Synchronization law for a Van der Pol array

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We explore the transition to in-phase synchronization in globally coupled oscillator arrays, and compare results for van der Pol arrays with Josephson junction arrays. Our approach yields in each case an analytically tractable iterative map; the resulting stability formulas are simple because the expansion procedure identifies natural parameter groups. A third example, an array of Duffing-Van der Pol oscillators, is found to be of the same fundamental type as the van der Pol arrays, but the Josephson arrays are fundamentally different owing to the absence of self-resonant interactions.

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I. INTRODUCTION

The study of globally coupled oscillators arises in various areas of physics [1, 2], including optics [3], superconducting electronics [4], and mechanics [5]. They are also used in modeling biological systems such as the brain [6, 7] and firefly populations [8]. Globally coupled oscillators are of some interest from a general theoretical perspective as well: the all-to-all coupling endows the dynamics with a symmetry which makes its analysis unusually tractable.

In this paper, we explore an analytic approach that has recently led to dramatic progress in the study of Josephson junction arrays [9, 10]. Traditionally, Josephson arrays are treated as belonging to two separate classes, depending on whether or not the junctions have negligible capacitance. The new analysis derives accurate stability conditions for both classes, expanding the success of other methods for zero capacitance junctions [11, 12]. In addition the method leads to surprisingly simple expressions for the transition boundary. In effect, the structure of the calculation identifies natural parameter groups, so that an apparently opaque formula in terms of the original system parameters is made transparent. The result is a unified stability formula for the two classes of Josephson arrays.

We examine whether this success can be extended to other oscillator arrays. We apply the method to a globally coupled array of van der Pol oscillators. The result for the stability boundary of the in-phase state is indeed simple and in agreement with numerical simulations; nevertheless, its structure is fundamentally different than the corresponding Josephson one. We identify the source of the distinction as a *self-resonant interaction* which is absent in the Josephson problem. We also analyze an array of Duffing-Van der Pol oscillators, which has self-resonant interactions, and get results virtually identical to the van der Pol array.

II. VAN DER POL OSCILLATORS

Balthazar Van der Pol originally derived his equation to describe the dynamics of an electronic valve oscillator implemented with a triode vacuum tube [13]. Although this particular technology has long since lost its relevance, the van der Pol equation has not. Besides its status as a fundamental nonlinear oscillator, it is widely used to model the behavior of a number of different systems in various areas of science. Van der Pol himself suggested that this oscillator can be used to model the beating of the human heart [14] and since then it has been a favorite choice in modeling biological phenomena [15]. In a very recent example [18], a variant of van der Pol equation (known as FitzHugh-Nagumo equation [16, 17]) has been used to describe a behavior of synaptically coupled neurons. Meanwhile, the most common use of the van der Pol oscillator is in engineering, where it is used extensively, for instance in the study of vibrations [1, 19].

In this section we study synchronization in an array of globally coupled Van der Pol oscillators. We solve the equations of motion perturbatively and use this scheme to derive an iterated map. A synchronized solution is a fixed point of that map, and by studying the stability of the fixed point we derive an analytic formula for the stability of in-phase state.

We assume that the oscillators are identical and globally coupled by a passive linear load typified by an inductor-resistor-capacitor combination (Fig. 1). The equations of motion for such a system take the form:

$$\ddot{v}_k + \epsilon(1 - v_k^2)\dot{v}_k + v_k = \dot{q} \tag{1}$$

$$\mu_1\ddot{q} + \mu_2\dot{q} + q = \frac{\eta}{N}\sum_{j=1}^{N} v_j \tag{2}$$

We assume that ϵ and η are small parameters of the same order and $\eta = \epsilon\kappa$, where $\kappa \sim O(1)$. For a concrete physical picture, we can think of v_k as the voltage across the

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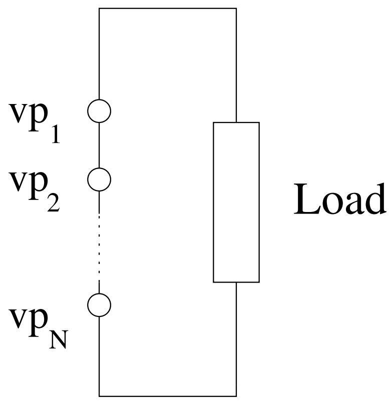

FIG. 1: A global coupling scheme for a series array of N Van der Pol oscillators (vp). The coupling is weak and the load can be described by linear equations of motion.

k^{th} oscillator and q as the charge on the coupling capacitor; the parameters μ_1 and μ_2 are then proportional to the coupling inductance and resistance, respectively. This is by no means a general way to couple oscillators globally. Various coupling schemes have been proposed in literature (see e.g. Ref. [2] and references therein).

