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Triple Collision in the Quasi-Homogeneous Collinear Three-Body Problem

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We generalize and improve the result of F. Diacu (*J. Differential Equations* **128** (1996), 58–77) concerning the dynamics of three-points particles moving on the line under the influence of a quasihomogeneous potential $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$, respectively. For $b > 2$ we found a set of positive measure of collision orbits that do not tend to form asymptotically a central configuration.

1. INTRODUCTION

In 1996, F. Diacu [2] studied the collision–ejection orbits in a quasi-homogeneous collinear three-body problem; that is, he studied the set of triple collision orbits for three point particles whose dynamics is determined by the potential $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$. He shows that the dynamics close to triple collision is dominated by the potential V, and then he considers the following cases: $b < 2$, $b = 2$, and $b > 2$.

Here we have used the same techniques as Diacu [2]: regularization of binary collisions, the McGehee blow-up of the singularity due to triple collision, and topological arguments to analyze the flow given by the respective differential equations. We have also considered in our analysis the same cases as Diacu.

For $b < 2$, we found analogous results (see Diacu [2]), but our proof is simpler. The analysis and results in the cases $b = 2$ and $b > 2$ differ from those given in [2].

If $b = 2$, regularization of the singularities due to binary collisions creates a large set of equilibrium points. An orbit which reaches double collision dies at one point of that set. To obtain more information on the global flow, we introduce a new reparametrization of binary collisions. In the resulting equations the equilibria corresponding to double collisions disappear. Unfortunately the vector field obtained is only continuous on those points that

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correspond to double collisions. We are not able to prove the existence of a positive measure set of triple collision orbits in this case.

The case $b>2$ contains our main results. We characterize the orbits for a large set of initial conditions, whose Lebesgue measure is infinity. The corresponding orbits for these initial data tend to triple collision with two particles colliding infinitely many times. The pair of particles that are colliding never collide with the third one. Of course, asymptotically the three particles collide. This is a significant difference with respect to the behavior of the particles moving under the influence of a classical Newtonian potential.

2. EQUATIONS OF MOTION AND GENERAL ASPECTS

The quasihomogeneous n-body problem is a generalization of the classical Newtonian n-body problem of celestial mechanics, where the goal is to describe the motion of n-point particles $m_1, m_2, ..., m_n$ moving in $\mathbb{R}^3$ under the action of a quasihomogeneous potential W, which is the sum of two homogeneous potentials U and V,

$$W(\mathbf{q}) = U(\mathbf{q}) + V(\mathbf{q}),$$

where

$$U(\mathbf{q}) = \sum_{1\leqslant i<j\leqslant n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^a},$$

$$V(\mathbf{q}) = \sum_{1\leqslant i<j\leqslant 3} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^b},$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n), \qquad \mathbf{q}_i \in \mathbb{R}^3, \quad 0 \leqslant a < b.$$

Similarly to the classical Newtonian n-body problem, the equations of motion for the quasihomogeneous n-body problem are given by

$$M\ddot{\mathbf{q}} = \nabla W(\mathbf{q}), \tag{2.1}$$

with $M = \mathrm{diag}(m_1, m_1, m_1, ..., m_n, m_n, m_n)$, or

$$\begin{aligned} m_i \ddot{\mathbf{q}}_i &= \frac{\partial W}{\partial \mathbf{q}_i}(\mathbf{q}), \\ &= \sum_{j\neq i}^{n} \left[\frac{-a m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^{a+2}} (\mathbf{q}_i - \mathbf{q}_j) + \frac{-b m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^{b+2}} (\mathbf{q}_i - \mathbf{q}_j) \right], \\ & \quad i = 1, ..., n \end{aligned} \tag{2.2}$$

Let $\Delta_{ij} = \{\mathbf{q} \mid \mathbf{q}_i = \mathbf{q}_j\}$ and $\Delta = \bigcup_{1 \leqslant i < j \leqslant n} \Delta_{ij}$; then the vector field given by (2.2) is defined in $[(\mathbb{R}^3)^n \backslash \Delta] \times T\mathbb{R}^{3n}$ where $T\mathbb{R}^{3n}$ is the tangent bundle of $\mathbb{R}^{3n}$. The equations of motion (2.2) may be written as a first order system of differential equations in Hamiltonian form, with Hamiltonian function

$$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}\mathbf{p}^t M^{-1}\mathbf{p} - W(\mathbf{q}) = T - W(\mathbf{q}), \tag{2.3}$$

where $\mathbf{q} = M\dot{\mathbf{q}}$ and the kinetic energy T is given by $T = \frac{1}{2}\mathbf{p}^t M^{-1}\mathbf{p}$.

Analogous to the Newtonian case, the quasihomogeneous n-body problem has the classical 10 first integrals. In order to get the Lagrange–Jacobi identity for this problem we define:

Definition 2.1. The moment of inertia of the system (2.2) is given by

$$I = \tfrac{1}{2} \sum_{i=1}^{n} m_i \, |\mathbf{q}_i|^2.$$

Let us observe that for fixed j,

$$\begin{aligned}\sum_{i=1}^{n} m_i \, |\mathbf{q}_i - \mathbf{q}_j|^2 &= \sum_{i=1}^{n} m_i \, |\mathbf{q}_i|^2 - 2\mathbf{q}_j \cdot \sum_{i=1}^{n} m_i \mathbf{q}_i + |\mathbf{q}_j|^2 \sum_{i=1}^{n} m_i \\ &= 2I + |\mathbf{q}_j|^2 \, \hat{M},\end{aligned}$$

where $\hat{M} = m_1 + m_2 + \cdots + m_n$ is the total mass of the system. After multiplying by m_i, and adding all the terms in the above equation we get

$$\sum_{i,\, j=1}^{n} m_i m_j \, |\mathbf{q}_i - \mathbf{q}_j|^2 + 4\hat{M}I. \tag{2.4}$$

Thus total collision occurs if and only if $I = 0$.

Now we differentiate the expression for the momentum of inertia I twice:

$$\begin{aligned}\dot{I} &= \sum_{i=1}^{n} m_i \dot{\mathbf{q}}_i \cdot \mathbf{q}_i, \\ \ddot{I} &= \sum_{i=1}^{n} m_i \ddot{\mathbf{q}}_i \cdot \mathbf{q}_i + \sum_{i=1}^{n} m_i \, |\dot{\mathbf{q}}_i|^2.\end{aligned}$$

Finally, using the equations of motion (2.2), the expression for the kinetic energy T, and the energy relation $T - U - V = h$, we get the Lagrange–Jacobi identity for quasihomogeneous potentials:

$$\ddot{I} = (2 - a) \, U(\mathbf{q}) + (2 - b) \, V(\mathbf{q}) + 2h. \tag{2.5}$$

Let us observe that for some quasihomogeneous potentials with $b > 2$ it is possible to have $\ddot{I} < 0$ for $h > 0$, implying that it is possible to have all motions on this level of energy bounded even for positive energy h. This is the first huge difference from the classical Newtonian case where the motions are unbounded for $h > 0$.

An interesting problem concerns the time in which an orbit reaches total collision in a quasihomogeneous problem. The next result shows that this time is always finite.

PROPOSITION 2.1. *In the quasihomogeneous n-body problem the total collision always occurs in finite time.*

Proof. We divide the proof into two cases:

(i) $b > 2$. In this case $\ddot{I} < 0$ close enough to total collision, and therefore the function I is concave down and non-negative: The total collision should occur in finite time.

