Dirac delta function
delta function!Dirac

Chapter 14. Transporting densities

Solution 14.1: Integrating over Dirac delta functions. *(a) Whenever $h(x)$ crosses 0 with a nonzero velocity ($\det \partial_x h(x) \neq 0$), the delta function contributes to the integral. Let $x_0 \in h^{-1}(0)$. Consider a small neighborhood V_0 of x_0 so that $h : V_0 \to V_0$ is a one-to-one map, with the inverse function $x = x(h)$. By changing variable from x to h, we have*

$$\begin{aligned}\int_{V_0} dx\, \delta(h(x)) &= \int_{h(V_0)} dh\, |\det \partial_h x|\, \delta(h) = \int_{h(V_0)} dh\, \frac{1}{|\det \partial_x h|} \delta(h) \\ &= \frac{1}{|\det \partial_x h|_{h=0}} \,.\end{aligned}$$

Here, the absolute value $|\cdot|$ is taken because delta function is always positive and we keep the orientation of the volume when the change of variables is made. Therefore all the contributions from each point in $h^{-1}(0)$ add up to the integral

$$\int_{\mathbb{R}^d} dx\, \delta(h(x)) = \Sigma_{x \in h^{-1}(0)} \frac{1}{|\det \partial_x h|} \,.$$

Notice that if $\det \partial_x h = 0$, then the delta function integral is not well defined.

(b) The formal expression can be written as the limit

$$F := \int_{\mathbb{R}} dx\, \delta(x^2) = \lim_{\sigma \to 0} \int_{\mathbb{R}} dx\, \frac{e^{-\frac{x^4}{2\sigma}}}{\sqrt{2\pi\sigma}} \,,$$

by invoking the approximation given in the exercise. The change of variable $y = x^2/\sqrt{\sigma}$ gives

$$F = \lim_{\sigma \to 0} \sigma^{-3/4} \int_{\mathbb{R}^+} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi y}} = \infty \,,$$

where $\mathbb{R}^+$ represents the positive part of the real axis. So, the formal expression does not make sense. Notice that x^2 has a zero derivative at $x = 0$, which invalidates the expression in (a).

(Yueheng Lan)

Solution 14.2: Derivatives of Dirac delta functions. *We do this problem just by direct evaluation. We denote by Ω_y a sufficiently small neighborhood of y. (a)*

$$\begin{aligned}\int_{\mathbb{R}} dx\, \delta'(y) &= \Sigma_{x \in y^{-1}(0)} \int_{\Omega_y} dy \det\left(\frac{dy}{dx}\right)^{-1} \delta'(y) \\ &= \Sigma_{x \in y^{-1}(0)} \frac{\delta(y)}{|y'|}\Big|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \frac{\delta(y)}{y'^2}(-y'')\frac{1}{y'} \\ &= \Sigma_{x \in y^{-1}(0)} \frac{y''}{|y'| y'^2} \,,\end{aligned}$$

where the absolute value is taken to take care of the sign of the volume.

(b)

$$
\begin{aligned}
\int_{\mathbb{R}} dx\, \delta^{(2)}(y) &= \Sigma_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, \frac{\delta^{(2)}(y)}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} \frac{\delta'(y)}{|y'|}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \frac{\delta'(y)}{y'^2}(-y'')\frac{1}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} \frac{y''\delta(y)}{|y'|y'^2}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \delta(y)\frac{d}{dx}(\frac{y''}{y'^3})\frac{1}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} - \int_{\Omega_y} dy\, \delta(y)\left(\frac{y'''}{y'^3} - 3\frac{y''^2}{y'^4}\right)\frac{1}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} \left(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3}\right)\frac{1}{|y'|}\,.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\int_{\mathbb{R}} dx\, b(x)\delta^{(2)}(y) &= \Sigma_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, b(x)\frac{\delta^{(2)}(y)}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} \frac{b(x)\delta'(y)}{|y'|}|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \delta'(y)\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} - \delta(y)\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'}|_{-\epsilon}^{\epsilon} + \int_{\Omega_y} dy\, \delta(y)\frac{d}{dx}(\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'})\frac{1}{y'} \\
&= \Sigma_{x\in y^{-1}(0)} \frac{1}{|y'|}\frac{d}{dx}(\frac{b'}{y'^2} - \frac{by''}{y'^3})) \\
&= \Sigma_{x\in y^{-1}(0)} \frac{1}{|y'|}\left[\frac{b''}{y'^2} - \frac{b'y''}{y'^3} - 2\frac{b'y''}{y'^3} + b(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3})\right] \\
&= \Sigma_{x\in y^{-1}(0)} \frac{1}{|y'|}\left[\frac{b''}{y'^2} - 3\frac{b'y''}{y'^3} + b(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3})\right]\,.
\end{aligned}
$$