It is convenient to rewrite Equation (1) by introducing polar coordinates:

$$\begin{aligned} v_k &= r_k \cos\theta_k \\ \dot{v}_k &= -r_k \sin\theta_k \end{aligned}$$

so instead of N second order equations (1) we get $2N$ first order equations

$$\begin{aligned} \dot{r}_k &= \epsilon\frac{r_k}{2}\left(1-\frac{r_k^2}{4}\right) - \epsilon\frac{r_k}{2}\cos(2\theta_k) \\ &+\epsilon\frac{r_k^3}{8}\cos(4\theta_k) - \sin(\theta_k)\dot{q} \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{\theta}_k &= 1+\frac{\epsilon}{2}\left(1-\frac{r_k^2}{2}\right)\sin(2\theta_k) \\ &-\epsilon\frac{r_k^2}{8}\sin(4\theta_k) - \frac{\cos(\theta_k)}{r_k}\dot{q} \end{aligned} \tag{4}$$

We expand solutions of this system in terms of ϵ: $r_k = r_k^{(0)} + \epsilon r_k^{(1)} + \epsilon^2 r_k^{(2)} + \cdots$, $\theta_k = \theta_k^{(0)} + \epsilon\theta_k^{(1)} + \epsilon^2\theta_k^{(2)} + \cdots$, and $q = q^{(0)} + \epsilon q^{(1)} + \epsilon^2 q^{(2)} + \cdots$. Convergence of this expansion at time scale of order one is guaranteed by the Poincaré expansion theorem [20]. We need accuracy to this order (and not for large t) in order to construct an iterative map. The zeroth order steady state solution is:

$$r_k^{(0)} = r_{0k}, \quad \theta_k^{(0)} = t + \theta_{0k}, \quad \text{and } q^{(0)} = 0. \tag{5}$$

where r_{0k} and θ_{0k} are integration constants. For simplicity we drop the transient part of the solution $q^{(0)}$ as we are interested only in the stability of steady state, in-phase solution.

To first order in ϵ we find from (2)

$$q^{(1)} = \frac{\kappa}{NZ}\sum_j r_{0j}\cos(t+\theta_{0j}-\delta) \tag{6}$$

and when we substitute this into the expressions for $r_k^{(1)}$ and $\theta_k^{(1)}$ we obtain:

$$\begin{aligned} \dot{r}_k^{(1)} &= \frac{r_{0k}}{2}\left(1-\frac{r_{0k}^2}{4}\right) \\ &+\frac{\kappa}{2NZ}\sum_j r_{0j}\cos(\theta_{0k}-\theta_{0j}+\delta) + P_{2\pi} \end{aligned} \tag{7}$$

$$\dot{\theta}_k^{(1)} = \frac{\kappa}{2NZ}\sum_j \frac{r_{0j}}{r_{0k}}\sin(\theta_{0j}-\theta_{0k}-\delta) + P_{2\pi} \tag{8}$$

where we use the symbol $P_{2\pi}$ to denote terms which are 2π-periodic; as we shall see, we don't need to be more specific. The constants Z and δ can be interpreted as the impedance and phase shift of the (linear) load:

$$Z = \sqrt{(1-\mu_1)^2 + \mu_2^2}, \qquad \delta = \arctan\frac{\mu_2}{1-\mu_1} \tag{9}$$

Equations (7, 8) can be integrated directly, so the solution through first order in ϵ is:

$$\begin{aligned} r_k(t) &= r_{0k} + \epsilon\pi\frac{r_{0k}}{2}\left(1-\frac{r_{0k}^2}{4}\right)t \\ &+\epsilon\pi\frac{\kappa}{2NZ}\sum_j r_{0j}\cos(\theta_{0k}-\theta_{0j}+\delta)t \\ &+\epsilon P_{2\pi} + O(\epsilon^2) \end{aligned} \tag{10}$$

$$\begin{aligned} \theta_k(t) &= \theta_{0k} + t + \epsilon\pi\frac{\kappa}{2NZ}\sum_j \frac{r_{0j}}{r_{0k}}\sin(\theta_{0j}-\theta_{0k}-\delta)t \\ &+\epsilon P_{2\pi} + O(\epsilon^2) \end{aligned} \tag{11}$$

