Newtonian dynamics

7

You might think that the strangeness of contracting flows, flows such as the Rössler flow of Fig. 2.4 is of concern only to chemists; real physicists do Hamiltonians, right? Not at all - while it is easier to visualize aperiodic dynamics when a flow is contracting onto a lower-dimensional attracting set, there are plenty examples of chaotic flows that do preserve the full symplectic invariance of Hamiltonian dynamics. The truth is, the whole story started with Poincaré's restricted 3-body problem, a realization that chaos rules also in general (non-Hamiltonian) flows came much later.

Here we briefly review parts of classical dynamics that we will need later on; symplectic invariance, canonical transformations, and stability of Hamiltonian flows. We discuss billiard dynamics in some detail in Chapter 8.

7.1 Hamiltonian flows

(P. Cvitanović and L.V. Vela-Arevalo)

An important class of flows are Hamiltonian flows, given by a Hamiltonian $H(q,p)$ together with the Hamilton's equations of motion

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$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \tag{7.1}$$

with the $2D$ phase space coordinates x split into the configuration space coordinates and the conjugate momenta of a Hamiltonian system with D degrees of freedom (dof):

⇒ Section ??

$$x = (\mathbf{q},\mathbf{p})\,, \qquad \mathbf{q} = (q_1, q_2, \ldots, q_D)\,, \qquad \mathbf{p} = (p_1, p_2, \ldots, p_D)\,. \tag{7.2}$$

The energy, or the value of the Hamiltonian function at the state space point $x = (\mathbf{q},\mathbf{p})$ is constant along the trajectory $x(t)$,

$$\begin{aligned} \frac{d}{dt}H(\mathbf{q}(t),\mathbf{p}(t)) &= \frac{\partial H}{\partial q_i}\dot{q}_i(t) + \frac{\partial H}{\partial p_i}\dot{p}_i(t) \\ &= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0\,, \end{aligned} \tag{7.3}$$

so the trajectories lie on surfaces of constant energy, or *level sets* of the Hamiltonian $\{(q,p) : H(q,p) = E\}$. For 1-dof Hamiltonian systems this is basically the whole story.

Example 7.1 Unforced undamped Duffing oscillator:
When the damping term is removed from the Duffing oscillator (2.7), the system can be written in Hamiltonian form with the Hamiltonian

$$H(q,p) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{q^4}{4}. \tag{7.4}$$

This is a 1-dof Hamiltonian system, with a 2-dimensional state space, the plane (q,p). The Hamilton's equations (7.1) are

$$\dot{q} = p\,, \qquad \dot{p} = q - q^3\,. \tag{7.5}$$

For 1-dof systems, the 'surfaces' of constant energy (7.3) are simply curves in the phase plane (q,p), and the dynamics is very simple: the curves of constant energy *are* the trajectories, as shown in Fig. 7.1.

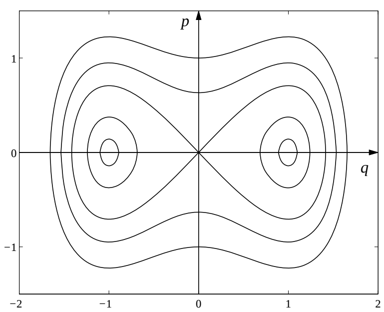

Fig. 7.1 Phase plane of the unforced, undamped Duffing oscillator. The trajectories lie on level sets of the Hamiltonian (7.4).

Example 6.1

Chapter ??

Thus all 1-dof systems are *integrable*, in the sense that the entire phase plane is foliated by curves of constant energy, either periodic – as is the case for the harmonic oscillator (a 'bound state')–or open (a 'scattering trajectory'). Add one more degree of freedom, and chaos breaks loose.

Example 7.2 Collinear helium:
In Chapter **??**, we shall apply the periodic orbit theory to the quantization of helium. In particular, we will study *collinear helium*, a doubly charged nucleus with two electrons arranged on a line, an electron on each side of the nucleus. The Hamiltonian for this system is

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_1 + r_2}\,. \tag{7.6}$$

Collinear helium has 2 dof, and thus a 4-dimensional phase space $\mathcal{M}$, which energy conservation reduces to 3 dimensions. The dynamics can be projected onto the 2-dimensional configuration plane, the (r_1, r_2), $r_i \geq 0$ quadrant, Fig. 7.2. It looks messy, and, indeed, it will turn out to be no less chaotic than a pinball bouncing between three disks. As always, a Poincaré section will be more informative than this rather arbitrary projection of the flow.