(ii) $0 \leqslant a < b \leqslant 2$. We prove this case by contradiction. Suppose that total collision occurs in infinite time, so we have

$$U(\mathbf{q}) \to \infty \qquad \text{and} \qquad V(\mathbf{q}) \to \infty \qquad \text{as} \quad t \to \infty.$$

Then, from the Lagrange–Jacobi identity (2.5), we have that

$$\ddot{I} \to \infty, \qquad \text{if} \quad t \to \infty.$$

Thus, there exists $t_1 > 0$ such that $\ddot{I} > 1$ for $t > t_1$. Integrating this inequality twice we have

$$I \geqslant t^2/2 + ct + d, \qquad c, d \text{ constants}, \quad t > t_1.$$

Taking limits on both sides of the above inequality, we obtain

$$I \to \infty \qquad \text{as} \quad t \to \infty.$$

This is a contradiction, since at infinity there is total collision with $I \to 0$ as $t \to \infty$.

Now suppose that for a given initial condition the corresponding orbit reaches total collision at time t_0. Another interesting problem here is the behavior of the moment of inertia I along this orbit when $t \to t_0^+$. The moment of inertia I represents the size of the system. We know that at total collision $\lim_{t \to t_0^+} I = 0$. We are interested in determining the order of this limit. To this end we define

DEFINITION 2.2. Let $\mathbf{q}_0 \in \mathbb{R}^{3n}$, the configuration $\mathbf{q}_0$ is called a central configuration if there exists a scalar function $r(t) > 0$ such that $\mathbf{q}(t) = r(t)\,\mathbf{q}_0$ is a solution of the quasihomogeneous n-body problem.

Let us observe that central configurations are invariant under rotations and homotheties, allowing us to define equivalence classes with respect to these transformations. In the Newtonian case it is well known that total collision and escapes to infinity always occur by central configuration; that is, for an orbit that goes to total collision (ejection), the final configuration tends asymptotically to a central configuration. Is the same result valid for a quasihomogeneous problem? To address this question we define

$$Q = \frac{\dot{I}^2}{I^{1-b/2}}.$$

If

$$\lim_{t \to t_0} Q = \lim_{t \to 0} \frac{\dot{I}^2}{I^{1-b/2}}$$

exists and is positive, let $\lim_{t \to t_0} Q = \mu > 0$. Then, close enough to total collision, we have

$$\dot{I}^2 \sim \mu I^{1-b/2},$$

which implies that

$$\dot{I} \sim \sqrt{\mu}\, I^{(1/2)(1-b/2)}.$$

Solving this separable differential equation, we get

$$t - t_0 \sim \frac{4}{2+b} \frac{1}{\sqrt{\mu}} I^{(2+b)/4},$$

or equivalently

$$I \sim A(t - t_0)^{4/(2+b)}.$$

Let us observe that in the Newtonian case ($a = 0$ and $b = 1$), the above $\lim_{t \to t_0} Q$ always exists and we have in this case $I \sim (t - t_0)^{2/3}$, which is a classical result in celestial mechanics [8].

Unfortunately, for a quasihomogeneous problem, the above limit does not exist in general. Intuitively we can see that its existence depends on the configuration of the particles at the moment of total collision. Later in this paper we will show that for the quasihomogeneous collinear three body problem this limit in general does not exist for an orbit that reaches total collision. When the limit does exist, then total collision (ejection) occurs by central configuration and the order of the respective limit is $I \sim A(t-t_0)^{4/(2+b)}$. Our main result in this paper states that for any $b \geqslant 1$ the measure (in the Lebesgue sense) of the set of initial conditions which reaches total collisions (ejection) by central configuration is zero, whereas the set of initial conditions where the orbits reach total collision and $\lim_{t \to t_0} Q$ does not exist has positive measure. In summary, we will show that most of the orbits reach total collision (ejection) on a configuration which is far from a central configuration.

3. TRIPLE COLLISION

In this section we will use the blow-up method introduced by R. McGehee [5] to study the dynamics of the orbits that reach triple collision and the orbits that pass very close to triple collision in the quasihomogeneous collinear three-body problem. This is a particular case of the quasihomogeneous three-body problem where we suppose that the particles are moving on a line, so $q_i \in \mathbb{R}$, and then $\mathbf{q} \in \mathbb{R}^3$. The equations of motion (2.2) and the energy relation (2.3) hold for this case with

$$\Delta = \{\mathbf{q}\colon q_1 = q_2, \text{ or } q_2 = q_3\},$$

and the mass matrix is given by $M = \operatorname{diag}(m_1, m_2, m_3)$.

If we define $\mathbf{p} = M\dot{\mathbf{q}}$, the equations of motion can be written as a first-order system of differential equations in Hamiltonian form,

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} = M^{-1}\mathbf{p},$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} = \nabla W(\mathbf{q}),$$

where

$$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}\mathbf{p}^t M^{-1}\mathbf{p} - W(\mathbf{q}) = T - W(\mathbf{q}),$$

and $W(\mathbf{q}) = U(\mathbf{q}) + V(\mathbf{q})$, (remember that U and V are homogeneous functions of degree $-a$ and $-b$, respectively).

Now we define the planes:

$$\begin{aligned} Q &= \{\mathbf{q} \in \mathbb{R}^3 \backslash \Delta\colon m_1 q_1 + m_2 q_2 + m_3 q_3 = 0\}, \\ P &= \{\mathbf{p} \in \mathbb{R}^3\colon p_1 + p_2 + p_3 = 0\}. \end{aligned} \tag{3.6}$$

From here on, we will work in the cartesian product $Q \times P$ intersected with a fixed level of energy h given by

$$\tfrac{1}{2}\mathbf{p}^{t} M^{-1} \mathbf{p} - W(\mathbf{q}) = T - W(\mathbf{q}) = h. \tag{3.7}$$

Introducing McGehee coordinates [5], we obtain

$$\begin{aligned} r &= \sqrt{\mathbf{q}^{\mathrm{T}} M \mathbf{q}}, \\ v &= r^{b/2} \mathbf{p}^{\mathrm{T}} \mathbf{s}, \\ \mathbf{s} &= \frac{1}{r}\, \mathbf{q}, \\ \mathbf{u} &= r^{b/2}(\mathbf{p} - y M \mathbf{s}). \end{aligned}$$

After the time parametrization $dt/d\tau = r^{(b/2)/1}$, the system of differential equations becomes

$$\begin{aligned} r' &= rv, \\ v' &= \frac{b}{2}\, v^2 + \mathbf{u}^{\mathrm{T}} M^{-1} \mathbf{u} - a r^{b-a} U(\mathbf{s}) - b V(\mathbf{s}), \\ \mathbf{s}' &= M^{-1} \mathbf{u}, \\ \mathbf{u}' &= \left(\frac{b}{2} - 1\right) v \mathbf{u} - (\mathbf{u}^{\mathrm{T}} M^{-1} \mathbf{u})\, \mathbf{s} \\ &\quad + r^{b-a}(\nabla U(\mathbf{s}) + a U(\mathbf{s})\, M \mathbf{s}) + (\nabla V(\mathbf{s}) + b V(\mathbf{s})\, M \mathbf{s}), \end{aligned} \tag{3.8}$$

where ($'$) means differentiation with respect to the new time τ. The energy relation (3.7) goes over to

$$\tfrac{1}{2}(\mathbf{u}^{\mathrm{T}} M^{-1} \mathbf{u} + v^2) - r^{b-a} U(\mathbf{s}) - V(\mathbf{s}) = h r^{b}. \tag{3.9}$$

Equations (3.8) and (3.9) are of class C^1 at $r = 0$ if $b - a \geqslant 1$. Moreover, if $a, b \in \mathbb{N} \cup \{0\}$, then the flow is analytic at $r = 0$. Thus, in these cases we have extended the equations of motion to include triple collision.