Since $r_{0k} = r_k(0) + O(\epsilon)$ and $\theta_{0k} = \theta_k(0) + O(\epsilon)$ we may use this solution to construct a $2N$-dimensional map, which is correct up to order ϵ^2. An iteration of the map corresponds to a translation in time from $t=0$ to some $t=T$. Choosing $T = 2\pi + O(\epsilon)$, the map assumes the form:

$$\begin{aligned} r_k(T) &= r_k(0) + \epsilon\pi r_k(0)\left(1-\frac{r_k(0)^2}{4}\right) \\ &+\epsilon\pi\frac{\kappa}{NZ}\sum_j r_j(0)\cos(\theta_k(0)-\theta_j(0)+\delta) \\ &+O(\epsilon^2) \end{aligned} \tag{12}$$

$$\begin{aligned} \theta_k(T) &= \theta_k(0) + 2\pi \\ &+\epsilon\pi\frac{\kappa}{NZ}\sum_j \frac{r_j(0)}{r_k(0)}\sin(\theta_j(0)-\theta_k(0)-\delta) \\ &+O(\epsilon^2) \end{aligned} \tag{13}$$

Note that to this order $r_k(T)$ and $\theta_k(T)$ are functions of $\{r_j(0)\}$ and $\{\theta_j(0)\}$, but do not depend on $q(0)$ or $\dot{q}(0)$. We therefore drop equations for the map $q(0) \to q(T)$ and $\dot{q}(0) \to \dot{q}(T)$ from further consideration.

To make further analytic progress, we want to choose the period of strobing carefully. If $T = 2\pi + \epsilon\pi\kappa/Z \sin\delta + O(\epsilon^2)$, then the in-phase synchronized state is a fixed point of the map. Setting $r_k = r$ and $\theta_k = \theta$ for all $k = 1, \ldots, N$, we find that

$$r = 2\sqrt{1 + \kappa/Z \cos\delta} \tag{14}$$

while θ can assume any value on $[0, 2\pi]$. (The freedom in θ does not reflect the existence of a family of periodic solutions; rather, it amounts to fixing the value of the phase at time $t = 0$.) The next step is to determine the stability of the fixed point. We calculate the Jacobi matrix of the map and find its eigenvalues at this point. The eigenvalues are readily found to be, up to $O(\epsilon^2)$:

$$\lambda_1 = 1, \quad \lambda_2 = 1 - 2\pi\epsilon\left(1 + \frac{\kappa}{Z}\cos\delta\right) \tag{15}$$

and

$$\begin{aligned} \lambda_{i,i+1} &= 1 - \pi\epsilon\left(1 + 2\frac{\kappa}{Z}\cos\delta\right) \\ &\pm\pi\epsilon\sqrt{\left(1 + \frac{\kappa}{Z}\cos\delta\right)^2 - \left(\frac{\kappa}{Z}\sin\delta\right)^2} \end{aligned} \tag{16}$$

with $i = 3, 5, \ldots, 2N-1$. The unity eigenvalue λ_1 corresponds to the requirement that the orbit is neutrally stable with respect to perturbations along the trajectory in phase space. The synchronized solution is stable when all the other eigenvalues of the map have magnitude smaller than one.

Note that the expressions for the eigenvalues are independent of the parameter N. This is a direct consequence of the way we have scaled the parameters in the governing Eqs.(1-2). It follows that the change of stability of the in-phase state will occur for the same parameter values regardless of the number of coupled oscillators.

To check our results we calculate numerically the Floquet multiplier of the synchronized solution of the original nonlinear system Eqs. (1, 2). In Fig. 2 we plot the numerically determined Floquet multiplier of the in-phase state as a function of the load parameter μ_1, keeping the other parameters fixed, and compare it with the largest eigenvalue determined from Eqs.(15,16). The difference between the two is of order $O(\epsilon^2)$, so there is agreement within the expected error.

We can write down a condition for the stability boundary of the in-phase state as follows. We first note that eigenvalue λ_2 corresponds to perturbations within the symmetric sub-space, while the degenerate eigenvalues Eq. (16) corresponds to perturbations which introduce phase differences. We get the condition for a symmetry breaking instability by setting the modulus of the latter equal to one, which yields (after some algebra):

