mathematical methods - week 8

Discrete Fourier transform

Georgia Tech PHYS-6124

Homework HW #8 due Wednesday, October 16, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 8.1 *Laplacian is a non-local operator*	4 points
Exercise 8.2 *Lattice Laplacian diagonalized*	8 points

Total of 12 points = 100 % score.

edited October 23, 2019

Week 8 syllabus

Monday, October 7, 2019

Discretization of continuum, lattices, discrete derivatives, discrete Fourier transforms.

Mon Applied math version: how to discretize derivatives: *ChaosBook* Appendix A24 Deterministic diffusion,
Sects. A24.1 to A24.1.1 *Lattice Laplacian*.

Wed A periodic lattice as the simplest example of the theory of finite groups:
Sects. A24.1.2 to A24.3.1.
Example A24.2 *Projection operators for discrete Fourier transform.*
Example A24.3 *'Configuration-momentum' Fourier space duality.*

Fri Sect. A24.4 Fourier transform as the limit of a discrete Fourier transform.

Optional reading

- A theoretical physicist's version of the above notes: *Quantum Field Theory - a cyclist tour*, Chapter 1 *Lattice field theory* motivates discrete Fourier transforms by computing a free propagator on a lattice.

Exercises

8.1. **Laplacian is a non-local operator.**
While the Laplacian is a simple tri-diagonal difference operator, its inverse (the "free" propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2\mathbf{1}-\Delta} = \frac{1}{m^2}\sum_{n=0}^{\infty}\frac{1}{m^{2n}}\Delta^n\,. \tag{8.1}$$

As Δ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, show that Δ^2 is:

$$\Delta^2 = \frac{1}{a^4}\left[\begin{array}{ccccccc} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & \ddots & & & \\ & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{array}\right]. \tag{8.2}$$

What Δ^3, Δ^4, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise 8.2.

8.2. **Lattice Laplacian diagonalized.** Insert the identity $\sum P^{(k)} = \mathbf{1}$ wherever you profitably can, and use the shift matrix eigenvalue equation to convert shift σ matrices into scalars. If $\mathbf{M}$ *commutes* with σ, then $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}^{(k)} \delta_{kk'}$, and the matrix $\mathbf{M}$ acts as a multiplication by the scalar $\tilde{M}^{(k)}$ on the kth subspace. Show that for the 1-dimensional version of the lattice Laplacian (**??**) the projection on the kth subspace is

$$(\varphi_k^\dagger \cdot \Delta \cdot \varphi_{k'}) = \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right) - 1\right)\delta_{kk'}\,. \tag{8.3}$$

In the kth subspace the propagator is simply a number, and, in contrast to the mess generated by (8.1), there is nothing to evaluating:

$$\varphi_k^\dagger \cdot \frac{1}{m^2\mathbf{1} - \Delta} \cdot \varphi_{k'} = \frac{\delta_{kk'}}{m^2 - \frac{2}{(ma)^2}\left(\cos 2\pi k/N - 1\right)}\,, \tag{8.4}$$

where k is a site in the N-dimensional dual lattice, and $a = L/N$ is the lattice spacing.