The total collision manifold Λ is defined by setting $r = 0$ in the energy relation (3.9),

$$\Lambda = \{(r, v, \mathbf{s}, \mathbf{u}) \colon r = 0,\ \tfrac{1}{2}(\mathbf{u}^{\mathrm{T}} M^{-1} \mathbf{u} + v^2) - V(\mathbf{s}) = 0\}. \tag{3.10}$$

From the equations (3.8), $r = 0$ implies $r' = 0$; that is, the total collision manifold is invariant under the flow given by the system (3.8), and from (3.9) we obtain that Λ is independent of the energy level h, in other words we are gluing the same border Λ to all levels of energy h. The set of orbits ending in total collision corresponds to the set of orbits asymptotic to Λ. Also by continuity of the flow with respect to initial conditions the knowledge of the orbits on Λ can be used to understand the behavior of orbits for r small (orbits passing close to triple collision).

4. REDUCTION OF COORDINATES

Before regularizing the singularities due to double collisions, it is convenient to reduce the coordinates in order to work with a vector field defined in a subset of $\mathbb{R}^4$. In this section we will follow the ideas of McGehee [5].

Let $S^3 = \{\mathbf{q} \in \mathbb{R}^3 \mid \mathbf{q}^t M \mathbf{q} = 1\}$ be the unit sphere in the metric induced by the mass matrix M, and let $S_Q = S^3 \cap Q$. In the quasihomogeneous collinear three-body problem we will consider that the configuration $\mathbf{q} = (q_1, q_2, q_3)^t$ satisfies $q_1 \leqslant q_2 \leqslant q_3$, and then we will study the configuration space in S_Q restricted to this order. In other words, we are restricting our study to an arc S_1 on S_Q. This arc has extremes at the points $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, where $a_1 = a_2 < a_3$ and $b_1 < b_2 = b_3$. These points $\mathbf{a}$ and $\mathbf{b}$ correspond to double collisions.

$$S_1 = \{\mathbf{s} \in S_Q : a_1 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant b_3\}.$$

Using the same computations given in [5], we can construct an analytic diffeomorphism S between the interval $[-1, 1]$ and S_1 such that $S(-1) = \mathbf{a}$ and $S(1) = \mathbf{b}$. $S\colon [-1, 1] \to S_1$ is defined by

$$S(s) = \frac{1}{\sin 2\lambda} ([\sin \lambda(1-s)]\, \mathbf{a} + [\sin \lambda(1+s)]\, \mathbf{b}), \tag{4.11}$$

where

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$A = \frac{1}{m_1 + m_2 + m_3} A_1 M + \left(\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3} \right)^{1/2} M^{-1} A_2, \qquad \text{and}$$

$$\sin 2\lambda = \langle A\mathbf{a}, \mathbf{b} \rangle.$$

To reduce the McGehee coordinates we define the new variables

$$s = S^{-1}(\mathbf{s}), \qquad u = \mathbf{s}^{\mathrm{T}} A^{\mathrm{T}} \mathbf{u}$$

and the new potentials

$$\tilde{U}\colon (-1,1) \to \mathbb{R} \qquad\qquad \tilde{V}\colon (-1,1) \to \mathbb{R}$$
$$s \mapsto U(S(s)) \qquad\qquad s \mapsto V(S(s)).$$

The analytic expressions for these functions in their respective domains are given by

$$\begin{aligned}\tilde{U}(s) &= \sin^a 2\lambda \left[\frac{m_1 m_2}{[(b_2-b_1)\sin\lambda(1+s)]^a} + \frac{m_2 m_3}{[(a_3-a_2)\sin\lambda(1-s)]^a} \right. \\ &\quad \left. + \frac{m_1 m_3}{[(b_2-b_1)\sin\lambda(1+s)+(a_3-a_2)\sin\lambda(1-s)]^a} \right], \\ \tilde{V}(s) &= \sin^b 2\lambda \left[\frac{m_1 m_2}{[(b_2-b_1)\sin\lambda(1+s)]^b} + \frac{m_2 m_3}{[(a_3-a_2)\sin\lambda(1-s)]^b} \right. \\ &\quad \left. + \frac{m_1 m_3}{[(b_2-b_1)\sin\lambda(1+s)+(a_3-a_2)\sin\lambda(1-s)^b]} \right].\end{aligned} \tag{4.12}$$

Finally the equations (3.8) in the new reduced coordinates are given by the following system defined in a subset of $\mathbb{R}^4$,

$$\begin{aligned} r' &= rv, \\ v' &= \frac{b}{2} v^2 + u^2 - a r^{b-a} \tilde{U}(s) - b \tilde{V}(s), \\ s' &= \frac{1}{\lambda} u, \\ u' &= \left(\frac{b}{2} - 1\right) vu + r^{b-a} \frac{1}{\lambda} \frac{d\tilde{U}}{ds} + \frac{1}{\lambda} \frac{d\tilde{V}}{ds}, \end{aligned} \tag{4.13}$$

and the energy relation (3.9) goes over

$$\tfrac{1}{2}(v^2+u^2) - r^{b-a}\tilde{U}(s) - \tilde{V}(s) = hr^b. \tag{4.14}$$

5. REGULARIZATION OF DOUBLE COLLISIONS

The vector field given by (4.13) still has singularities; The potentials $\tilde{U}$ and $\tilde{V}$ are not defined at $s=1$ and $s=-1$, which correspond to double collisions. In this section we will regularize those kind of singularities by a technique introduced by Sundman [10]. Physically this corresponds to an elastic bounce. Since the degrees of homogeneity $-a$ and $-b$ of the potentials U and V satisfy $0 \leqslant a < b$; the potential V "dominates" at double collisions.

The Sundman regularization (or more precisely, Sundman type regularization), consists in general of defining a new tangential velocity together with a rescaling of time. That is, let

$$w = \phi(s)\, u, \qquad \frac{d\tau}{d\sigma} = \phi(s), \tag{5.15}$$

where the function $\phi(s)$ must satisfy the necessary conditions for the equations (4.13), written in the new variable w and the new time σ, to be at least of class C^1.

The system (4.13) using (5.15) goes over to

$$\begin{aligned}
r' &= \phi(s)\, rv,\\
v' &= \left(\frac{b}{2} - 1\right) \phi(s)\, v^2 + (2-a)\, r^{b-a} \phi(s)\, \tilde{U}(s) + (2-b)\, \phi(s)\, \tilde{V}(s) + 2hr^b \phi(s),\\
s' &= \frac{1}{\lambda}\, w, \qquad\qquad (5.16)\\
w' &= \left(\frac{b}{2} - 1\right) \phi(s)\, vw - \frac{1}{\lambda}\, \phi(s) \frac{d\phi}{ds}\, v^2 + \frac{1}{\lambda}\, r^{b-a} \left[2\phi(s) \frac{d\phi}{ds}\, \tilde{U}(s) + \phi^2(s) \frac{d\tilde{U}}{ds} \right]\\
&\quad + \frac{1}{\lambda} \left[2\phi(s) \frac{d\phi}{ds}\, \tilde{V}(s) + \phi^2(s) \frac{d\tilde{V}}{ds} \right] + 2hr^b \phi(s) \frac{d\phi}{ds},
\end{aligned}$$

where ($'$) now means differentiation with respect to the new time σ.