$$2\frac{\kappa}{Z}\cos^2\delta + 2\cos\delta + \frac{\kappa}{Z} = 0 \tag{17}$$

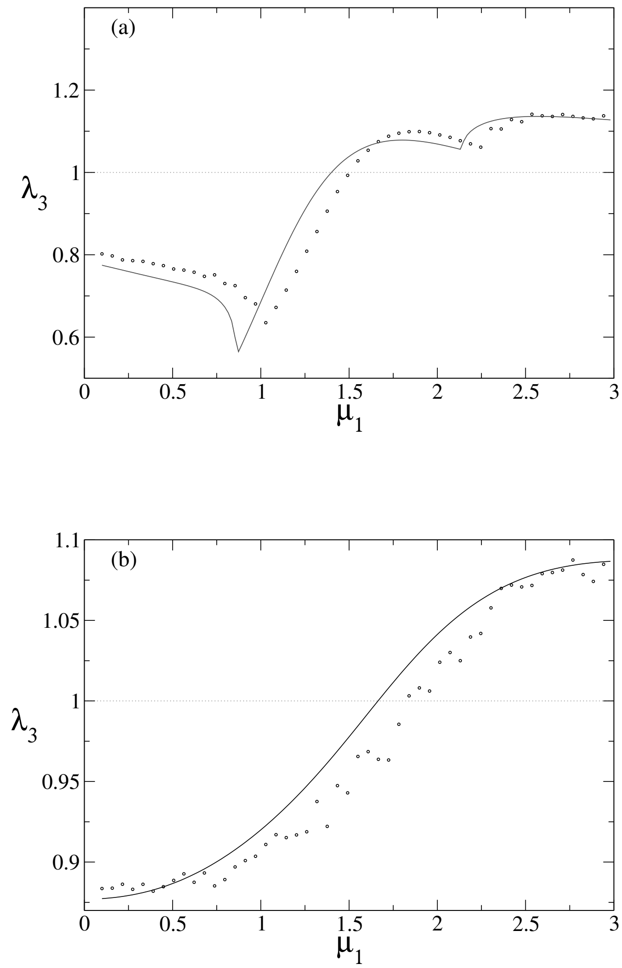

FIG. 2: An in-phase state becomes unstable as the modulus of the leading eigenvalue becomes larger than one. Dots represent a numerical estimate of the eigenvalue, while the solid line is its approximate analytical value (Eq. 16). Parameters are set to $\epsilon = 0.1$, $\kappa = 1$ and (a) $\mu_2 = 0.8$, (b) $\mu_2 = 1.5$.

if the eigenvalue is real, and

$$1 + 2\frac{\kappa}{Z}\cos^2\delta = 0 \tag{18}$$

if the eigenvalue is complex.

III. DISCUSSION

This same approach can be used to study different physical problems. As a first example we consider a simple variation of the system we just analyzed, which can be implemented using a tunnel diode polarized so that its operational point lies on the negative resistance part of its voltage-current curve [21]. This oscillator is quantitatively described by the van der Pol equation. If we

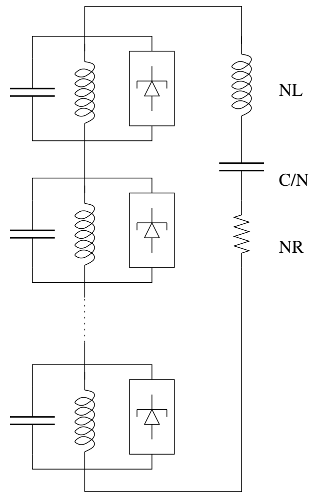

FIG. 3: Globally coupled Van der Pol oscillators.

couple a series array of these oscillators with a parallel RLC load (Fig.3) the governing circuit equations become:

$$c\ddot{u}_k - (\alpha - 3\beta u_k^2)\dot{u}_k + \frac{1}{l}u_k = \ddot{Q}_L \tag{19}$$

$$NL\ddot{Q}_L + NR\dot{Q}_L + \frac{N}{C}Q_L = \sum_j u_j \tag{20}$$

where N is number of coupled oscillators, Q_L is the charge on the load, NR, NL and C/N are resistance, inductance and capacitance of the load, respectively, l and c are inductance and capacitance of individual Van der Pol oscillators, and α and β are parameters characterizing the tunnel diodes. With the substitutions $t \to \tau = t/(lc)$, $u_k \to v_k = \sqrt{3\beta/\alpha}u_k$, and

$$Q_L \to q = \frac{1}{c}\sqrt{\frac{3\beta}{\alpha}}q$$

these equations are brought into dimensionless form:

$$\ddot{v}_k + \epsilon(1 - v_k^2)\dot{v}_k + v_k = \ddot{q} \tag{21}$$

$$\mu_1\ddot{q} + \mu_2\dot{q} + q = \frac{\eta}{N}\sum_{j=1}^{N} v_j \tag{22}$$

where $\epsilon = \alpha l$, $\eta = C/c$, $\mu_1 = LC/(lc)$ and $\mu_2 = RC/\sqrt{lc}$. These equations are almost the same as Eqs.(1,2), except that $\ddot{q}$ appears on the right hand side of Eq.(21) rather than $\dot{q}$. This minor structural difference leads to a significant change in the synchronization properties of the array. Assuming that ϵ and η are small and repeating the previous procedure we readily derive the eigenvalues of the in-phase solution:

