Synchronization by Nonlinear Frequency Pulling

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We analyze a model for the synchronization of nonlinear oscillators due to reactive coupling and nonlinear frequency pulling motivated by the physics of arrays of nanoscale oscillators. We study the model for the mean field case of all-to-all coupling, deriving results for the onset of synchronization as the coupling or nonlinearity increase, and the fully locked state when all the oscillators evolve with the same frequency.

In the last decade we have witnessed exciting technological advances in the fabrication of nanoelectromechanical systems (NEMS). Such systems are being developed for a host of nanotechnological applications, as well as for basic research in the mesoscopic physics of phonons and the general study of the behavior of mechanical degrees of freedom at the interface between the quantum and the classical worlds [1, 2]. Among the outstanding features of nanomechanical resonating elements is the fact that at these dimensions their normal frequencies are extremely high—recently exceeding the 1GHz mark [3]—facilitating the design of ultra-fast mechanical devices. Since with diminishing size output signals diminish as well, there is a need to use the coherent response in large arrays of coupled nanomechanical resonators (like the ones that were recently fabricated [4, 5]) for signal enhancement and noise reduction. One potential obstacle for achieving such coherent response is the fundamental problem of the irreproducibility of NEMS devices. Clearly, as the size of a resonating beam or cantilever decreases to the point that its width is only that of a few dozen atoms, any misplaced atomic cluster dramatically can change the normal frequency or any other property of the resonator. Thus, it is almost inevitable that an array of nanomechanical resonators will contain a distribution of normal frequencies. Here we propose to overcome this potential difficulty by making use of another typical feature of nanomechanical resonators—their tendency to behave nonlinearly at even modest amplitudes. We shall demonstrate here that systems of coupled nonlinear nanomechanical resonators (like the one we studied recently [6]) can self-synchronize to one common frequency through the dependence of their frequencies on the amplitude of oscillation.

The synchronization of systems of coupled oscillators that have a distribution of individual frequencies is important in many disciplines of science [7, 8]. The coherent oscillations can be used to enhance the sensitivity of detectors or the power output from sources, as proposed here. Synchronization is also important in biological phenomena, for example the collective behavior in populations of animals, such as the synchronized flashing of fire flies, and the coherent oscillations observed in the brain.

Although synchronization is often put forward as an example of the importance of understanding a nonlinear phenomenon, the intuition for the phenomenon, and indeed the subsequent mathematical discussion, can often be developed in terms of simple linear ideas. Even the famous example of Huygens' clocks can largely be understood [9] in terms of a linear coupling of the two pendulums through the common support. The larger damping of the symmetric mode (coming from the larger, dissipative motion of the support) compared with the antisymmetric mode tends to lead to a synchronized state with the two pendulums oscillating antiphase. The nonlinearity in the system is present in the individual motion of each pendulum in the mechanism to sustain the oscillations, and to reach a full description of the range of possible states must be included in the analysis. However, even without this drive the oscillators would still become synchronized through the faster decay of the even mode, albeit in a slowly decaying state. A second important feature of the model describing the two pendulums, and of many other models used to show synchronization, is that the essential coupling between the oscillators is dissipative, whereas in many physical situations the coupling is mainly reactive.

In our example of the coupled array of nanomechanical oscillators we expect predominantly reactive coupling terms coming from the elastic forces between the oscillators. Furthermore synchronization will arise from the intrinsically nonlinear effect of the frequency pulling of one oscillator by another. Thus, in this paper, we propose and analyze a model for synchronization involving *reactive coupling* between the oscillators, which then leads to synchronization through *nonlinear frequency pulling*. As well as being directly applicable to nanomechanical systems, we believe this model emphasizing the tuning of the individual oscillator frequencies through reactive couplings, rather than the mode-dependent dissipation mechanism described above, is a more accurate paradigm for synchronization in diverse systems including, for example, lasers coupled through evanescent fields, and biological organisms where the coupling is the perception of neighbors' behavior (e.g. in the synchronized flashing of fireflies).