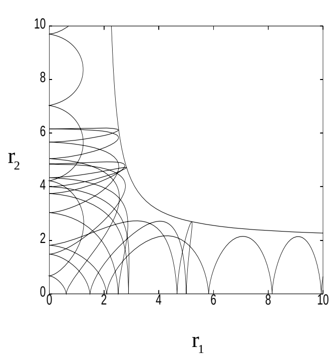

Fig. 7.2 A typical collinear helium trajectory in the $[r_1, r_2]$ plane; the trajectory enters along the r_1-axis and then, like almost every other trajectory, after a few bounces escapes to infinity, in this case along the r_2-axis.

Note an important property of Hamiltonian flows: if the Hamilton equations (7.1) are rewritten in the $2D$ phase space form $\dot{x}_i = v_i(ssp)$, the divergence of the velocity field v vanishes, namely the flow is incompressible. The symplectic invariance requirements are actually more stringent than just the phase space volume conservation, as we shall see in the next section.

7.2 Stability of Hamiltonian flows

Hamiltonian flows offer an illustration of the ways in which an invariance of equations of motion can affect the dynamics. In the case at hand, the *symplectic invariance* will reduce the number of independent stability eigenvalues by a factor of 2 or 4.

7.2.1 Canonical transformations

The equations of motion for a time-independent, D-dof Hamiltonian (7.1) can be written

$$\dot{x}_i = \omega_{ij} H_j(x) \,, \qquad \omega = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \,, \quad H_j(x) = \frac{\partial}{\partial x_j} H(x) \,, \tag{7.7}$$

where $x = (\mathbf{q}, \mathbf{p}) \in \mathcal{M}$ is a phase space point, $H_k = \partial_k H$ is the column vector of partial derivatives of H, $\mathbf{I}$ is the $[D \times D]$ unit matrix, and ω the $[2D \times 2D]$ *symplectic* form

$$\omega^T = -\omega \,, \qquad \omega^2 = -\mathbf{1} \,. \tag{7.8}$$

To stress the peculiar properties of Hamiltonian flows, we change the notation slightly, and denote the fundamental matrix (4.6) by $M^t(x)$. The evolution of M^t is again determined by the stability matrix A, (4.9):

$$\frac{d}{dt} M^t(x) = A(x) M^t(x) \,, \qquad A_{ij}(x) = \omega_{ik} \, H_{kj}(x) \,, \tag{7.9}$$

where the matrix of second derivatives $H_{kn} = \partial_k \partial_n H$ is called the ***Hessian matrix***. From the symmetry of H_{kn} it follows that

$$A^T \omega + \omega A = 0 \,. \tag{7.10}$$

This is the defining property for infinitesimal generators of *symplectic* (or canonical) transformations, transformations which leave the symplectic form ω invariant.

Symplectic matrices are by definition linear transformations that leave the (antisymmetric) quadratic form $x_i \omega_{ij} y_j$ invariant. This immediately implies that any symplectic matrix satisfies

$$Q^T \omega Q \, = \, \omega \,, \tag{7.11}$$

and – when Q is close to the identity $Q = \mathbf{1} + \delta t A$ – it follows that that A must satisfy (7.10).

In group language this means that the property (7.11) defines the symplectic group $Sp(2D)$, just as the Lie group of orthogonal matrices $O(d)$ is defined by linear transformations that preserve the (symmetric) quadratic form $x^2 = x_i \delta_{ij} x_j$, The symplectic Lie algebra $sp(2D)$ follows by writing $Q = \exp(\delta t A)$ and linearizing $Q = 1 + \delta t A$. This yields (7.10) as the defining property of infinitesimal symplectic transformations.