$$\lambda_1 = 1, \quad \lambda_2 = 1 - 2\pi\epsilon\left(1 + \frac{\kappa}{Z}\sin\delta\right) \tag{23}$$

and

$$\lambda_{i,i+1} = 1 - \pi\epsilon\left(1 + 2\frac{\kappa}{Z}\sin\delta\right) \pm \pi\epsilon\sqrt{\left(1 + \frac{\kappa}{Z}\sin\delta\right)^2 - \left(\frac{\kappa}{Z}\cos\delta\right)^2} \tag{24}$$

Since by definition the phase angle $\delta \in [0, \pi]$, all eigenvalues except for λ_1 are strictly smaller than one. Consequently, the in-phase state is always stable, a prediction which is verified by our extensive numerical simulations. This is reminiscent of the coupled pendulum problem originally studied by Christiaan Huygens who found that the clocks always synchronized in anti-phase [5, 22].

On the other hand, if in the previous example we replace the Van der Pol oscillators with Duffing-Van der Pol oscillators, we do obtain a dynamical transition. The Duffing-Van der Pol equation,

$$\ddot{v}_k - \epsilon(1 - v_k^2)\dot{v}_k + v_k - \epsilon\alpha v_k^3 = 0 \tag{25}$$

has an additional cubic term, which in turn leads to an additional term in eigenvalues

$$\lambda_{i,i+1} = 1 - \pi\epsilon\left(1 + 2\frac{\kappa}{Z}\sin\delta\right) \pm \pi\epsilon\sqrt{\left(1 + \frac{\kappa}{Z}\sin\delta\right)^2 - \left(\frac{\kappa}{Z}\cos\delta\right)^2 + \alpha A} \tag{26}$$

Here $A = 3\kappa/Z(2\cos\delta + \sin 2\delta)$. The structure of the calculation is identical to the one of the last section, and we omit the details here. As expected, the spectrum of eigenvalues (24) is just a special case of the new spectrum (26), when $\alpha = 0$. We tested these results numerically, and confirmed that eigenvalues (26) predict correctly the phase transition (Fig. 4).

These results suggest that the van der Pol and Duffing-Van der Pol arrays are members of a larger class obeying the same in-phase stability rules. On the other hand, as we now show, the Josephson junction arrays appear to belong to a fundamentally different class.

IV. JOSEPHSON JUNCTION ARRAYS

Figure 5 is a circuit schematic for a current-biased series array of Josephson junctions shunted by a parallel load. The equations of motion in dimensionless form are [9]:

$$\beta\ddot{\phi}_k + \dot{\phi}_k + b\sin\phi_k + \dot{Q}_L = 1 \tag{27}$$

$$\mu_1\ddot{Q}_L + \mu_2\dot{Q}_L + Q_L = \alpha\sum_j \dot{\phi}_j \tag{28}$$

The parameter β is proportional to the junction capacitance.

