the following system of ordinary differential equations,

$$\dot{Q} = A\,Q + Q\,A^\top + \Delta\,, \tag{13.1}$$

in which $\{Q, A, \Delta\} = \{Q(t), A(t), \Delta(t)\}$ are $[d\times d]$ matrix functions of time t through their dependence on a deterministic trajectory, $A(t) = A(x(t))$, etc., with stability matrix A and noise covariance matrix Δ given, and density covariance matrix Q sought. The superscript $(\ )^\top$ indicates the transpose of the matrix. Find the solution $Q(t)$, by taking the following steps:

(a) Write the solution in the form $Q(t) = J(t)[Q(0) + W(t)]J^\top(t)$, with Jacobian matrix $J(t)$ satisfying

$$\dot{J}(t) = A(t)\,J(t)\,, \qquad J(0) = \mathbf{1}\,, \tag{13.2}$$

with $\mathbf{1}$ the $[d\times d]$ identity matrix. The Jacobian matrix at time t

$$J(t) = \hat{T} e^{\int_0^t d\tau\, A(\tau)}\,, \tag{13.3}$$

where $\hat{T}$ denotes the 'time-ordering' operation, can be evaluated by integrating (13.2).

(b) Show that $W(t)$ satisfies

$$\dot{W} = \frac{1}{J}\,\Delta\,\frac{1}{J^\top}\,, \qquad W(0) = 0\,. \tag{13.4}$$

(c) Integrate (13.1) to obtain

$$Q(t) = J(t)\left[Q(0) + \int_0^t d\tau\,\frac{1}{J(\tau)}\,\Delta(\tau)\,\frac{1}{J^\top(\tau)}\right] J^\top(t)\,. \tag{13.5}$$

(d) Show that if $A(t)$ commutes with itself throughout the interval $0 \le \tau \le t$ then the time-ordering operation is redundant, and we have the explicit solution $J(t) = \exp\left\{\int_0^t d\tau\, A(\tau)\right\}$. Show that in this case the solution reduces to

$$Q(t) = J(t)\;Q(0)\;J(t)^\top + \int_0^t d\tau'\; e^{\int_{\tau'}^t d\tau\, A(t)}\;\Delta(\tau')\; e^{\int_{\tau'}^t d\tau\, A^\top(t)}\,. \tag{13.6}$$

(e) It is hard to imagine a time dependent $A(t) = A(x(t))$ that would be commuting. However, in the neighborhood of an equilibrium point x^* one can approximate the stability matrix with its time-independent linearization, $A = A(x^*)$. Show that in that case (13.3) reduces to

$$J(t) = e^{t\,A}\,,$$

and (13.6) to what?