UNIVERSALITY FOR PERIOD n-TUPLINGS IN COMPLEX MAPPINGS

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The theory of period doublings for real iterative mappings is generalized to period n-tupling for complex iterative mappings. We find an infinity of universal functions associated with different sequences of period n-tuplings.

There is much experimental, numerical and theoretical evidence that infinite sequences of period doublings occur commonly in dissipative dynamical systems. In the universality theory for one-dimensional iterative mappings developed by Feigenbaum [1], variation of a single parameter drives the system through a bifurcation sequence. In this letter we extend this theory to the period n-tuplings found in iterations of complex analytic functions [2–6] parameterized by pairs of (real) parameters. A complex function can be viewed as a two-dimensional mapping. Iterative two-dimensional mappings are important in studies of dynamical systems [7,8], and the scaling numbers that we predict might be observable. Period n-tuplings are common in hamiltonian systems [9] and are also seen in nonlinear oscillator simulations [10,11], but it is not known whether realistic dynamical systems which can be modelled by complex iterations exist. Even if that is not the case, we find the complex universality rewarding for the insight it gives into the standard universality theory.

The first important difference between the real and complex cases is that the latter allows infinite sequences of period n-tuplings, rather than just period doublings. We give the corresponding universality equations and universal scaling numbers. Mandelbrot [6] has conjectured that Feigenbaum's [12,13] δ is one

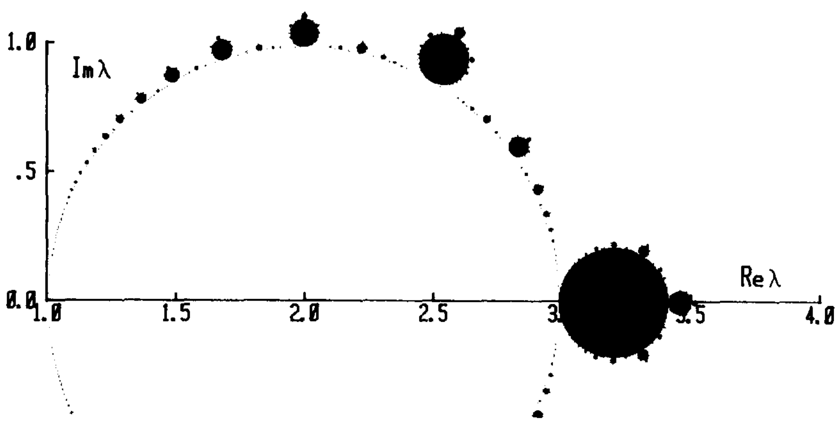

Fig. 1. A region in the complex λ plane where the Julia mapping, eq. (1), has stable cycles. Inside the big semicircle (left open for clarity) iterations converge to a fixed point. The full region has two symmetry axes, Re λ = 1 and Im λ = 0, so only one quarter is shown. The usual bifurcation sequence is on the real axis. See Mandelbrot [6] for detailed scans of this region.

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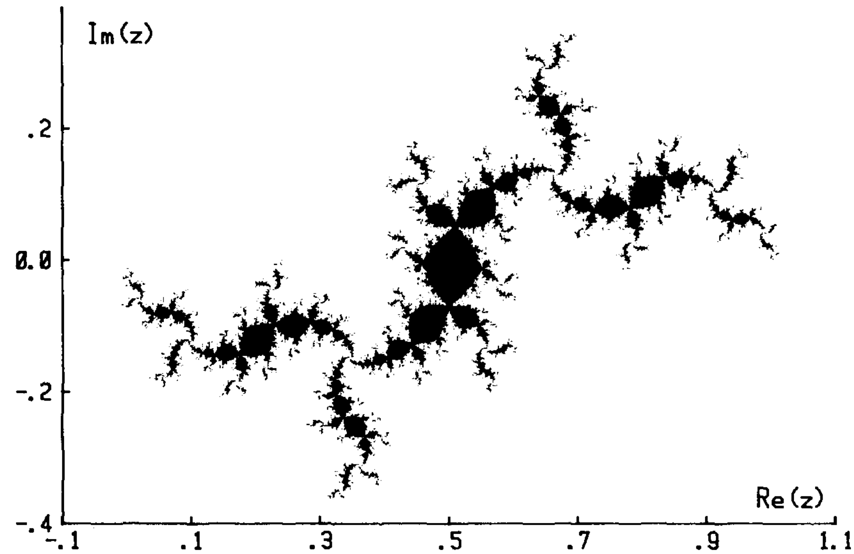