Consider now a smooth nonlinear change of variables of form $y_i = h_i(x)$, and define a new function $K(x) = H(h(x))$. Under which conditions does K generate a Hamiltonian flow? In what follows we will use the notation $\tilde{\partial}_j = \partial / \partial y_j$: by employing the chain rule we have that 7.2, page 97

$$\omega_{ij} \partial_j K \, = \, \omega_{ij} \frac{\partial h_l}{\partial x_j} \tilde{\partial}_l H \tag{7.12}$$

By virtue of (7.1) $\tilde{\partial}_l H = -\omega_{lm}\dot{y}_m$, so that, again by employing the chain rule, we obtain

$$\omega_{ij}\partial_j K \;=\; -\omega_{ij}\frac{\partial h_l}{\partial x_j}\omega_{lm}\frac{\partial h_m}{\partial x_n}\dot{x}_n \tag{7.13}$$

The right hand side simplifies to $\dot{x}_i$ (yielding Hamiltonian structure) only if

$$-\omega_{ij}\frac{\partial h_l}{\partial x_j}\omega_{lm}\frac{\partial h_m}{\partial x_n} \;=\; \delta_{in} \tag{7.14}$$

or, in compact notation, by defining $(\partial h)_{ij} = \frac{\partial h_i}{\partial x_j}$

$$-\omega(\partial h)^T\omega(\partial h) \;=\; \mathbf{1} \tag{7.15}$$

which is equivalent to the requirement that ∂h is symplectic. h is then called a *canonical transformation*. We care about canonical transformations for two reasons. First (and this is a dark art), if the canonical transformation h is very cleverly chosen, the flow in new coordinates might be considerably simpler than the original flow. Second, Hamiltonian flows themselves are a prime example of canonical transformations.

Example 6.1

Example 7.3 Hamiltonian flows are canonical:
For Hamiltonian flows it follows from (7.10) that $\frac{d}{dt}\left(M^T\omega M\right) = 0$, and since at the initial time $M^0(x_0) = \mathbf{1}$, M is a symplectic transformation (7.11). This equality is valid for all times, so a Hamiltonian flow $f^t(x)$ is a canonical transformation, with the linearization $\partial_x f^t(x)$ a symplectic transformation (7.11): For notational brevity here we have suppressed the dependence on time and the initial point, $M = M^t(x_0)$. By elementary properties of determinants it follows from (7.11) that Hamiltonian flows are phase space volume preserving:

$$|\det M| = 1\,. \tag{7.16}$$

Actually it turns out that for symplectic matrices (on any field) one always has $\det M = +1$.

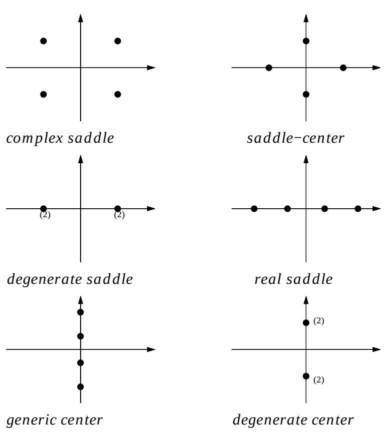

Fig. 7.3 Stability exponents of a Hamiltonian equilibrium point, 2-dof.

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Section 4.3.1
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7.2.2 Stability of equilibria of Hamiltonian flows

For an equilibrium point x_q the stability matrix A is constant. Its eigenvalues describe the linear stability of the equilibrium point. In the case of Hamiltonian flows, from (7.10) it follows that the characteristic polynomial of A for an equilibrium x_q satisfies

$$\begin{aligned}\det(A-\lambda\mathbf{1}) &= \det(\omega^{-1}(A-\lambda\mathbf{1})\omega) \;=\; \det(-\omega A\omega-\lambda\mathbf{1}) \\ &= -\det(A^T+\lambda\mathbf{1}) \;=\; -\det(A+\lambda\mathbf{1})\,.\end{aligned} \tag{7.17}$$

A is the matrix (7.10) with real matrix elements, so its eigenvalues (the stability exponents of (4.27)) are either real or come in complex pairs. Symplectic invariance implies in addition that if λ is an eigenvalue, then $-\lambda$, λ^* and $-\lambda^*$ are also eigenvalues. Distinct symmetry classes of the stability exponents of an equilibrium point in a 2-dof system are displayed in Fig. 7.3. It is worth noting that while the linear stability of equilibria in a Hamiltonian system always respects this symmetry, the nonlinear stability can be completely different.