The vector field given by (5.16) will be at least of class C^1 if all the functions defined in (5.16) are at least of class C^1; in this way we have several choices for the function ϕ, for example

$$\phi(s) = \frac{1}{\tilde{V}(s)}, \qquad \phi(s) = (1 - s^2)^b, \qquad \phi(s) = \frac{(1 - s^2)^{b/2}}{[2\tilde{V}(s)]^{1/2}}.$$

Nevertheless, the dynamics given by the equations (5.16) using each of these functions ϕ are qualitatively the same. We can easily check that the vector field (5.16) is differentiable only if $b \geqslant 2$, or $b = 1$ and $a = 0$. If $1 < b < 2$, then (5.16) is only continuous.

In order to simplify the computations we choose

$$\phi(s) = \frac{(1 - s^2)^b}{W(s)^{1/2}},$$

where

$$\begin{aligned} W(s) &= 2(1-s^2)^b\,\tilde{V}(s) \\ &= \frac{2}{\lambda^b}\sin^b 2\lambda \left[\frac{m_1 m_2 (1-s)^b}{\left[(b_2-b_1)\,\frac{\sin\lambda(1+s)}{\lambda(1+s)} \right]^b} + \frac{m_2 m_3 (1+s)^b}{\left[(a_3-a_2)\,\frac{\sin\lambda(1-s)}{\lambda(1-s)} \right]^b} \right. \\ &\quad \left. + \frac{m_1 m_3 (1-s^2)^b\,\lambda^b}{[(b_2-b_1)\sin\lambda(1+s)+(a_3-a_2)\sin\lambda(1-s)]^b} \right]. \end{aligned}$$

With this choice of the function $\phi(s)$, the system (5.16) can be written

$$\begin{aligned} r' &= \frac{(1-s^2)^b}{W(s)^{1/2}}\, rv, \\ v' &= \left(\frac{b}{2}-1\right) W(s)^{1/2} \left[\frac{(1-s^2)^b}{W(s)}\, v^2 - 1 \right] \\ &\quad + \frac{(1-s^2)^b}{W(s)^{1/2}}\,[(2-a)\, r^{b-a}\tilde{U}(s) + 2hr^b], \\ s' &= \frac{1}{\lambda}\, w, \\ w' &= -\frac{b}{\lambda}\, s(1-s^2)^{b-1} + \frac{2}{\lambda}\, bs\, \frac{(1-s^2)^{2b-1}}{W(s)}\,[v^2 - 2r^{b-a}\tilde{U}(s) - 2hr^b] \\ &\quad + \frac{1}{2\lambda}\frac{W'(s)}{W(s)}\,[(1-s^2)^b - w^2] + \left(\frac{b}{2}-1\right)\frac{(1-s^2)^b}{w(s)^{1/2}}\, vw \\ &\quad + \frac{r^{b-a}}{\lambda}\,\frac{(1-s^2)^{2b}}{W(s)}\,\frac{d\tilde{U}}{ds}, \end{aligned} \tag{5.17}$$

and the energy relation (4.14) goes over to

$$w^2 - (1-s^2)^b + \frac{(1-s^2)^{2b}}{W(s)}\,[v^2 - 2r^{b-a}\tilde{U}(s) - 2hr^b] = 0. \tag{5.18}$$

We remark that the vector field (5.17) is of class C^1 only if $b>2$ and $b-a \geqslant 1$; if moreover we have $a, b \in \mathbb{N} \cup \{0\}$, then the system (5.17) is analytic. If $1<b<2$ and $b-a \geqslant 1$, we can only guarantee that the vector field (5.17) is continuous, so we do not have necessarily uniqueness of the solutions.

5.1. *The Special Case* $b=2$

For $b=2$, all points with coordinates $(0, v, \pm 1, 0)$ which correspond to double collisions are equilibrium points for the global flow given by (5.17). That is, in the special case $b=2$, when we regularize double collisions we are creating a large set of equilibrium points. So in this case we actually have not regularized the double collisions, i.e., an orbit which reaches double collisions dies at that point.

For $b=2$, using the energy relation, it is not difficult to check that from (5.16) the variable v' can be written as

$$v' = v^2 + u^2 - ar^{b-a}\tilde{U}(s) - 2\tilde{V}(s) = (2-a)\, r^{2-a}\phi(s)\ \tilde{U}(s) + 2hr^2\phi(s),$$

so in this particular case, in order to guarantee the differentiability of the function v' it is enough to define the function ϕ as

$$\phi(s) = \frac{(1-s^2)^{3/2}}{W(s)^{1/2}}. \tag{5.19}$$

The vector field (5.16) goes over to

$$\begin{aligned} r' &= \frac{(1-s^2)^{3/2}}{W(s)^{1/2}}\, rv, \\ v' &= \frac{(1-s^2)^{3/2}}{W(s)^{1/2}}\, [(2-a)\, r^{2-a}\tilde{U}(s) + 2hr^2], \\ s' &= \frac{1}{\lambda}\, w, \\ w' &= -\frac{s}{\lambda} + \frac{3s}{\lambda}\, \frac{(1-s^2)^2}{W(s)}\, [v^2 - 2\tilde{U}(s)\, r^{2-a} - 2hr^2] \\ &\quad + \frac{W'(s)}{2\lambda W(s)}\, [1 - s^2 - w^2] + \frac{r^{2-a}}{\lambda}\, \frac{d\tilde{U}}{ds}\, \frac{(1-s^2)^3}{W(s)}, \end{aligned} \tag{5.20}$$

and the energy relation takes the form

$$w^2 + \frac{(1-s^2)^3}{W(s)}\, v^2 = (1-s^2) + \frac{(1-s^2)}{W(s)}\, [2\tilde{U}(s)\, r^{2-a} + 2hr^2]. \tag{5.21}$$

The differentiability of the above vector field depends on the degree of homogeneity $-a$ of the function U. If $0 \leqslant a \leqslant \frac{1}{2}$, then the vector field is differentiable, but if $\frac{1}{2} < a \leqslant 1$, then the equations (5.20) are only continuous at $s = \pm 1$, and therefore we cannot guarantee the uniqueness of the orbits that pass across double collision. Let us observe that the important case $a = 1$ and $b = 2$, known as the Manev problem, corresponds to the last affirmation. We will come back to this case later in this paper.