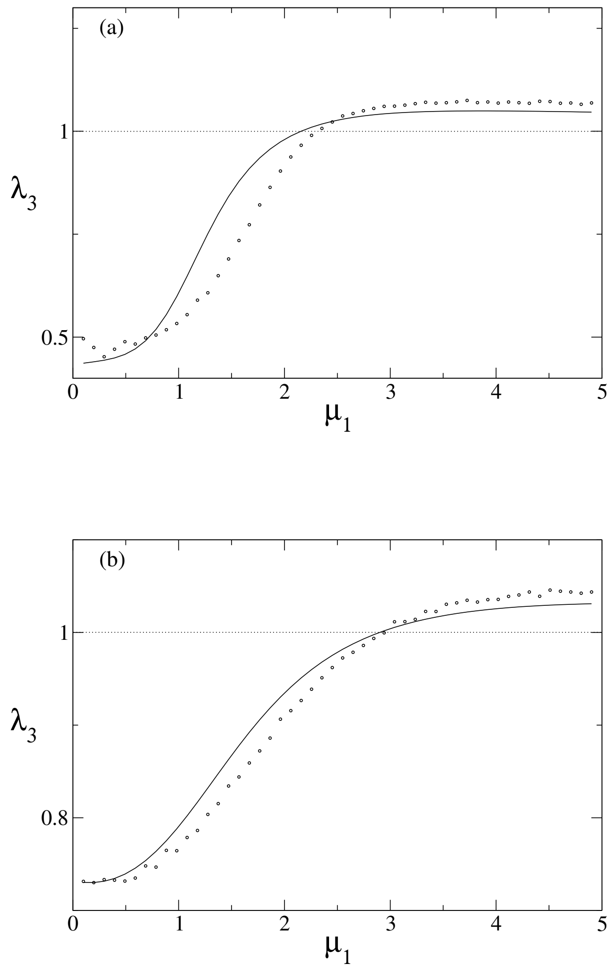

FIG. 4: Results for the Duffing-Van der Pol array, showing analytic (solid line) and numerically determined Floquet multipliers (dots) for the in-phase state. Parameters are set to $\epsilon = 0.1$, $\kappa = 1$ and (a) $\mu_2 = 0.8$, (b) $\mu_2 = 1.5$. Note that small parameter approximation breaks down as parameter μ_2 becomes smaller.

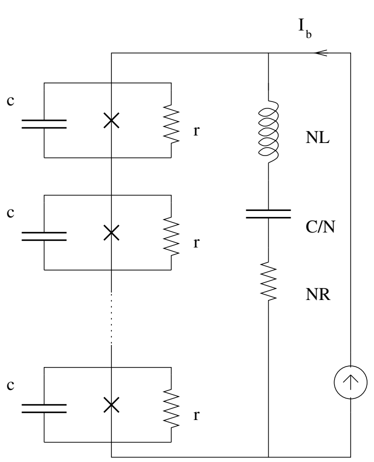

FIG. 5: Globally coupled Josephson junction oscillators.

It is precisely this problem where Chernikov and Schmidt originally applied the perturbation expansion we will use. It was also for this problem where the extension of their calculation to produce an iterative map was first sketched, in Ref. [10]; it is in the latter that the simple rendering of the stability formula was first noted. Unfortunately, neither reference provides a sufficiently systematic derivation of the synchronization condition for our purposes. We therefore present a streamlined but complete derivation which (i) emphasizes how the method identifies key parameter groups; and (ii) allows us to identify important differences with the van der Pol calculation.

As we will see, one obvious difference is that we need to take the expansion out to second order to get non-trivial results for the Josephson array. Even so, the in-phase stability condition turns out to be *simpler* than for the van der Pol array.

Taking b as the small parameter, we introduce the expansions

$$\phi_k = \phi_k^{(0)} + b\phi_k^{(1)} + b^2\phi_k^{(2)} + \ldots \quad (29)$$

$$Q_L = Q_L^{(0)} + bQ_L^{(1)} + b^2Q_L^{(2)} + \ldots \quad (30)$$

To zeroth order equations (27) and (28) are

$$\beta\ddot{\phi}_k^{(0)} + \dot{\phi}_k^{(0)} + \dot{Q}_L^{(0)} = 1 \quad (31)$$

$$\mu_1\ddot{Q}_L^{(0)} + \mu_2\dot{Q}_L^{(0)} + Q_L^{(0)} = \alpha\sum_j \dot{\phi}_j^{(0)} \quad (32)$$

which have solutions $Q_L^{(0)} = Q_0$ and $\phi_k^{(0)} = t + \theta_k$, where Q_0 and θ_k are constants. In first order, the differential equations are

$$\beta\ddot{\phi}_k^{(1)} + \dot{\phi}_k^{(1)} + \dot{Q}_L^{(1)} = -\sin(t + \theta_k) \quad (33)$$

$$\mu_1\ddot{Q}_L^{(1)} + \mu_2\dot{Q}_L^{(1)} + Q_L^{(1)} = \alpha\sum_j \dot{\phi}_j^{(1)} \quad (34)$$

The steady state solution to these equations is periodic, so we can write it in the form:

$$\phi_k^{(1)} = A_k \sin t + B_k \cos t \quad (35)$$

$$Q_L^{(1)} = C \sin t + D \cos t \quad (36)$$

By substituting these two solutions into (33) and (34) we obtain a system of algebraic equations for the coefficients:

$$\beta A_k + B_k = -D + \cos\theta_k \quad (37)$$

$$-A_k + \beta B_k = C + \sin\theta_k \quad (38)$$

and

$$(1-\mu_1)C - \mu_2 D = -\alpha\sum_j B_j \quad (39)$$

$$\mu_2 C + (1-\mu_1)D = \alpha\sum_j A_j \quad (40)$$

We note that the left hand side of Eq. (37-38) can be understood as a rotation of a vector (A_k, B_k) through some angle $\zeta = \arccos(\beta/G)$, where $G = \sqrt{1+\beta^2}$. Similarly, after proper normalization by Z, equations (39-40) can be interpreted as a rotation of the vector (C, D) by an angle δ, where Z and δ are defined like in (9). Solving these equations for the constants A_k and B_k yields, after some algebra (*cf* Ref. [9], Appendix)