7.3 Symplectic maps

A stability eigenvalue $\Lambda = \Lambda(x_0, t)$ associated to a trajectory is an eigenvalue of the monodromy matrix M. As M is symplectic, (7.11) implies that

$$M^{-1} = -\omega M^T \omega \,, \tag{7.18}$$

so the characteristic polynomial is reflexive, namely it satisfies

$$\begin{aligned} \det(M - \Lambda \mathbf{1}) &= \det(M^T - \Lambda \mathbf{1}) \;=\; \det(-\omega M^T \omega - \Lambda \mathbf{1}) \\ &= \det(M^{-1} - \Lambda \mathbf{1}) \;=\; \det(M^{-1}) \det(\mathbf{1} - \Lambda M) \\ &= \Lambda^{2D} \det(M - \Lambda^{-1} \mathbf{1}) \,. \end{aligned} \tag{7.19}$$

Hence if Λ is an eigenvalue of M, so are $1/\Lambda$, Λ^* and $1/\Lambda^*$. Real (non-marginal, $|\Lambda| \neq 1$) eigenvalues always come paired as Λ, $1/\Lambda$. The Liouville conservation of phase space volumes (7.16) is an immediate consequence of this pairing up of eigenvalues. The complex eigenvalues come in pairs Λ, Λ^*, $|\Lambda| = 1$, or in loxodromic quartets Λ, $1/\Lambda$, Λ^* and $1/\Lambda^*$. These possibilities are illustrated in Fig. 7.4.

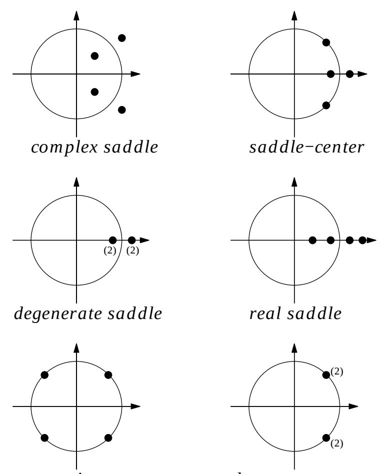

Fig. 7.4 Stability of a symplectic map in $\mathbb{R}^4$.

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Example 7.4 Hamiltonian Hénon map, reversibility:
By (4.41) the Hénon map (3.15) for $b = -1$ value is the simplest 2-d orientation preserving area-preserving map, often studied to better understand topology and symmetries of Poincaré sections of 2 dof Hamiltonian flows. We find it convenient to multiply (3.16) by a and absorb the a factor into x in order to bring the Hénon map for the $b = -1$ parameter value into the form

$$x_{i+1} + x_{i-1} = a - x_i^2 \,, \quad i = 1, ..., n_p \,, \tag{7.20}$$

The 2-dimensional *Hénon map* for $b = -1$ parameter value

$$\begin{aligned} x_{n+1} &= a - x_n^2 - y_n \\ y_{n+1} &= x_n \,. \end{aligned} \tag{7.21}$$

is Hamiltonian (symplectic) in the sense that it preserves area in the $[x, y]$ plane.
For definitiveness, in numerical calculations in examples to follow we shall fix (arbitrarily) the stretching parameter value to $a = 6$, a value large enough to guarantee that all roots of $0 = f^n(x) - x$ (periodic points) are real.

Example 7.5 2-dimensional symplectic maps:
In the 2-dimensional case the eigenvalues (5.2) depend only on tr M^t

$$\Lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}\, M^t \pm \sqrt{(\mathrm{tr}\, M^t - 2)(\mathrm{tr}\, M^t + 2)}\right) \,. \tag{7.22}$$

The trajectory is elliptic if the *stability residue* $|\mathrm{tr}\, M^t| - 2 \leq 0$, with complex eigenvalues $\Lambda_1 = e^{i\theta t}$, $\Lambda_2 = \Lambda_1^* = e^{-i\theta t}$. If $|\mathrm{tr}\, M^t| - 2 > 0$, λ is real, and the trajectory is either

hyperbolic $\quad \Lambda_1 = e^{\lambda t}, \quad \Lambda_2 = e^{-\lambda t}$, or $\quad$ (7.23)

inverse hyperbolic $\quad \Lambda_1 = -e^{\lambda t}, \quad \Lambda_2 = -e^{-\lambda t}$. $\quad$ (7.24)