Fig. 2. The basin of attraction for the superstable 9 cycle arising from two successive 2/3 trifurcations (superstability means that the extremum $z = 1/2$ is a cycle point). In each successive trifurcation the central region is scaled down and rotated by the universal scaling number $-\alpha$. In this plot the scaled down basin of attraction for the superstable 3 cycle is visible in the center. See ref. [6] for many beautiful plots of basins of attraction.

of the infinity of possible δs, and that their existence and universality are consequences of the self-similarity of the region plotted in fig. 1. Our results fully support Mandelbrot's conjecture.

The second important difference between the two cases is in the nature of the basins of attraction. For the real case the basin is trivial: the unit interval is always mapped into itself. In the complex case the basin of attraction for a stable cycle is an area with a fractal boundary [6] (fig. 2), and our generalization of Feigenbaum's α plays two roles: it describes the trajectory splitting, just as in the real case, but it also characterizes the size of the basin of attraction. The difference between the real and the complex cases is especially dramatic for the chaotic bands [13]. In the real case their basin of attraction is again the unit interval, and the reverse bifurcations [13] are as easy to observe as the period doublings. We find that in the complex case the basin of attraction is shrunk to a fractal line (fig. 3), so that the chaotic bands have little chance of being observed experimentally.

Our starting point is iterations of the Julia–Fatou [2–6] type

$$z_{k+1} = \lambda z_k(1 - z_k)\,, \tag{1}$$

with complex z_k and complex parameter λ. By the usual universality-theory arguments we expect that the results will be the same for any function with a quadratic maximum. A k cycle satisfies $z_k = z_0$, and it is stable if

$$|\mathrm{d}z_k/\mathrm{d}z_0| < 1. \tag{2}$$

Whenever

$$\mathrm{d}z_k/\mathrm{d}z_0 = \exp(2\pi i m/n) \tag{3}$$

(m and n relatively prime integers) the k cycle becomes unstable and branches into an nk cycle, which then becomes stable. The main region in the complex λ plane for which stable cycles exist is plotted in fig. 1. This cactus-like region is self-reproducing because a stable cycle of any length gets unstable in the same fashion. If λ_j are the successive critical parameter values for m/n period n-tuplings, the limit

$$\delta_{m/n} = \lim_{j\to\infty} (\lambda_j - \lambda_{j-1})/(\lambda_{j+1} - \lambda_j)$$

exists. The magnitude of $\delta_{m/n}$ gives the relative size of two successive "cactii" while the phase gives their relative angle. $|\delta_{m/n}| \to C_m n^2$ asymptotically as $n \to \infty$, where (at least for small m) C_m is a number

