Chapter 5. Cycle stability

coordinate!change
stability!exact
‖PRIVATE equilibrium!stability
stroboscopic map
map!stroboscopic
‖PRIVATE limit cycle!stability

Solution 5.1: Driven damped harmonic oscillator limit cycle. *Driven damped harmonic oscillator stability is discussed in Chapter 4 of Tél and Gruiz [1.11].*

Solution 5.2: A limit cycle with analytic stability exponent. *The 2-d flow (5.18) is cooked up so that $x(t) = (q(t), p(t))$ is separable (check!) in polar coordinates $q = r\cos\phi\,, \quad p = r\sin\phi$:*

$$\dot{r} = r(1-r^2)\,, \qquad \dot{\phi} = 1\,. \tag{S.9}$$

In the (r, ϕ) coordinates the flow starting at any $r > 0$ is attracted to the $r = 1$ limit cycle, with the angular coordinate ϕ wraping around with a constant angular velocity $\Omega = 1$. The non–wandering set of this flow consists of the $r = 0$ equilibrium and the $r = 1$ limit cycle.

equilibrium stability: *As the change of coordinates is defined everywhere except at the the equilibrium point ($r = 0$, any ϕ), the equilibrium stability matrix (4.28) has to be computed in the original (q, p) coordinates,*

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{S.10}$$

The eigenvalues are $\lambda = \mu \pm i\,\nu = 1 \pm i$, indicating that the origin is linearly unstable, with nearby trajectories spiralling out with the constant angular velocity $\Omega = 1$. The Poincaré section ($p = 0$, for example) return map is in this case also a stroboscopic map, strobed at the period (Poincaré section return time) $T = 2\pi/\Omega = 2\pi$. The radial stability multiplier per one Poincaré return is $|\Lambda| = e^{\mu T} = e^{2\pi}$.

Limit cycle stability: *From (S.9) the stability matrix is diagonal in the (r, ϕ) coordinates,*

$$A = \begin{bmatrix} 1-3r^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{S.11}$$

The vanishing of the angular $\lambda_\theta = 0$ eigenvalue is due to the rotational invariance of the equations of motion along ϕ direction. The expanding $\lambda_r = 1$ radial eigenvalue of the equilibrium $r = 0$ confirms the above equilibrium stability calculation. The contracting $\lambda_r = -2$ eigenvalue at $r = 1$ decreases the radial deviations from $r = 1$ with the radial stability multiplier $\Lambda_r = e^{\mu T} = e^{-4\pi}$ per one Poincaré return. This limit cycle is very *attracting.*

Stability of a trajectory segment: *Multiply (S.9) by r to obtain $\frac{1}{2}\dot{r^2} = r^2 - r^4$, set $r^2 = 1/u$, separate variables $du/(1-u) = 2\,dt$, and integrate: $\ln(1-u) - \ln(1-u_0) = -2t$. Hence the $r(r_0, t)$ trajectory is*

$$r(t)^{-2} = 1 + (r_0^{-2} - 1)e^{-2t}\,. \tag{S.12}$$

The $[1\times 1]$ fundamental matrix

$$J(r_0, t) = \left.\frac{\partial r(t)}{\partial r_0}\right|_{r_0 = r(0)}\,. \tag{S.13}$$

satisfies (4.9)

$$\frac{d}{dt}J(r,t) = A(r)\,J(r,t) = (1-3r(t)^2)\,J(r,t)\,, \qquad J(r_0,0) = 1\,.$$

This too can be solved by separating variables $d(\ln J(r,t)) = dt - 3r(t)^2 dt$, *substituting (S.12) and integrating. The stability of any finite trajectory segment is:*

$$J(r_0,t) = (r_0^2 + (1-r_0^2)e^{-2t})^{-3/2}e^{-2t}\,. \tag{S.14}$$

On the $r = 1$ *limit cycle this agrees with the limit cycle multiplier* $\Lambda_r(1,t) = e^{-2t}$*, and with the radial part of the equilibrium instability* $\Lambda_r(r_0,t) = e^t$ *for* $r_0 \ll 1$*.*

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Solution 5.3: The other example of a limit cycle with analytic stability exponent. *Email your solution to ChaosBook.org and G.B. Ermentrout.*

Solution 5.4: Yet another example of a limit cycle with analytic stability exponent. *Email your solution to ChaosBook.org and G.B. Ermentrout.*