7.3.1 The standard map

Truth is rarely pure, and never simple.
Oscar Wilde

Given a smooth function $g(x)$, the map

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \\ y_{n+1} &= y_n + g(x_n) \end{aligned} \tag{7.25}$$

is an area-preserving map. The corresponding monodromy matrix is

$$M(x,y) = \begin{pmatrix} 1+g'(x) & 1 \\ g'(x) & 1 \end{pmatrix} \tag{7.26}$$

$\det M = 1$ so the map preserves areas, moreover one can easily check that M is symplectic. In particular one can consider x as an angle, and y as the conjugate angular momentum, with a function g periodic, with period 2π. The phase space of the map is thus the cylinder $S_1 \times \mathbf{R}$ (S_1 being the 1-torus): by taking (7.25) $mod\ 2\pi$ the map can be reduced on the 2-torus S_2. Note that the mapping provides a stroboscopic view of the flow generated by a pulsed Hamiltonian

$$H(x,y;t) = \frac{1}{2}y^2 + G(x)\delta_1(t) \tag{7.27}$$

where δ_1 denotes the periodic delta function

$$\delta_1(t) = \sum_{m=-\infty}^{\infty} \delta(t-m) \tag{7.28}$$

and

$$G'(x) = -g(x)\,. \tag{7.29}$$

The *standard map* corresponds to the choice $g(x) = k\sin(x)$: the corresponding map will be denoted by A. When $k = 0$ angular momentum is conserved, and orbits are pure rotations, motion is periodic or quasiperiodic according to y_0 being rational or irrational; invariant tori are straight lines in the (x,y) phase plane.

Despite the simple structure of the standard mapping, a complete description of its dynamics for arbitrary values of the nonlinear parameter k is fairly complex: small k regime falls within KAM scheme, yet any perturbative regime fails very soon as k increases. It turns out that interesting features of this map, including transition to *global chaos* (destruction of the last deformed invariant torus), may be tackled by detailed investigation of the stability of periodic orbits. A compact index of stability of a Q-periodic orbit is provided by the *residue*:

$$R_Q = \frac{1}{4}\left(2 - \mathrm{tr}\, M^Q\right)\;; \tag{7.30}$$

as a matter of fact if $R_Q \in (0,1)$ the orbit is elliptic, $R_Q > 1$ corresponds to hyperbolic orbits and, finally, $R_Q < 0$ marks the case of inverse hyperbolic orbits.

For $k = 0$ all orbits with $y_0 = P/Q$ are periodic with period Q (and winding number P/Q): since x_0 is arbitrary, actually they are organized in families (and they all have $R_Q = 0$). As soon as k increases there is only a finite number of such orbits that survive, according to Poincaré-Birkhoff theorem, half of them are elliptic, and half hyperbolic. If we further vary k in such a way that the residue of the elliptic Q cycle go to 1 a bifurcation takes place, and two or more periodic orbits of higher period are generated.

In practice the search for remarkable classes of periodic orbits for the standard map takes advantage of an important symmetry property: A can be written as the product of two *involutions* T_1 and T_2 (involution means that the square of the map is the identity):

$$A = T_2 \cdot T_1 \tag{7.31}$$

where

$$T_1 \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} -x & y - k\sin x \end{pmatrix} \tag{7.32}$$

and

$$T_2 \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} -x + y & y \end{pmatrix} . \tag{7.33}$$

Now define symmetry lines $\mathcal{L}_1$ and $\mathcal{L}_2$ as the set of fixed points of the corresponding involution: $\mathcal{L}_1$ consists of the lines $x = 0, \pi$, $\mathcal{L}_2$ of $x = y/2 \; mod\,(2\pi)$. There are deep connections between symmetry lines and periodic orbits: we just give an example with the following statement: if $(x_0, y_0) \in \mathcal{L}_1$ and $A^M(x_0, y_0) \in \mathcal{L}_1$ (*i.e.* they are both fixed points of T_1), then (x_0, y_0) is a periodic point of period $2M$. As a matter of fact