$$\begin{aligned} A_k &= G^{-1}\cos(\theta_k+\zeta) \\ &\quad - \frac{\alpha}{GZP}\sum_j \cos(\theta_j+\xi-\delta+\zeta) \end{aligned} \tag{41}$$

$$\begin{aligned} B_k &= G^{-1}\sin(\theta_k+\zeta) \\ &\quad - \frac{\alpha}{GZP}\sum_j \sin(\theta_j+\xi-\delta+\zeta) \end{aligned} \tag{42}$$

where

$$P = \sqrt{\left(\beta + \frac{N\alpha}{Z}\cos\delta\right)^2 + \left(1 + \frac{N\alpha}{Z}\sin\delta\right)^2} \tag{43}$$

and

$$\xi = \arctan\frac{1 + N\alpha/Z\sin\delta}{\beta + N\alpha/Z\cos\delta} \tag{44}$$

Therefore, the first order solutions are

$$\begin{aligned} \phi_k^{(1)} &= G^{-1}\sin(t+\theta_k+\zeta) \\ &\quad - \frac{\alpha}{GZP}\sum_j \sin(t+\theta_j+\gamma+\zeta) \end{aligned} \tag{45}$$

$$Q_L^{(1)} = \frac{\alpha}{ZP}\sum_j \cos(t+\theta_j+\gamma) \tag{46}$$

where we substituted $\gamma = \xi - \delta$. From the Eq. (34) and its solution (46) we infer that γ has a physical interpretation as the characteristic voltage phase shift of the load. That is, if one removes the nonlinearity of the Josephson junction (setting $b = 0$) and probes the system with a unit amplitude sinusoidal current, the voltage oscillations across the load will lag by γ radians (Fig. 6). Also, by definition ζ is the voltage phase shift on a single junction, due to a sinusoidal driving current, when $b = 0$ (Fig. 7). Finally, the second order equations of motion are

$$\beta\ddot{\phi}_k^{(2)} + \dot{\phi}_k^{(2)} + \dot{Q}_L^{(2)} = -\cos(t+\theta_k)\phi_k^{(1)} \tag{47}$$

$$\mu_1\ddot{Q}_L^{(2)} + \mu_2\dot{Q}_L^{(2)} + Q_L^{(2)} = \alpha\sum_j \dot{\phi}_j^{(2)} \tag{48}$$

Substituting $\phi_k^{(1)}$ into (47) and solving yields

$$\begin{aligned} \phi_k^{(2)} &= -\frac{t}{2G}\sin\zeta \\ &\quad + t\frac{\alpha}{2GZP}\sum_j \sin(\theta_j-\theta_k+\gamma+\zeta) \\ &\quad + P_{2\pi}^{(2)} \end{aligned} \tag{49}$$

Putting this all together, we have the solution for ϕ_k:

$$\begin{aligned} \phi_k &= t + bP_{2\pi}^{(1)} - b^2\frac{t}{2G}\sin\zeta \\ &\quad - b^2\frac{t\alpha}{2GZP}\sum_j \sin(\theta_j-\theta_k+\gamma+\zeta) \\ &\quad + b^2P_{2\pi}^{(2)} + O(b^3) \end{aligned} \tag{50}$$

From this expression, we immediately deduce the N-dimensional map $\phi_k(0) \to \phi_k(T)$, good through order b^2:

$$\begin{aligned} \phi_k(T) &= \phi_k(0) + T - b^2\frac{\pi}{G}\sin\zeta \\ &\quad + b^2\frac{\pi\alpha}{GZP}\sum_j \sin(\phi_j(0)-\phi_k(0)+\gamma+\zeta) \\ &\quad + O(b^3) \end{aligned}$$

where we have used the fact that $\theta_k = \phi_k(0) + O(b)$ and $T = 2\pi + O(b^2)$. The in-phase state of the oscillator array corresponds to the symmetric fixed point of the map with $\phi_k(0) = \phi_k(T) = \phi_\star$ for all k. It is straightforward to to find the eigenvalues of the Jacobi matrix evaluated at this fixed point, with result

$$\lambda_1 = 1, \quad \lambda_{i>1} = 1 - b^2\frac{\pi\alpha}{ZPG}\cos(\gamma+\zeta) + O(b^3) \tag{51}$$

As a consequence, the stability condition for the in-phase state takes on the very simple form

$$\cos(\gamma+\zeta) > 0 \tag{52}$$

Numerical simulations [10] show that this condition accurately describes the onset of synchronization for $b < 1$. By substituting expressions for angles γ and ζ one recovers the synchronization condition quoted by [9]. Moreover, by setting parameter $\beta = 0$ we obtain the well known synchronization condition for noncapacitive junctions [12, 24, 25].