(Yueheng Lan)

Solution 14.3: $\mathcal{L}^t$ generates a semigroup. *Every "sufficiently good" transformation f^t in state space $\mathcal{M}$ is associated with a Perron-Frobenius operator $\mathcal{L}^t$ which is when acting on a function $\rho(x)$ in $\mathcal{M}$*

$$
\mathcal{L}^t \cdot \rho(x) = \int_{\mathcal{M}} dy\, \delta(x - f^t(y))\rho(y)\,.
$$

In some proper function space $\mathcal{F}$ on $\mathcal{M}$, the one parameter family of operators $\{\mathcal{L}^t\}_{t\in\mathbb{R}^+}$ generate a semigroup. Let's check this statement. For any $t_1, t_2 > 0$ and $\rho \in \mathcal{F}$, the product "$\circ$" of two operators is defined as usual

$$
(\mathcal{L}^{t_1} \circ \mathcal{L}^{t_2}) \cdot \rho(y) = \mathcal{L}^{t_1} \cdot (\mathcal{L}^{t_2} \cdot \rho)(y)\,.
$$

So, we have

$$\begin{aligned}
(\mathcal{L}^{t_1} \circ \mathcal{L}^{t_2})(y,x) &= \int_{\mathcal{M}} dz\, \mathcal{L}^{t_1}(y,z)\mathcal{L}^{t_2}(z,x) \\
&= \int_{\mathcal{M}} dz\, \delta(y - f^{t_1}(z))\delta(z - f^{t_2}(x)) \\
&= \delta(y - f^{t_1}(f^{t_2}(x))) \\
&= \delta(y - f^{t_1+t_2}(x)) \\
&= \mathcal{L}^{t_1+t_2}(y,x)\,,
\end{aligned}$$

where the semigroup property $f^{t_1}(f^{t_2}(x)) = f^{t_1+t_2}(x)$ *of* f^t *has been used. This proves the claim in the title.*

(Yueheng Lan)

Solution 14.5: Invariant measure. *Hint: We do (a),(b),(c),(d) for the first map and (e) for the second.*

(a) The partition point is in the middle of $[0,1]$. *If the density on the two pieces are two constants* ρ_0^A *and* ρ_0^B, *respectively, the Perron-Frobenius operator still leads to the piecewise constant density*

$$\rho_1^A = \frac{1}{2}(\rho_0^A + \rho_0^B)\,, \qquad \rho_1^B = \frac{1}{2}(\rho_0^A + \rho_0^B)\,.$$

Notice that in general if a finite Markov partition exists and the map is linear on each partition cell, a finite-dimensional invariant subspace which is a piecewise constant function can always be identified in the function space.

(b) From the discussion of (a), any constant function on $[0,1]$ *is an invariant measure. If we consider the invariant probability measure, then the constant has to be* 1.