6. GLOBAL FLOW ON THE TRIPLE COLLISION MANIFOLD

In Section 3 we defined the total collision manifold Λ. In our case we will call it, for simplicity, the *triple collision manifold*. In the regularized and reduced McGehee coordinates (r, v, s, w) this manifold will be further denoted by

$$\Lambda = \left\{ (r, v, s, w) \colon r = 0,\ w^2 - (1 - s^2)^b + \frac{(1 - s^2)^{2b}}{W(s)} v^2 = 0 \right\}. \tag{6.22}$$

We have seen that Λ is invariant under the flow given by (5.17). Taking $r = 0$ in that equations we get a vector field defined on Λ,

$$\begin{aligned} v' &= \left(\frac{b}{2} - 1\right) W(s)^{1/2} \left[\frac{(1 - s^2)^b}{W(s)} v^2 - 1 \right], \\ s' &= \frac{1}{\lambda} w, \\ w' &= -\frac{b}{\lambda} s(1 - s^2)^{b-1} + \frac{2}{\lambda} bs \frac{(1 - s^2)^{2b-1}}{W(s)} v^2 \\ &\quad + \frac{1}{2\lambda} \frac{W'(s)}{W(s)} [(1 - s^2)^b - w^2] + \left(\frac{b}{2} - 1\right) \frac{(1 - s^2)^b}{W(s)^{1/2}} vw. \end{aligned} \tag{6.23}$$

6.1. *Equilibrium Points*

For $b \neq 2$, the equilibrium points on Λ given by the vector field (6.23) have the form $(0, \pm v_c, s_c, w)$ with

$$v_c = \sqrt{2\tilde{V}(s_c)}, \qquad \frac{d\tilde{V}}{ds}(s_c) = 0, \qquad w = 0.$$

We can check that these are all the equilibrium points on Λ for the global flow given by (5.17). In other words, all equilibrium points of the equations (5.17) are the points above defined located on Λ. For $b = 2$, in addition to these equilibrium points, we also have all points $(0, v, \pm 1, 0)$ which correspond to double collisions.

By straightforward computations and using the fact that the vectors $\{\mathbf{s}_0, A\mathbf{s}_0\}$ form a basis for the plane Q, where $\mathbf{s}_0 = S(s_0)$, we can check that

$$\frac{d\tilde{V}}{ds}(s_0) = 0 \Leftrightarrow M^{-1}\nabla V(\mathbf{s}_0) = \mu\mathbf{s}_0. \tag{6.24}$$

So equilibrium points on the triple collision manifold Λ correspond to vectors $(\mathbf{q},\mathbf{p})$, in which the configuration vector $\mathbf{q}$ is parallel to the vector $M^{-1}\nabla V(\mathbf{q})$. In other words, at triple collision, the potential V predominates on the behavior. In the homogeneous case, that is, when $a = 0$, this corresponds to a central configuration. Since we will be interested in these kinds of configurations, we define:

Definition 6.1. A configuration $\mathbf{q}_0$ is called a quasi-central configuration if the vectors $\mathbf{q}_0$ and $M^{-1}\nabla V(\mathbf{q})$ are parallel.

Of course, as for the central configurations, these new configurations are considered equivalence classes with respect to rotations and homotheties. F. Diacu in his paper [2] takes the above definition to be like the definition of central configuration for the quasihomogeneous case. We consider that is convenient to take both degrees of homogeneity $-a$ and $-b$ of the potentials U and V at the same time in the definition of central configuration. (See Definition 2.1.)

Differentiating the function $\tilde{V}$ twice we get

$$\frac{d^2\tilde{V}}{ds^2}(s) = \lambda^2[AS(s)]^{\mathrm{T}} D^2V(S(s))[AS(s)] + b\lambda^2 V(S(s)).$$

This function is positive because D^2V is positive definite, and therefore the function $\tilde{V}$ is a convex function with $\lim_{s\to\pm 1}\tilde{V} = +\infty$. Then we can affirm that $\tilde{V}$ has a unique critical point on the interval $[-1, 1]$, and, therefore, there exists only one quasi-central configuration in the quasihomogeneous collinear three-body problem. (See Fig. 1.)

6.2. *Linearization Around the Equilibrium Points*

Consider the vector field given by (5.17), the energy relation (5.18), and the equilibrium point $A_0 = (0, -v_c, s_c, 0)$, where $v_c = \sqrt{2\tilde{V}(s_c)}$ and $(d\tilde{V}/ds)(s_c) = 0$. The analysis of the Jacobian matrix at the equilibrium point A_0 depends on the exponent of the coordinate r. On this basis we have divided the analysis into three cases.

Case I. $b > 1$ and $b - a > 1$.

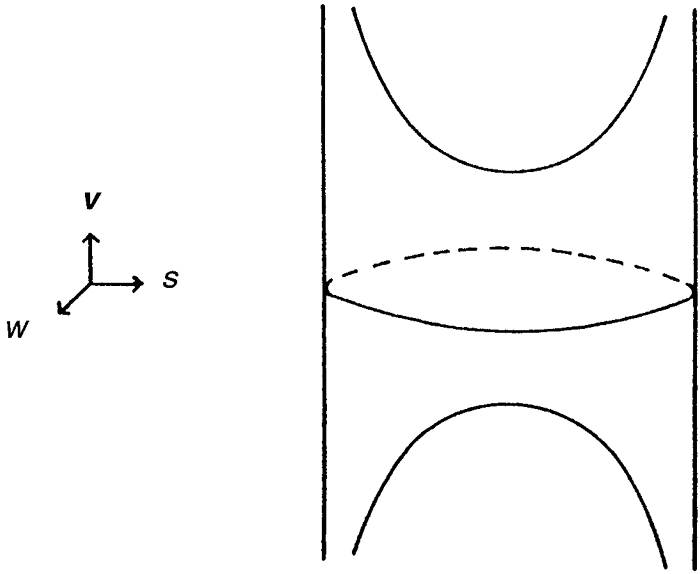

FIG. 1. The triple collision manifold.

In this case, when we calculate the Jacobian matrix, it is possible to eliminate all terms which have as factor some power of the variable r, getting the following expression for the Jacobian matrix:

$$\begin{pmatrix} -(1-s_c^2)^{b/2} & 0 & 0 & 0 \\ 0 & -(b-2)(1-s_c^2)^{b/2} & 0 & 0 \\ 0 & 0 & 0 & 1/\lambda \\ 0 & -\dfrac{4bs_c(1-s_c^2)^{b-1}}{\lambda v_c} & (1-s_c^2)^b \dfrac{\tilde{V}''(s_c)}{\lambda v_c^2} & \left(1-\dfrac{b}{2}\right)(1-s_c^2)^{b/2} \end{pmatrix}.$$

From the energy relation (5.18), we have that the level of energy h is given by

$$F(r, v, s, w) = w^2 - (1-s^2)^b + \frac{(1-s^2)^b}{2\tilde{V}(s)}\left[v^2 - 2r^{b-a}\tilde{U}(s) - 2hr^b\right] = 0.$$

Then the tangent space of this manifold at A_0 is

$$T_{A_0}F = \{(\rho, \gamma, \sigma, \chi) : \nabla F(A_0) \cdot (\rho, \gamma, \sigma, \chi) = 0\} = \{(\rho, \gamma, \sigma, \chi) : \gamma = 0\}.$$

The linear part of the vector field (5.17) restricted to the space $T_{A_0}F$ is given by

$$\bar{J}=\begin{pmatrix} -(1-s_c^2)^{b/2} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\lambda \\ 0 & 0 & (1-s_c^2)^b \dfrac{\widetilde{V}''(s_c)}{\lambda v_c^2} & \left(1-\dfrac{b}{2}\right)(1-s_c^2)^{h/2} \end{pmatrix}.$$

A basis for $T_{A_0}F$ is formed by the vectors

$$\xi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\qquad \xi_2=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\qquad \xi_3=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

and a representative of $\bar{J}$ in this basis is the matrix

$$J^*=\begin{pmatrix} (-1-s_c^2)^{b/2} & 0 & 0 \\ 0 & 0 & 1/\lambda \\ 0 & (1-s_c^2)^b \dfrac{\widetilde{V}''(s_c)}{\lambda v_c^2} & \left(1-\dfrac{b}{2}\right)(1-s_c^2)^{b/2} \end{pmatrix}.$$

Thus ξ_1 is an eigenvector with eigenvalue $-(1-s_c^2)^{b/2}<0$. The other two eigenvalues are computed by solving the quadratic polynomial

$$P(m)=m^2+\left(\frac{b}{2}-1\right)(1-s_c^2)^{b/2}\,m-(1-s_c^2)^b\,\frac{\widetilde{V}''(s_c)}{\lambda^2 v_c^2}=0,$$

which has one positive and one negative root. Therefore, we can conclude that the equilibrium point A_0 is hyperbolic. Let B_0 be the equilibrium point $B_0=(0, v_c, s_c, 0)$. We can repeat the same analysis for the Jacobian matrix at B_0, getting that B_0 is also hyperbolic with two positive eigenvalues and one negative eigenvalue.