Important advances in the understanding of synchronization have come from studying a simple model [10] often known as the Kuramoto model [11]. In this model, the oscillators are represented as phase variables, which in the absence of coupling simply advance at a rate that is constant in time, but with some dispersion of frequencies over the different oscillators. The coupling is included as infinite-range, or all-to-all coupling, so that the model is represented by the equations of motion for the N oscillators (with the dot denoting a time derivative)

$$\dot{\theta}_m = \omega_m + \frac{K}{N}\sum_{n=1}^{N}\sin(\theta_n - \theta_m), \qquad m = 1\ldots N. \quad (1)$$

Here the ω_m are the individual oscillator frequencies taken from a distribution $g(\omega)$, and K is a positive coupling constant. The synchronization is captured by a nonzero value of a complex order parameter ψ

$$\psi = R\,e^{i\Theta} = \frac{1}{N}\sum_{m=1}^{N} z_m = \frac{1}{N}\sum_{m=1}^{N} r_m e^{i\theta_m}, \quad (2)$$

with the magnitude $r_m = 1$ for the Kuramoto model,

The Kuramoto equation shows rich behavior, including, in the large N limit, a sharp synchronization transition at a value of the coupling constant $K = K_c$ [11], which depends on the frequency distribution of the uncoupled oscillators $g(\omega)$. The transition is from an unsynchronized state with $\psi = 0$ in which the oscillators run at their individual frequencies, to a synchronized state with $\psi \neq 0$ in which a finite fraction of the oscillators lock to a single frequency, $\Omega = \dot{\Theta}$, equal to the mean frequency of the locked oscillators. The transition at K_c has many of the features of a second order phase transition, with universal power laws and critical slowing down [11], and a diverging response to an applied force [12].

Equation (1) is an abstraction from the equations describing most real oscillator systems, leaving out many important physical features. A natural generalization is to include the magnitude of the oscillations as dynamical variables [7], while adding nonlinearities and considering reactive as well as dissipative coupling. Thus we are led to the model

$$\dot{z}_m = i(\omega_m - \alpha|z_m|^2)z_m + (1-|z_m|^2)z_m + \frac{K+i\beta}{N}\sum_{n=1}^{N}(z_n - z_m). \quad (3)$$

The behavior including just nonlinear saturation and dissipative coupling (i.e. setting $\alpha = \beta = 0$) was analyzed by Matthews et al. [13]. We will instead study the case of reactive coupling ($\beta \neq 0$, $K = 0$) and allow for nonlinear frequency pulling ($\alpha \neq 0$). This model then has two parameters: α the strength of the imaginary nonlinear term which yields the frequency pulling, and β the reactive coupling strength. In addition the probability distribution $g(\omega)$ of the ω_m must be specified. We will study the case of positive α and β; for a symmetric distribution $g(\omega)$ the results are the same changing the sign of both α and β.

The main focus of this paper is analyzing the behavior of (3), but first we want to show how such an equation might arise from the equations of motion of realistic nonlinear coupled nanomechanical resonators. A possible set of equations describing such a system of N coupled resonators (similar to the system we studied recently in a different context [6]) is

$$\ddot{x}_m + (1+\delta_m)x_m - \nu(1-x_m^2)\dot{x}_m - a x_m^3 - D[x_m - \tfrac{1}{2}(x_{m+1} + x_{m-1})] = 0. \quad (4)$$

The first two terms describe uncoupled harmonic oscillators, where the coordinate x_m measures the position of the m^{th} nanomechanical cantilever or beam, oscillating in its fundamental mode of vibration. We suppose the uncoupled oscillators have a linear frequency that is near unity (by an appropriate scaling of time) so that $\delta_m \ll 1$. The third term is a *negative* linear damping, which represents some unspecified energy source to sustain the oscillations, and positive nonlinear damping, so that the oscillation amplitude saturates at a finite value. For an example of such effects in a micromechanical oscillator see [14]. The saturation value is chosen to be of order unity by an appropriate scaling of the displacements x_m. The first three terms comprise a set of uncoupled *van der Pohl oscillators*. The term $a x_m^3$ is a reactive nonlinear term that leads to an amplitude dependent shift of the resonant frequency, observed experimentally in many nanomechanical resonators [15, 16]. With $\nu = 0$ this would give us a set of uncoupled *Duffing oscillators*. The final term is a coupling between the oscillators, depending on the difference of the displacements. This is a reactive term, typical for either elastic or electrostatic interaction between resonators that conserves the energy of the system. We have written the equation with nearest neighbor coupling for simplicity. In fact, both elastic and electrostatic coupling may be long range, with a decay rate that depends on the details of the geometry, elastic support etc. (e.g. proportional to the inverse of the separation for long charged beam oscillators interacting electrostatically). Others [17] have considered nonlinear oscillators coupled through their velocities; this is a dissipative coupling that would lead to $K \neq 0$ in the amplitude-phase reduction.