$$\begin{aligned} A^{2M}(x_0, y_0) &= A^{M-1}T_2T_1A^{M-1}T_2T_1(x_0, y_0) \\ &= A^{M-1}T_2A^{M-1}T_2(x_0, y_0) \end{aligned} \tag{7.34}$$

by the fixed point property. Now the involution property implies

$$T_2A = T_1 \qquad AT_1 = T_2 \tag{7.35}$$

and thus

$$AT_2AT_2 = AT_1T_2 = \mathbf{1} \tag{7.36}$$

and

$$A^PT_2A^PT_2 = A^{P-1}T_2A^{P-1}T_2 \tag{7.37}$$

from which it easily follows that (x_0, y_0) belongs to a $2M$ cycle.

7.3.2 Poincaré invariants

Let C a region in the phase space and $V(0)$ its volume. Denoting the flow of the Hamiltonian system by $f^t(x)$, the volume of C after a time

t is $V(t) = f^t(C)$, and using (7.16) we derive the *Liouville theorem*:

$$\begin{aligned} V(t) &= \int_{f^t(C)} dx = \int_C \left\| \frac{\partial f^t(x')}{\partial x} \right\| dx' \\ & \int_C \det(M) dx' = \int_C dx' = V(0)\,, \end{aligned} \tag{7.38}$$

Hamiltonian flows preserve phase space volumes.

The symplectic structure of Hamilton's equations buys us much more than the 'incompressibility,' or the phase space volume conservation. Consider the symplectic product of two infinitesimal vectors

$$\begin{aligned} (\delta x, \delta\hat{x}) &= \delta x^T \omega \delta\hat{x} = \delta p_i \delta\hat{q}_i - \delta q_i \delta\hat{p}_i \\ &= \sum_{i=1}^{D} \{\text{oriented area in the } (q_i, p_i) \text{ plane}\}\,. \end{aligned} \tag{7.39}$$

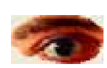
Time t later we have

$$(\delta x', \delta\hat{x}') = \delta x^T M^T \omega M \delta\hat{x} = \delta x^T \omega \delta\hat{x}\,.$$

This has the following geometrical meaning. We imagine there is a reference phase space point. We then define two other points infinitesimally close so that the vectors δx and $\delta\hat{x}$ describe their displacements relative to the reference point. Under the dynamics, the three points are mapped to three new points which are still infinitesimally close to one another. The meaning of the above expression is that the area of the parallelopiped spanned by the three final points is the same as that spanned by the inital points. The integral (Stokes theorem) version of this infinitesimal area invariance states that for Hamiltonian flows the D oriented areas $\mathcal{V}_i$ bounded by D loops $\Omega\mathcal{V}_i$, one per each (q_i, p_i) plane, are separately conserved:

$$\int_{\mathcal{V}} dp \wedge dq = \oint_{\Omega\mathcal{V}} p \cdot dq = \text{invariant}\,. \tag{7.40}$$

Morally a Hamiltonian flow is really D-dimensional, even though its phase space is $2D$-dimensional. Hence for Hamiltonian flows one emphasizes D, the number of the degrees of freedom.



in depth:
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In theory there is no difference between theory and practice. In practice there is.
Yogi Berra

Further reading

Hamiltonian dynamics literature. If you are reading this book, in theory you already know everything that is in this chapter. In practice you do not. Try this: Put your right hand on your heart and say: "I understand why nature prefers symplectic geometry". Honest? We make an attempt in Section **??**. Out there there are about 2 centuries of accumulated literature on Hamilton, Lagrange, Jacobi etc. formulation of mechanics, some of it excellent. In context of what we will need here, we make a very subjective recommendation–we enjoyed reading Percival and Richards [10] and Ozorio de Almeida [11].

Symplectic. The term symplectic–Greek for twining or plaiting together–was introduced into mathematics by Hermann Weyl. 'Canonical' lineage is church-doctrinal: Greek 'kanon,' referring to a reed used for measurement, came to mean in Latin a rule or a standard.

The sign convention of ω. The overall sign of ω, the symplectic invariant in (7.7), is set by the convention that the Hamilton's principal function (for energy conserving flows) is given by $R(q, q', t) = \int_q^{q'} p_i dq_i - Et$. With this sign convention the action along a classical path is minimal, and the kinetic energy of a free particle is positive.