Case II. $b>1$ and $b-a=1$.

In this case, when we derive r^{b-a}, it is not possible to eliminate the corresponding term, and then the tangent space to the manifold corresponding to the level of energy h at the equilibrium point A_0 is

$$T_{A_0}F=\{(\rho, \gamma, \sigma, \chi)\colon \widetilde{U}(s_c)\,\rho+v_c\gamma=0\}.$$

As in Case I, this tangent vector space has dimension 3; a basis for T_{A_0} is given by

$$\xi_1 = \begin{pmatrix} -v_c \\ \widetilde{U}(s_c) \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and a representative of the linear part of (5.17) in this basis is

$$J^* = \begin{pmatrix} -(1-s_c^2)^{b/2} & 0 & 0 \\ 0 & 0 & 1/\lambda \\ -(1-s_c^2)^{b/2} \dfrac{\widetilde{U}(s_c)}{\lambda v_c^2} & (1-s_c^2)^b \dfrac{\widetilde{V}''(s_c)}{\lambda v_c^2} & \left(1-\dfrac{b}{2}\right)(1-s_c^2)^{b/2} \end{pmatrix}.$$

The eigenvalues of this matrix are $-(1-s_c^2)^{b/2}$ and the roots of

$$P(m) = m^2 + \left(\frac{b}{2} - 1\right)(1-s_c^2)^{b/2}\, m - (1-s_c^2)^b \frac{\widetilde{V}''(s_c)}{\lambda^2 v_c^2} = 0,$$

which is the same quadratic polynomial as in the above case, and therefore we have the same conclusions: The equilibrium point A_0 is hyperbolic, with two negative eigenvalues and one positive eigenvalue, for the equilibrium point B_0 we have one negative eigenvalue and two positive eigenvalues.

Case III. $a = 0$ and $b = 1$.

This case corresponds to the classical Newtonian case. Here we have that the potential $\widetilde{U}$ is a constant. The tangent space for the level of energy h at the equilibrium point A_0 is

$$T_{A_0}F = \{(\rho, \gamma, \sigma, \chi) \colon (\widetilde{U}(s_c) + h)\, \rho + v_c \gamma = 0\}.$$

In this case we work with the new energy $h' = \widetilde{U}(s_c) + h$, and then from the above expression $\gamma = -h'\rho/v_c$ on T_{A_0}, a basis for this vector space is formed by the vectors

$$\xi_1 = \begin{pmatrix} -v_c \\ h' \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and the respective matrix J^* can be written as

$$J^* = \begin{pmatrix} -(1-s_c^2)^{1/2} & 0 & 0 \\ 0 & 0 & 1/\lambda \\ 0 & (1-s_c^2)\,\dfrac{\tilde{V}''(s_c)}{\lambda v_c^2} & \dfrac{1}{2}(1-s_c^2)^{1/2} \end{pmatrix}.$$

Like in the two previous cases, the equilibrium point A_0 is hyperbolic, with two negative eigenvalues and one positive eigenvalue.

Let $W^{s(u)}_{A_0(B_0)}$ be the stable (unstable) submanifold associated to the equilibrium point $A_0(B_0)$. We summarize the previous analysis of the equilibrium points A_0 and B_0 in the following result.

THEOREM 6.1. *The equilibrium points A_0 and B_0 for the global flow of the quasihomogeneous collinear three-body problem are hyperbolic, both of them are on Λ, and*

$$\dim W^s_{A_0} = 2 \qquad \dim W^s_{B_0} = 1$$
$$\dim W^u_{A_0} = 1 \qquad \dim W^u_{B_0} = 2.$$

6.3. *Global Flow on Λ*

In order to describe the flow on Λ we start by giving a characteristic property of the flow on Λ. In this way we define

DEFINITION 6.2. A flow is called gradient-like with respect to one of its coordinates if this coordinate increases along every nonequilibrium solution.

With this definition we can now prove that, for the quasihomogeneous collinear three-body problem, the coordinate v plays a very important role if $b \neq 2$.

PROPOSITION 6.3. *If $b \neq 2$, then the flow on Λ is gradient-like with respect to the v coordinate.*

Proof. From the equations (6.23) and using (6.22) we get that for $s \neq \pm 1$

$$v' = \left(1 - \frac{b}{2}\right) W(s)^{1/2} \frac{w^2}{(1-s^2)^b}.$$

Then if $w \neq 0$ we have

$$\begin{array}{lll} \text{if} \quad 1 \leqslant b < 2, & \text{then} & v \text{ is increasing,} \\ \text{if} \quad b = 2, & \text{then} & v \text{ is constant, and} \\ \text{if} \quad b > 2, & \text{then} & v \text{ is decreasing.} \end{array}$$

For $w = 0$ we have that

$$v'' = 0, \qquad \text{and} \qquad v''' = (2-b)\,\frac{W(s)^{1/2}}{(1-s^2)^b}\,(w')^2,$$

but w' can be written as

$$w' = \left(\frac{v^2}{\tilde{V}(s)} - 2\right) bs(1-s^2)^{b-1} + \frac{1}{2}\,(1-s^2)\,\frac{d\tilde{V}/ds}{\tilde{V}(s)},$$

and again from (6.22) we get that $v^2 = 2\tilde{V}(s)$ if $w = 0$, then in this case we have that $v''' = 0$ iff $d\tilde{V}/ds = 0$, which correspond to equilibrium points.

When $s = \pm 1$, then we know that

$$v' = \left(1 - \frac{b}{2}\right) W(s)^{1/2},$$

getting the same results.

Therefore, we have proved that the nonequilibrium solutions increase along the coordinate v for $1 \leqslant b < 2$, and decrease if $b > 2$. This proves that the flow on Λ is gradient-like with respect to v if $1 \leqslant b < 2$ and with respect to $-v$ if $b > 2$. (See Fig. 2.)

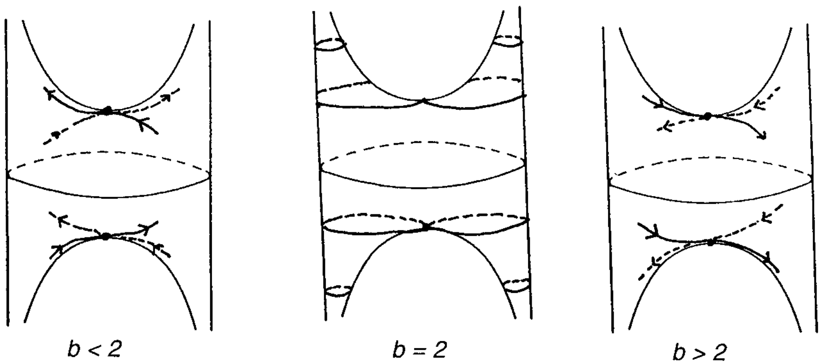

FIG. 2. Global flow on Λ.