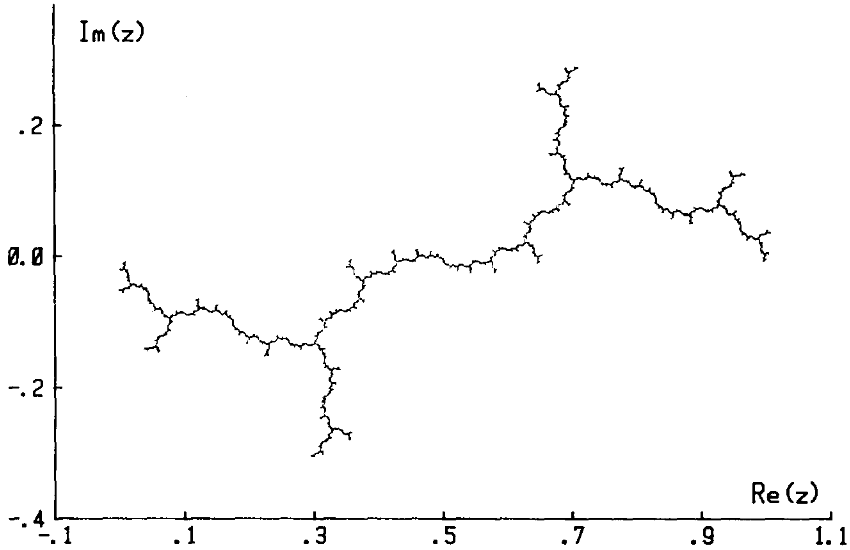

Fig. 3. The "basin of attraction" for a λ value such that eq. (1) maps the extremum $z = 1/2$ into the unstable fixed point $z = 0$ in 3 steps. This is a trifurcation sequence analogue of the last chaotic band for the real case.

close to 1. This n^2 behaviour was first observed by Mandelbrot [6].

Furthermore, any infinite sequence of m/n branchings defines a universal function $g(z) = g_{m/n}(z)$ and a universal scaling number $\alpha = \alpha_{m/n}$ satisfying a universal equation [‡1]

$$g(z) = -\alpha g^{(n)}(-z/\alpha) = -\alpha g(g(...g(-z/\alpha)...)) . \qquad (4)$$

Other itenararies of period n-tuplings define still other universal functions. We have computed polynomial approximations to $g(z)$, with up to 20 terms, in a way

[‡1] For $n = 2$ this is the universal equation of the bifurcation theory, derived by Cvitanović and Feigenbaum [1].

Table 1
Magnitude of δ and α, and phase of $-\alpha$ in units of 2π, for some sequences of m/n branchings. See also figs. 4 and 5.

m/n	$\lvert\delta\rvert$	$\lvert\alpha\rvert$	χ
1/2	4.6692	2.5029	−0.5
1/3	10.0908	3.1557	−0.3657
1/4	18.1298	3.4508	−0.3032
1/5	28.0371	3.5742	−0.2670
1/13	176.508	3.6839	−0.1783
2/5	23.9153	4.4459	−0.4184
2/13	170.943	5.3840	−0.2339

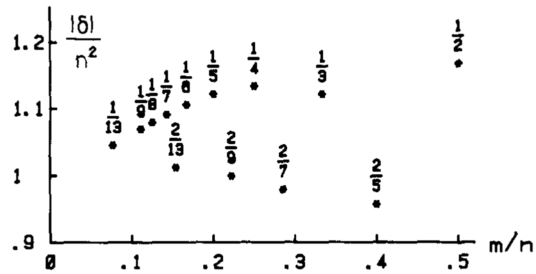

Fig. 4. Plot of $|\delta_{m/n}|/n^2$ versus m/n for the $1/n$ and $2/n$ sequences of period n-tuplings. $\delta_{1/2}$ is the Feigenbaum number δ for the bifurcation sequence.

similar to the one described in ref. [1]. These calculations also yield αs and δs. For example, the first few terms of the trifurcation universal function are

$$g_{1/3}(z) = 1 + (0.0547 + i0.7490)z^2$$

$$+ (-0.0244 + i0.0525)z^4 + ...$$

and some δs and αs are listed in table 1, and plotted in figs. 4 and 5. (More exhaustive tables will be given in a subsequent publication [14].) The phase of $\alpha_{m/n}$ is given by

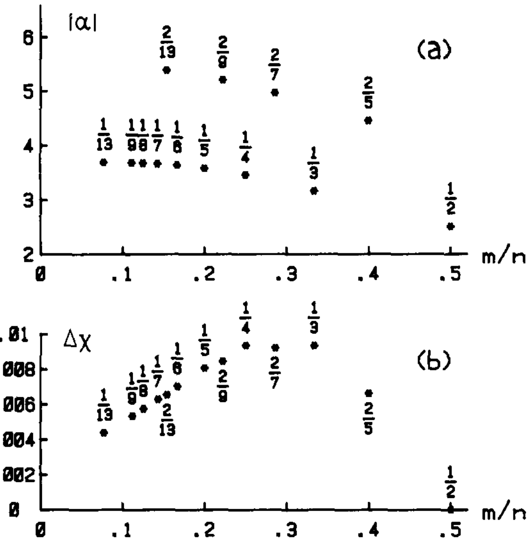

Fig. 5. (a) Plot of $|\alpha_{m/n}|$ versus m/n. This is the scaling number for trajectory splittings in successive period n-tuplings. (b) Plot of $\Delta\chi_{m/n}$, eq. (5), versus m/n.

$$-\alpha = |\alpha| \exp(2\pi\chi) ,$$

$$\chi = -1/8 - 3m/4n + \Delta\chi , \qquad (5)$$

where $\Delta\chi$ is a small deviation (fig. 5b).