(c) As the map is invariant in $[0,1]$ *(there is no escaping), the leading eigenvalue of* $\mathcal{L}$ *is always 1 due to the "mass" conservation.*

(d) Take a typical point on $[0,1]$ *and record its trajectory under the first map for some time (*10^5 *steps). Plot the histogram...ONLY* 0 *is left finally!! This happens because of the finite accuracy of the computer arithmetics. A small trick is to change the slope* 2 *to* 1.99999999. *You will find a constant measure on* $[0,1]$ *which is the natural measure. Still, the finite precision of the computer will make every point eventually periodic and strictly speaking the measure is defined only on subsets of lattice points. But as the resolution improves, the computer-generated measure steadily approaches the natural measure. For the first map, any small deviation from the constant profile will be stretched and smeared out. So, the natural measure has to be constant.*

(e) Simple calculation shows that α *is the partition point. We may use* $A\,,B$ *to mark the left and right part of the partition, respectively.* A *maps to* B *and* B *maps to the whole interval* $[0,1]$. *As the magnitude of the slope* $\Lambda = (\sqrt{5}+1)/2$ *is greater than* 1, *we may expect the natural measure is still piecewise constant with eigenvalue* 1. *The determining equation is*

$$\begin{pmatrix} 0 & 1/\Lambda \\ 1/\Lambda & 1/\Lambda \end{pmatrix} \begin{pmatrix} \rho^A \\ \rho^B \end{pmatrix} = \begin{pmatrix} \rho^A \\ \rho^B \end{pmatrix}\,,$$

which gives $\rho^B/\rho^A = \Lambda$.

Ulam!map, skew
Ulam!map, skew

For the second map, the construction of Exercise 13.6 is worth a look.

(Yueheng Lan)

Solution 14.7: Eigenvalues of the skew Ulam tent map Perron-Frobenius operator. *If we have density $\rho_n(x)$, the action of the Perron-Frobenius operator associated with $f(x)$ gives a new density*

$$\rho_{n+1}(x) = \frac{1}{\Lambda_0}\rho_n(x/\Lambda_0) + \frac{1}{\Lambda_1}\rho_n(1 - x/\Lambda_1)\,,$$

where $\Lambda_1 = \frac{\Lambda_0}{\Lambda_0 - 1}$. The eigenvalue equation is given by

$$\rho_{n+1}(x) = \lambda\rho_n(x)\,. \tag{S.47}$$

We may solve it by assuming that the eigenfunctions are N-th order polynomials $P(N)$ (check it). Indeed, detailed calculation gives the following results:

- *$P(0)$ gives $\lambda = 1$, corresponding to the expected leading eigenvalue.*
- *$P(1)$ gives $\lambda = \frac{1}{\Lambda_0^2} - \frac{1}{\Lambda_1^2} = \frac{2}{\Lambda_0} - 1$,*
- *$P(2)$ gives $\lambda = \frac{1}{\Lambda_0^3} + \frac{1}{\Lambda_1^3}$,*
- *$P(3)$ gives $\lambda = \frac{1}{\Lambda_0^4} - \frac{1}{\Lambda_1^4}$,*
- *The guess is that $P(N)$ gives $\lambda = \frac{1}{\Lambda_0^{N+1}} + (-1)^N \frac{1}{\Lambda_1^{N+1}}$.*

The final solution is that the piecewise linear function $\rho^A = -\Lambda_0\,, \rho^B = \Lambda_1$ gives the eigenvalue 0. If only the continuous functions are considered, this kind of eigenfunction of course should not be included.

(Yueheng Lan)

Solution 14.7: Eigenvalues of the skew Ulam tent map Perron-Frobenius operator. *The first few eigenvalues are*

$$\begin{aligned} e^{s_0} &= 1\,, \qquad e^{s_1} = \frac{2}{\Lambda_0} - 1 \\ e^{s_2} &= \frac{1}{4} + \frac{3}{4}\left(\frac{2}{\Lambda_0} - 1\right)^2\,, \qquad e^{s_3} = \frac{1}{2}\left(\frac{2}{\Lambda_0} - 1\right) + \frac{1}{2}\left(\frac{2}{\Lambda_0} - 1\right)^3 \ldots \end{aligned}$$

For eigenvectors (invariant densities for skew tent maps), see for example L. Billings and E.M. Bolt [14.10] .

Solution 14.10: $\mathcal{A}$ as a generator of translations. *If v is a constant in space, Taylor series expansion gives*

$$a(x + tv) = \Sigma_{k=0}^{\infty}\frac{1}{k!}(tv\frac{\partial}{\partial x})^k a(x) = e^{tv\frac{\partial}{\partial x}}a(x)\,.$$

(Yueheng Lan)