7. THE SET OF TRIPLE COLLISION–EJECTION ORBITS

In this section we will describe the set of all orbits which end or start at triple collisions. We will prove that for $1 \leqslant b < 2$, the set of orbits ending in triple collision for the quasihomogeneous collinear three-body problem corresponds to the stable manifold associated to the equilibrium point A_0. That is, in this case the collision orbits tend asymptotically to form a quasi-central configuration. Surprisingly this is not true for $b > 2$, where we will find a large set of triple collision orbits which tend to Λ, but far of the stable manifold associated to A_0. For $b = 2$ the regularization of double collisions is a big obstacle; in this case we are not able to show the existence of a positive measure set of triple collision orbits.

We being remarking that the global flow given by (5.17) is reversible; that is, if $\phi(\sigma) = (r, v, s, w)(\sigma)$ is a solution of the system (5.17), then $\bar{\phi}(\sigma) = (r, -v, s, -w)(-\sigma)$ is also a solution of (5.17). So, using this property, it is enough to study the orbits ending in triple collision, the ejection orbits are obtained by the reversibility of the flow.

We have seen in the last section that the flow on Λ depends on whether $1 \leqslant b < 2$, $b = 2$, or $b > 2$. Here, we will show that the same is true for the global flow given by (5.17).

7.1. *Case* $1 \leqslant b < 2$

Let us remember that in this case, the flow on Λ is gradient-like with respect to v. That is, the coordinate v is increasing along the nonequilibrium solutions. Now applying the theorem of continuity with respect to initial conditions, we obtain that the same is true for the orbits that are close enough to the triple collision manifold Λ. That is, for r small enough and $1 \leqslant b < 2$, the global flow of the quasihomogeneous collinear three-body problem is gradient-like with respect to the coordinate v. With this remark, we can prove the following result:

THEOREM 7.1. *Let C be the set of collision orbits in the quasihomogeneous collinear three-body problem. For $1 \leqslant b < 2$, any orbit in C reaches the triple collision manifold Λ along the stable submanifold associated to the equilibrium point A_0. The Lebesgue measure of the set C is* 0.

Proof. Let $\gamma(\sigma) = (r(\sigma), v(\sigma), s(\sigma), w(\sigma))$ be an orbit of the global flow given by (5.17), such that $\gamma(0) \notin \Lambda$ and $r(\sigma) \to 0$ as $\sigma \to \infty$. That is, $\gamma(\sigma) \to \Lambda$ as $\sigma \to \infty$, and therefore it is a triple collision orbit. We will prove that $\gamma(\sigma) \to A_0$. The proof is by contradiction: Suppose that $\gamma(\sigma)$ does not converge to any point on Λ. We affirm that, there exists $\sigma_1 > 0$ such that $v(\sigma_1) > 0$, because if $v(\sigma) \leqslant 0$ $\forall \sigma$, we can choose $T > 0$ such that $r(\sigma) < \varepsilon$ $\forall \sigma > T$, since this is a collision orbit and we know that the global flow is gradient-like with respect to v. With this hypothesis let

$$\alpha = \mathrm{Sup}\{v(\sigma): \sigma \geqslant T\}.$$

If $\lim_{\sigma\to\infty} v(\sigma)=\alpha$, then $\gamma(\sigma)$ should converge to a periodic orbit on Λ, since the functions $s(\sigma)$ and $w(\sigma)$ are bounded and $r(\sigma)\to 0$. This a contradiction, because the flow on Λ is gradient-like with respect to v, so on Λ there are no periodic orbits. Therefore, there exists $\sigma^* > T > 0$ such that $v(\sigma^*)=\alpha$. But then $v(\sigma) > v(\sigma^*)=\alpha$ for $\sigma > \sigma^*$, because the global flow is also gradient-like with respect to v if $r(\sigma)$ is small enough, which is a contradiction. This proves the affirmation.

Then there exists $\sigma_1 > T$ such that $v(\sigma_1) > 0$, and using again the fact that the global flow is gradient-like with respect to v for r small enough, we have that $v(\sigma) > 0\ \forall \sigma > \sigma_1$, and then from the equations (5.17) we get that

$$r'(\sigma) = r(\sigma)\, v(\sigma)\, \phi(s(\sigma)) > 0 \qquad \forall \sigma > \sigma_1,$$

but this is false, because our initial hypothesis was that $r(\sigma)\to 0$.

In this way we have proved that $\gamma(\sigma)$ converges to a point on Λ. This implies that $\gamma(\sigma) \subset W^s_{A_0} \cup W^s_{B_0}$ but $W^s_{B_0} \subset \Lambda$, this manifold is invariant under the flow, and $\gamma(0 \notin \Lambda$. Therefore $\gamma(\sigma) \subset W^s_{A_0}$, which proves the first par of the theorem $\gamma(\sigma) \to A_0$ as $\sigma \to \infty$.

The second part follows from the fact that $C \subset W^s_{A_0}$, and this is a 2-dimensional manifold, while the phase space has dimension 3.

Remark. Let E be the set of ejection orbits in the quasihomogeneous collinear three-body problem. Using the reversibility of the flow we can prove that the Lebesgue measure of the set E is zero, that is, $\mu(E)=0$, and therefore $\mu(C \cup E)=0$.

Another important consequence of the above theorem is that, for $1 \leqslant b < 2$, all collision or ejection orbits tend asymptotically to form a quasi-central configuration. This result was proved previously by F. Diacu [2], using the ideas of McGehee [5]. The proof presented here is simpler.

7.2. *Case* $b=2$

In this case, from the equations (5.17) we get that $v'=0$. Let us remember that here, using the classical regularization, we have created a large number of equilibrium points on Λ, corresponding to double collisions. In this case, all the orbits on Λ are homoclinic or heteroclinic. An orbit tending to Λ can die in double collision.

In order to obtain more information on the global flow close to Λ, we have used a new function $\phi(s)$ given by (5.19), the corresponding vector field (5.20) is only continuous at $s = \pm 1$. However, the advantage of this regularization is that the equilibrium points corresponding to double collisions have disappeared, and the vector field (5.20) has only the equilibrium points A_0 and B_0. In this case, the triple collision manifold is foliated by periodic orbits, except for the values of $v = \pm\sqrt{2\tilde{V}(s_c)}$, which correspond to a couple of homoclinic orbits. As in the previous case, let C and E be the sets of collision and ejection orbits, respectively, in the quasihomogeneous collinear three-body problem. We will prove the following result.

PROPOSITION 7.2. *For $b = 2$, any orbit in C tends to an periodic orbit on Λ, or to the corresponding equilibrium point, or to one of the two homoclinic orbits with $v < 0$.*

Proof. First, let us observe that from the equations (5.20) we have

$$v' = \frac{(1-s^2)^{3/2}}{W(s)^{1/2}} \left[(2-a)\, r^{2-a}\tilde{U}(s) + 2hr^2\right].$$

Since $a \leqslant 1$ the global flow outside Λ is gradient-like with respect to the coordinate v for r small enough regardless of the sign of h.