The significance of the universal equation is perhaps best illustrated by the associated n cycle. In fig. 6 we have plotted the first 30000 points obtained by iterating $g_{1/13}(z)$. The sequence $z_0 = 0 \to z_1 = g(z_0) \to ... \to z_{12}$ traces out a large "horseshoe". The sequence $z_0 \to z_{13} \to z_{26} \to ... \to z_{156}$ traces out a smaller horseshoe, and so forth. The universality equation (4) states that these horseshoes are similar under rescaling and rotation by $-\alpha_{1/13}$. χ, eq. (5), is the rotation angle. The horseshoes for $1/n, n \to \infty$, look very much alike, with all the extra cycle points accumulating between two unstable fixed points. In this way one can understand the existence of the $n \to \infty, m$ fixed limits for $\alpha_{m/n}$, see table 1 and fig. 5.

Plots of basins of attraction (sets of all initial points which converge towards a given stable cycle) provide another way of understanding the significance of the scaling number α. The basin of attraction for the $(2/3)^2$ cycle is shown in fig. 2. Its area is roughly $|\alpha_{2/3}|^2$ smaller than for the 2/3 basin of attraction. The self-

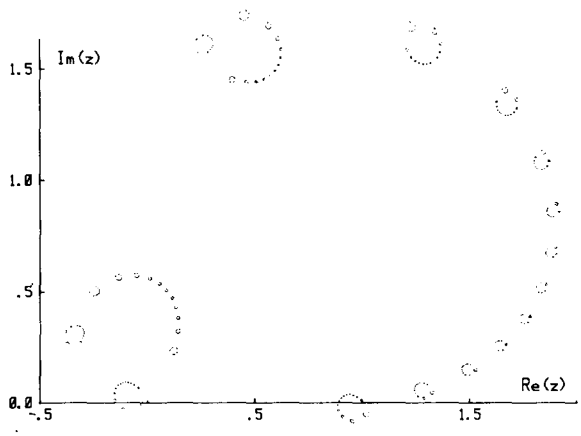

Fig. 6. The 1/13 "universal horseshoe": the first 30000 points $z_0 = 0, z_1, ...$ from the iteration of the 1/13 universal function. The smaller horseshoe at the origin is identical to the full horseshoe, except for a scaling and rotation by the complex number $-\alpha_{1/13}$.

similarity under rescaling and rotation by $-\alpha_{2/3} = -\alpha^*_{1/3}$ is already evident.

If the extremum maps into an unstable fixed point, then the attractor, in the real case [13], is one or more chaotic bands. Complex generalizations of the Misiurewicz [15] sequence of reverse bifurcations can also be studied. In the complex case the chaotic motion takes place along a fractal curve. An example is shown in fig. 3. It is associated with the 2/3 sequence of trifurcations, and the value of the parameter λ is determined by the conditions $z_0 = 1/2, z_2 = 1, z_3 = 0 =$ unstable fixed point, and Im $\lambda > 0$.

In summary, we have generalized the universality theory to iterations of complex analytic functions. An interesting aspect of this theory is the observation that the universal scaling numbers α for period n-tuplings approach $n \to \infty$ limits. An experimental observation of sequences of multifurcations would be made difficult both by the n^4 reduction of the area in parameter space for each successive n tupling, and by the $|\alpha|^2$ reduction of the basin of attraction, implying the need for very precise adjustment of initial conditions. Still, the experimental precision needed for observing one trifurcation is roughly the same as is required for observing two bifurcations, so several trifurcations could be observed.

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