Symmetries of the symbol square. For a more detailed discussion of symmetry lines see Refs. [4, 8, 49, 9].

Standard map. Standard maps have been very extensively studied (also in their quantum counterpart): as a starting point see the reviews Refs. [10, 11].

Exercises

(7.1) **Complex nonlinear Schrödinger equation.** Consider the complex nonlinear Schrödinger equation in one spatial dimension [1]:

$$i\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + \beta \phi |\phi|^2 = 0, \quad \beta \neq 0.$$

(a) Show that the function $\psi : \mathbb{R} \to \mathbf{C}$ defining the traveling wave solution $\phi(x,t) = \psi(x - ct)$ for $c > 0$ satisfies a second-order complex differential equation equivalent to a Hamiltonian system in $\mathbb{R}^4$ relative to the noncanonical symplectic form whose matrix is given by

$$w_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -c \\ 0 & -1 & c & 0 \end{bmatrix}.$$

(b) Analyze the equilibria of the resulting Hamiltonian system in $\mathbb{R}^4$ and determine their linear stability properties.

(c) Let $\psi(s) = e^{ics/2} a(s)$ for a real function $a(s)$ and determine a second order equation for $a(s)$. Show that the resulting equation is Hamiltonian and has heteroclinic orbits for $\beta < 0$. Find them.

(d) Find 'soliton' solutions for the complex nonlinear Schrödinger equation.

(Luz V. Vela-Arevalo)

(7.2) **Symplectic group/algebra**
Show that if a matrix C satisfies (7.10), then $\exp(sC)$ is a symplectic matrix.

(7.3) **When is a linear transformation canonical?**

(a) Let A be a $[n \times n]$ invertible matrix. Show that the map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $(\mathbf{q}, \mathbf{p}) \mapsto (A\mathbf{q}, (A^{-1})^T \mathbf{p})$ is a canonical transformation.

(b) If $\mathbf{R}$ is a rotation in $\mathbb{R}^3$, show that the map $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{R}\,\mathbf{q}, \mathbf{R}\,\mathbf{p})$ is a canonical transformation.

(Luz V. Vela-Arevalo)

(7.4) **Determinant of symplectic matrices.** Show that the determinant of a symplectic matrix is $+1$, by going through the following steps:

(a) use (7.19) to prove that for eigenvalue pairs each member has the same multiplicity (the same holds for quartet members),

(b) prove that the *joint* multiplicity of $\lambda = \pm 1$ is even,

(c) show that the multiplicities of $\lambda = 1$ and $\lambda = -1$ cannot be both odd. (Hint: write

$$P(\lambda) = (\lambda-1)^{2m+1}(\lambda+1)^{2l+1}Q(\lambda)$$

and show that $Q(1) = 0$).

(7.5) **Cherry's example.** What follows Refs. [2, 3] is mostly a reading exercise, about a Hamiltonian system that is *linearly stable* but *nonlinearly unstable*. Consider the Hamiltonian system on $\mathbb{R}^4$ given by

$$H = \frac{1}{2}(q_1^2+p_1^2) - (q_2^2+p_2^2) + \frac{1}{2}p_2(p_1^2-q_1^2) - q_1q_2p_1 .$$

(a) Show that this system has an equilibrium at the origin, which is linearly stable. (The linearized system consists of two uncoupled oscillators with frequencies in ratios 2:1).

(b) Convince yourself that the following is a family of solutions parametrized by a constant τ:

$$q_1 = -\sqrt{2}\frac{\cos(t-\tau)}{t-\tau}, \quad q_2 = \frac{\cos 2(t-\tau)}{t-\tau},$$
$$p_1 = \sqrt{2}\frac{\sin(t-\tau)}{t-\tau}, \quad p_2 = \frac{\sin 2(t-\tau)}{t-\tau}.$$

These solutions clearly blow up in finite time; however they start at $t = 0$ at a distance $\sqrt{3}/\tau$ from the origin, so by choosing τ large, we can find solutions starting arbitrarily close to the origin, yet going to infinity in a finite time, so the origin is *nonlinearly unstable*.

(Luz V. Vela-Arevalo)

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