mathematical methods - week 6

Cauchy - applications

Georgia Tech PHYS-6124

Homework HW #6 due Monday, September 30, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 6.1 *Complex integration* (a) 4; (b) 2; (c) 2; and (d) 3 points
Exercise 6.2 *Fresnel integral* 7 points

Bonus points
Exercise 6.3 *Cauchy's theorem via Green's theorem in the plane* 6 points

Total of 16 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited September 25, 2019

Week 6 syllabus

September 23, 2019

Mon Goldbart pages 3/10 - 3/30; 3/60 - 3/70 (Cauchy integral formula)

Wed Goldbart pages 3/80 - 3/110 (singularities; Laurent series)
Grigoriev pages 3.4 - 3.5b (evaluation of integrals)

Fri Grigoriev pages 3.4 - 3.5b (evaluation of integrals)
Goldbart pages 4/10 - 4/100 (linear response)

Optional reading

- Arfken and Weber [1] (click here) Chapter 6 sects. 6.3 - 6.4, on Cauchy contour integral
- Arfken and Weber [1] Chapter 6 sects. 6.5 - 6.8, on Laurent expansion, cuts, mappings
- Arfken and Weber [1] (click here) Chapter 7 sects. 7.1 - 7.2, on residues
- Stone and Goldbart [2] (click here) Chapter 17 sect. 17.2 - 17.4

Question 6.1. Henriette Roux had asked
Q You made us do exercise 4.5, but you did not cover this in class? What's up with that? I left it blank!
A Mhm. Check the discussion of this problem in the updated week 4 notes.

References

[1] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists: A Comprehensive Guide*, 6th ed. (Academic, New York, 2005).

[2] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge, 2009).

Exercises

6.1. **Complex integration.**

(a) Write down the values of $\oint_C (1/z)\, dz$ for each of the following choices of C:
(i) $|z| = 1$, (ii) $|z-2| = 1$, (iii) $|z-1| = 2$.
Then confirm the answers the hard way, using parametric evaluation.

(b) Evaluate parametrically the integral of $1/z$ around the square with vertices $\pm 1 \pm i$.

(c) Confirm by parametric evaluation that the integral of z^m around an origin centered circle vanishes, except when the integer $m = -1$.

(d) Evaluate $\int_{1+i}^{3-2i} dz\ \sin z$ in two ways: (i) via the fundamental theorem of (complex) calculus, and (ii) (bonus) by choosing any path between the end-points and using real integrals.

6.2. **Fresnel integral.**

We wish to evaluate the $I = \int_0^\infty \exp\left(ix^2\right) dx$. To do this, consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, R and $R\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from R to $R\exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty} \left(\int_0^R \exp\left(ix^2\right) dx - \mathrm{e}^{i\pi/4} \int_0^R \exp\left(-r^2\right) dr \right) = 0$$

and, from the well-known result of real integration $\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2$, deduce that $I = \mathrm{e}^{i\pi/4}\sqrt{\pi}/2$.

6.3. **Cauchy's theorem via Green's theorem in the plane.** Express the integral $\oint_C dz\, f(z)$ of the analytic function $f = u + iv$ around the simple contour C in parametric form, apply the two-dimensional version of Gauss' theorem (a.k.a. Green's theorem in the plane), and invoke the Cauchy-Riemann conditions. Hence establish Cauchy's theorem $\oint_C dz\, f(z) = 0$.