The complex amplitude equation (3) holds if the parameters ν, α, D, δ_m are sufficiently small. In this case the equations of motion are dominated by the terms describing simple harmonic oscillators at frequency one. We may then write

$$x_m \simeq z_m(t)e^{it} + c.c. + \cdots, \quad (5)$$

where $z_m(t)$ is slowly varying compared with the basic oscillation frequency of unity, and $\cdots$ are correction terms. Substituting (5) into the equations of motion (4) and requiring that secular terms proportional to e^{it} vanish yields the amplitude equations

$$2\dot{z}_m = (\nu + i\delta_m)z_m - (\nu + 3ia)\,|z_m|^2\, z_m - iD[z_m - \tfrac{1}{2}(z_{m+1} + z_{m-1})]. \quad (6)$$

With a rescaling of time $\bar{t} = \nu t/2$ (6) reduces to our model (3) except that in our model the nearest neighbor coupling is replaced by the all-to-all coupling convenient for theoretical analysis. Again we note that for nanomechanical oscillators the physical coupling may be long range, in which case the all-to-all coupling will be a better approximation.

Since we are interested in the behavior of (3) for a large number of oscillators, it is convenient to go to a continuum description, where we label the oscillators by their uncoupled linear frequency $\omega = \omega_j$ rather than the index j, $z_j \to z(\omega)$. Introducing the order parameter (2), the oscillator equations can be written in magnitude-phase form as

$$d_t\bar{\theta} = \bar{\omega} + \alpha(1 - r^2) + \beta R r^{-1}\cos\bar{\theta} \quad (7)$$

$$d_t r = (1 - r^2)r + \beta R \sin\bar{\theta} \quad (8)$$

where $\bar{\theta} = \theta - \Theta$ is the oscillator phase relative to that of the order parameter, and $\bar{\omega}$ is the bare oscillator frequency measured relative to Δ, which is the order parameter frequency $\Omega = \dot{\Theta}$, shifted by $-(\alpha + \beta)$

$$\bar{\omega} = \omega - \Delta;\ \ \Delta = \Omega + \alpha + \beta. \quad (9)$$

At each time t the oscillators are specified by $\rho(r, \bar{\theta}, \bar{\omega}, t)$, the distribution of oscillators at shifted frequency $\bar{\omega}$ over magnitude and phase values. The order parameter is given by the self-consistency condition

$$R = \left\langle re^{i\bar{\theta}}\right\rangle = \int d\bar{\omega}\bar{g}(\bar{\omega}) \int r\, dr\, d\bar{\theta}\rho(r, \bar{\theta}, \bar{\omega}, t)re^{i\bar{\theta}}. \quad (10)$$

where $\bar{g}(\bar{\omega})$ is the distribution of oscillator frequencies expressed in terms of the shifted frequency $\bar{\omega}$. Note that unlike the cases of the Kuramoto model and (3) with $\alpha = \beta = 0$ the imaginary part of this condition is not trivially satisfied even for the case of a symmetric distribution $g(\omega)$, and in fact serves to determine the frequency Ω of the order parameter. Furthermore, this frequency is not trivially related to the mean frequency of the oscillator distribution.

To uncover more fully the behavior of our model (3) we consider two issues: the existence of a fully locked state for large values of $\alpha\beta$; and the onset of synchronization, detected as the linear instability of the unsynchronized $R = 0$ state.

First we look at the fully locked solution for a bounded distribution of frequencies of width w. We define any state with an $O(1)$ magnitude of the order parameter R as *synchronized*. If *all* of the phases of a synchronized state are rotating at the order parameter frequency we call the state *fully locked*. The solutions are defined by setting $d_t r = 0$ which gives

$$(1 - r^2)r = -\beta R \sin\bar{\theta}, \quad (11)$$

and $d_t\bar{\theta} = 0$, which with (11) can be written

$$\bar{\omega} = F(\bar{\theta}) = \beta R r^{-1}(\alpha \sin\bar{\theta} - \cos\bar{\theta}), \quad (12)$$

where the solution to the cubic equation (11) for r is to be used to form the function of phase alone $F(\bar{\theta})$. The function $F(\bar{\theta})$ acts as the force pinning the locked oscillators to the order parameter. A particular oscillator, identified by its shifted frequency $\bar{\omega}$, may be locked to the order parameter if (12) has a solution $\bar{\theta} = F^{-1}(\bar{\omega})$ and if this solution is stable. The stability is tested by linearizing (7,8) about the solution. The fully locked solution will only exist if stable, locked solutions to (12) exist for *all* the oscillators in the distribution. In addition, the self consistency condition (10) must be satisfied.