Now we consider a collision orbit $\gamma(\sigma) = (r(\sigma), v(\sigma), s(\sigma), w(\sigma))$ that is, $\lim_{\sigma \to \infty} r(\sigma) = 0$. Again, from (5.20), we know that

$$r' = \frac{(1-s^2)^{3/2}}{W(s)^{1/2}}\, rv; \tag{7.25}$$

that is, r' has the same sign as v. Thus by the gradient-like condition, in order to have a collision orbit, the coordinate v must satisfy $v(\sigma) \leqslant 0\ \forall \sigma > T$. Then, since all coordinates along the orbit γ are bounded, the ω-limit of γ is a compact set, and by the gradient-like condition of the flow it must be an equilibrium point, or a periodic orbit, or one of the two homoclinic orbits with $v < 0$.

Let us observe here that we can have similar initial conditions with $v < 0$ and such that the corresponding orbit of the first one goes to collision, but the orbit of the second one does not. Remember from Eq. (7.25) that if for a positive time T we have $v(T) = 0$, then, for all $t > T$, $v(t) > 0$ by the gradient-like property of the flow with respect to the coordinate v; hence this orbit is not a collision orbit.

So in the case $b = 2$, at least using the techniques shown in this paper, we cannot prove the existence of a set of collision-ejected orbits of positive measure.

7.3. *Case $b > 2$*

In the case $b > 2$, the flow on the total collision manifold Λ is gradient-like with respect to the function $-v$, so the v coordinate decreases on Λ along nonequilibrium solutions. This will imply really different behavior for the collision orbits. First we will prove the following result.

THEOREM 7.3. *Let $b > 2$. If $\gamma(\sigma)$ is an orbit of the global flow of the quasi-homogeneous collinear three-body problem, which satisfies $r(\sigma_0)$ is small enough, and $v(\sigma_0) < 0$ for some time σ_0, then $\gamma(\sigma)$ is a triple collision orbit.*

Proof. Let us remember the Lagrange–Jacobi equation (2.5)

$$\ddot{I} = (2-a)\,U(\mathbf{q}) + (2-b)\,V(\mathbf{q}) + 2h.$$

From this equation we know that the moment of inertia I along $\gamma(\sigma)$ is concave down, because this orbit is close to triple collision. Using the works of Painlevé [7] and von Zeipel [6, 9], it is not difficult to prove that the orbit $\gamma(\sigma)$ only has singularities due to the collision of at least two particles, because along this orbit the positions of all the particles remain bounded. That is, in this problem we cannot have a non-collision singularity, and thus the function I along $\gamma(\sigma)$ is bounded. So, if we prove that the moment of inertia I is a decreasing function of real time along γ, then this orbit will necessarily be a triple collision orbit.

We know from the McGehee coordinates definition that

$$r = \sqrt{I}, \qquad v = r^{b/2}\dot{r}, \qquad \text{and} \qquad \frac{dt}{d\tau} = r^{b/2+1},$$

Thus, in the new time τ, we have $r' = rv$.

Remember also that we have regularized binary collisions using a positive function $\phi(s)$ and a new time $d\tau/d\sigma = \phi(s)$, giving $r' = \phi(s)\,rv$. Therefore, $\dot{r}$ and v have the same sign. Now, since

$$\dot{r} = \frac{1}{2\sqrt{I}}\,\dot{I},$$

we have that

$$v < 0 \Leftrightarrow \dot{I} < 0.$$

We shall now show that, on $\gamma(\sigma)$, the coordinate v remains negative after σ_0. The initial condition with $r(\sigma_0) \approx 0$ and $v(\sigma_0) < 0$ and the continuity of the flow with respect to initial conditions imply that we have $v' \leqslant 0$ for a finite time interval $[\sigma_0, \sigma_1]$; therefore

$$v(\sigma) \leqslant v(\sigma_0) < 0, \qquad \sigma \in [\sigma_0, \sigma_1].$$

Now observe that on this time interval $r'(\sigma) < 0$, and thus $r(\sigma) < r(\sigma_0)$, which implies that on this interval the orbit becomes closer and closer to Λ.

Thus we must have $v < 0$ for $\sigma > \sigma_0$. And for this orbit, we have $\dot{I} < 0$ in real time. Since I is a non-negative, concave-down, and decreasing function along $\gamma(\sigma)$, it must have a root corresponding to a triple collision; that is, $\gamma(\sigma)$ is a triple-collision orbit.

This theorem has really important consequences. The main one is given in the following result, which establishes that the set of triple collision orbits far from a quasi-central configuration is quite huge.

THEOREM 7.4. *For $b > 2$ the Lebesgue measure of the set of triple collision orbits C, has positive measure; moreover, $\mu(C) = \infty$.*

Proof. Let us remember that the coordinates of the equilibrium point A_0 are given by

$$A_0 = (0, -v_c, s_c, 0), \qquad \text{where} \quad v_c = \sqrt{2\tilde{V}(s_c)}, \qquad \text{and} \qquad \frac{d\tilde{V}}{ds}(s_c) = 0.$$

Let r_0^* be small enough, given by Theorem 7.3; that is, if $r_0 < r_0^*$ and $v_0 < 0$, the corresponding orbit is a triple collision orbit. Then the set

$$\Omega = \{(r, v, s, w) \colon 0 < r < r_0, v < -v_c\}$$

is an unbounded open set in the phase space of the vector field (5.17), and by the theorem, any point in Ω corresponds to a triple collision orbit.

Remark 1. The orbits with initial conditions in Ω spiral the left branch of Λ or the right branch of Λ depending on whether the respective initial condition is $s_0 < s_c$ or $s_0 < s_c$. (See Fig. 3.)

Physically, this means that the triple collision orbits with initial data in Ω tend to triple collision with two particles colliding infinitely many times. The particles which are colliding with each other never collide with the third particle, whereas the third particle is closer and closer to them. The fact that the middle particle is colliding with the left or the right particle depends on the position of the respective initial data with respect to the quasi-central configuration. This means that we have total control of the dynamics of the triple collision orbits starting in a large set of initial data.

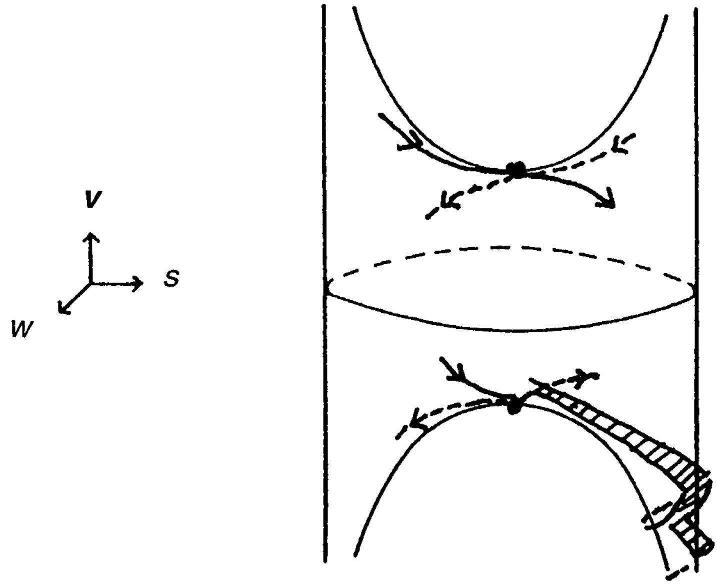

FIG. 3. Triple collision orbits for $b > 2$.

Remark 2. Since the equilibrium point A_0 is hyperbolic and the stable submanifold associated to it has dimension 2, the set of initial conditions getting triple collision asymptotically to a quasi-central configuration has Lebesgue measure zero.

Remark 3. A similar result corresponding to the set of ejection orbits E is obtained applying the reversibility of the flow.

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