It is worthwhile to consider this very simple result from another perspective. We emphasized that it is possible to give a direct physical interpretation of the phase angles. This means that the two constituents of the stability condition are determined by linear properties of the circuit, while the (crucial) role of the nonlinearity is to act as a catalyst. Furthermore, condition (52) does not depend on internal structure of individual oscillators, but only on phase shifts they induce when driven by a sinusoidal signal. It is natural to speculate that this law should apply to a broader class of oscillators.

V. CONCLUSION

The analytic scheme we have used is successful in capturing the stability properties of the in-phase state of various globally coupled arrays. It provides a clear and

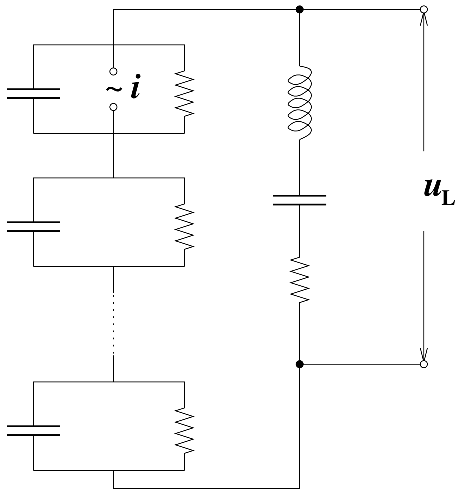

FIG. 6: Linear components of the full nonlinear problem in Fig. 5. This is equivalent circuit for the Eqs. (33-34). The angle γ is the phase shift by which the load voltageu_L lags behind the driving current i.

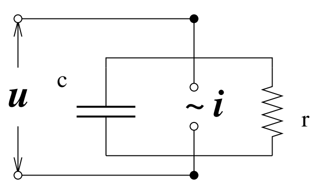

FIG. 7: The phase ζ as defined in (37-38) is the voltage phase shift on a single junction due to driving current, when $b = 0$.

straightforward analysis, and leads to simple looking results. The perturbation calculation can be relatively easily extended to higher orders in the small parameter. In certain cases this scheme may be more readily applied than standard methods, such as averaging. (Based on our personal experiences, this is certainly true for the examples studied here.) Nevertheless, there is no rigorous mathematical study of this approach, and it remains unclear what are its advantages (or shortcomings) relative to other analytic methods.

The different examples share much in common. In carrying out the calculation order by order, we are led to define certain parameter groups which are in some sense natural: the resulting formulas for the eigenvalues are rather simple functions of these parameter groups. In all cases the parameter groups can be interpreted physically as impedances or phase shifts.

Despite the similarities, there appear significant differences between the van der Pol and Josephson arrays. In the latter case we are led to an impressively simple stability condition involving only the phase shifts. The van der Pol problem is somewhat more subtle; the stability condition depends on both the phase shift and the impedance. There is no way to untangle these to yield a correspondingly simple stability condition. The two seem to fall into different classes.

In another sense, it is perhaps natural that the van der Pol and Josephson arrays behave differently. In his work on oscillator arrays [1], Blekhman underscores the distinction between vibrators and rotors. The class of vibrators is typified by the van der Pol oscillator, and the class of rotors is typified by the "overturning" Josephson junction. And indeed the Duffing-Van der Pol array (also vibrators) is described by the same stability law as the van der Pol array.

In the context of our calculations, the reason for this essential difference can be traced to the presence or absence of near-resonant interactions. In the van der Pol array, the zeroth order problem is just the undriven, undamped harmonic oscillator. These necessarily generate oscillations in the passive load at the system's resonant frequency. In contrast, the zeroth order problem of the Josephson array lacks any resonance. On the one hand this leads to non-trivial interactions only at second order, but it also leads to a separation of the roles of impedances and phase shifts. As one sees from Eq.(51), while the phase shifts (γ, δ, and ζ) directly determine the binary issue of whether or not the in-phase state is stable, the impedances (Z, P, and G) only affect the overall degree of stability.

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