For large values of $\alpha\beta$ the phases of the locked oscillators cover a narrow range of angles. The imaginary part of the self consistency condition (10) shows that the range of phases must be around $\bar{\theta} = 0$, and (12) becomes (note $r \simeq 1$ here)

$$\bar{\omega} \simeq -\beta R(1 - \alpha\bar{\theta}). \quad (13)$$

The imaginary part of the self-consistency condition reduces to $\langle\bar{\theta}\rangle = 0$ (the average is over the distribution of frequencies), and the real part to simply $R \simeq 1$. Finally, averaging (13) over the distribution of frequencies fixes the order parameter frequency $\Omega \simeq \langle\omega\rangle - \alpha$. This construction remains valid for large β, so that unlike the case studied by Matthews et al. [13], "amplitude death" does not necessarily occur at large values of the coupling constant. The extension of this calculation to find the boundary of the fully locked state will be presented elsewhere.

We now turn our attention to the initial onset of partial synchronization from the unsynchronized state. This is calculated by linearizing the distribution ρ around the unsynchronized distribution which is a uniform phase distribution at $r = 1$, and seeking the parameter values at which deviations from the uniform phase distribution begin to grow exponentially. Care is needed in the analysis due to the important role the magnitude perturbations play.

Introducing the small expansion parameter ε characterizing the small deviations from the unsynchronized state, we write

$$\rho(r, \theta, \bar{\omega}, t) \simeq (2\pi r)^{-1}\delta[r - 1 - \varepsilon r_1(\bar{\theta}, \bar{\omega}, t)][1 + \varepsilon f_1(\bar{\theta}, \bar{\omega}, t)],$$

as well as $R \simeq \varepsilon R_1$, with $r_1, f_1, R_1 \propto e^{\lambda t}$ with λ the growth rate of the linear instability. With this expansion ρ remains normalized to linear order in ε providing the average of f_1 over $\bar{\theta}$ is zero. The dynamical equations (7-8) at $O(\varepsilon)$ lead to the explicit solutions $r_1 = R_1(A\cos\bar{\theta} + B\sin\bar{\theta})$ with

$$A = -\beta\frac{\bar{\omega}}{\bar{\omega}^2 + (\lambda + 2)^2},\ \ B = \beta\frac{(\lambda + 2)}{\bar{\omega}^2 + (\lambda + 2)^2},$$

and $f_1 = R_1(C\cos\bar{\theta} + D\sin\bar{\theta})$, with

$$C = \beta\frac{2\alpha(\lambda^2 + 2\lambda - \bar{\omega}^2) - \bar{\omega}[\bar{\omega}^2 + (\lambda + 2)^2]}{\left(\bar{\omega}^2 + \lambda^2\right)\left[\bar{\omega}^2 + (\lambda + 2)^2\right]},$$

$$D = \beta \frac{4\alpha\bar{\omega}(\lambda+1) + \lambda[\bar{\omega}^2 + (\lambda+2)^2]}{(\bar{\omega}^2+\lambda^2)\left[\bar{\omega}^2 + (\lambda+2)^2\right]}.$$

The self-consistency condition (10) to first order in ε gives

$$1 = \frac{1}{2}\int d\bar{\omega}\, \bar{g}(\bar{\omega})\, [(A+C) + i(B+D)], \qquad (14)$$

We evaluate (14) at the onset of instability where the growth rate $\lambda \to 0$ (it is not sufficient to put $\lambda = 0$ since some terms of the integrals then diverge). We have evaluated the integrals analytically for uniform, triangular, and Lorentzian distributions of bare frequencies. Here we present results for the triangular distribution, for which the resulting equation for the critical values of α, β and the order parameter frequency at onset must be solved numerically.

Figure 1 shows comprehensive results for a triangular distribution with full width $w = 2$. Panels (a-c) show the magnitude of the order parameter R as a function of β for constant α scans from numerical simulations of (3) for 1000 oscillators and $K = 0$. Limits of the unsynchronized state are consistent with the linear stability analysis. For the largest value shown, $\alpha = 0.9$, the low β transition $\beta = \beta_{c1} \simeq 0.8$ is weakly hysteretic, whereas the large β transition $\beta < \beta_{c2} = 3.7$ is continuous. The state growing for $\beta < \beta_{c2}$ is a novel state with $R \neq 0$, but with no oscillator frequency locked to the order parameter, which has a frequency outside of the band of shifted oscillator frequencies. For $\beta > 1.8$ there is also a state with R close to unity in which all or most of the oscillators are locked to the order parameter. For smaller $\alpha = 0.42$ there is a stable small R state for $\beta_{c1} < \beta < \beta_{c2}$, as well as a large R solution. For $\alpha = 0$ the large R synchronized state persists down to $\beta > 1.6$, whilst the unsynchronized state remains linearly stable for all β (panel c). Panel (d) shows the phase diagram, including results from the simulations as well as the linear stability analysis of the unsynchronized and fully locked state. Over a large portion of the α, β plane there are two stable solutions—a large R synchronized state, and either the unsynchronized state (hatched region) or a small R state (cross hatched region)—leading to hysteresis for continuous parameter scans. Over the dotted portion only a synchronized state is stable, and over the unshaded region only the unsynchronized state is stable.

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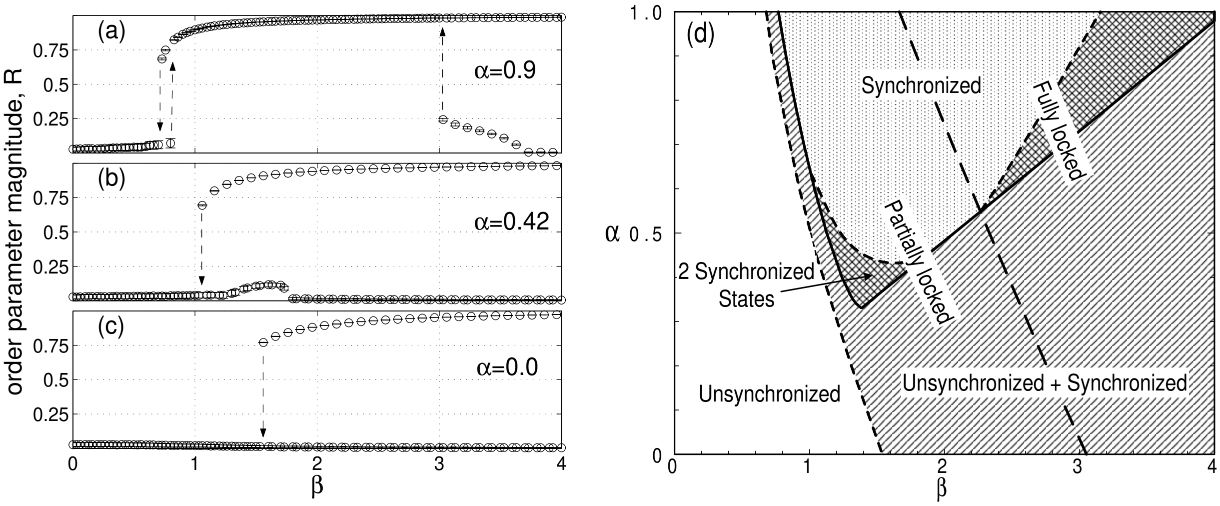

FIG. 1: Results for a triangular distribution of full width $w = 2$. Panels (a)-(c) show the order parameter R found in numerical simulations of 1000 oscillators for sweeps increasing and decreasing β and for three representative values of α. The symbols are time-averaged values and the error bars are the standard deviations in R over the averaging time. Panel (d) shows the phase diagram deduced from sweeps at many values of α, and numerical calculations of the linear instabilities: solid line - linear instability of the unsynchronized state; short-dashed line - saddle-node line deduced from jumps of R in the numerical simulations (denoted by arrows in panels a-c); long dashed line - linear instability of the fully locked solution (the large R solution is fully locked to the right of this line).