Chapter 11

Elastodynamics

11.1 Overview

In the previous chapter we considered elastostatic equilibria in which the forces acting on elements of an elastic solid were balanced so that the solid remained at rest. When this equilibrium is disturbed, the solid will undergo acceleration. This is the subject of elastodynamics. In this chapter (Secs. 11.2 and 11.3), we derive the equations of motion for elastic media, paying particular attention to the underlying conservation laws. We do this in a manner which will allow us easily to modify our equations when we consider fluid dynamics in the following chapter. The simplest examples of elastodynamics involve waves. We will show that there are two distinct wave modes that propagate in a uniform, isotropic solid, *longitudinal* waves and *shear* waves, and both are nondispersive (their phase speeds are independent of frequency).

A major use of elastodynamics is in structural engineering (Sec. 11.4). Here we discuss the types of waves that propagate on bars, rods and beams and find that the boundary conditions at the surfaces make the waves dispersive. We also return briefly to the problem of bifurcation and show how, by changing the parameters controlling an equilibrium, a linear wave can be made to grow exponentially in time, thereby rendering the equilibrium unstable.

A second application of elastodynamics is to seismology (Sec. 11.5). The earth is mostly a solid body through which waves can propagate. The waves can be excited naturally by earthquakes or artificially using man-made explosions. Understanding how waves propagate through the earth is important for locating the sources of earthquakes and for analyzing the structure of the earth. We briefly describe some of the wave modes that propagate through the earth and some of the inferences about the earth's structure that have been drawn from studying their propagation. In the process, we gain some experience in applying the tools of geometric optics to new types of waves, and we learn how rich can be the Green's function for elastodynamic waves, even when the medium is as simple as a homogeneous half space.

Finally (Sec. 11.6), we return to physics to consider the quantum limit of our classical approach. We contrast the classical description of waves with their quantum mechanical counterparts, specialising to quantised vibrations in an elastic solid, phonons.

11.2 Conservation Laws

Prior to stating the fundamental laws of elastodynamics, we should first explain a general approach to classical physics which unifies many of the topics we treat in this course. This approach is based on *conservation laws*, and is the Newtonian version of the special relativistic ideas we developed in Sec. 1.12.

In non-relativistic continuum mechanics, three basic quantities are conserved: mass, momentum and energy. The most elementary conservation law involves mass. Consider a continuous substance with mass density $\rho(\mathbf{x}, t)$, and a small elementary volume $\mathcal{V}$, fixed in space (i.e., fixed in some Newtonian reference frame). If the matter moves, then there will be a flow of mass, a *mass flux* across each element of surface $d\mathbf{\Sigma}$ on the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Provided that there are no sources or sinks of matter, the total rate of change of the mass residing within $\mathcal{V}$ will be given by the net rate of mass transport across $\partial\mathcal{V}$. Now the rate at which mass moves across a unit area is the *mass flux*, $\rho\mathbf{v}$, where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field. We can therefore write

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho dV = -\int_{\partial\mathcal{V}} \rho\mathbf{v}\cdot d\mathbf{\Sigma}. \tag{11.1}$$

(Remember that the surface is fixed in space.) Invoking Gauss' theorem, we obtain

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho dV = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) dV. \tag{11.2}$$

As this must be true for arbitrary small volumes $\mathcal{V}$, we can abstract the differential equation of mass conservation:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0. \tag{11.3}$$

(Even if this equation is not familiar in the context of continuum mechanics, its analogue, the equation of electrical charge conservation, ought to be familiar.) Writing the conservation equation in this manner where we monitor the changing density at a given location in space, rather following a moving element, is called the *Eulerian* approach.

Equation (11.3) is our model. It says that there is a quantity, in this case *mass*, with a certain density, in this case ρ, and a certain flux, in this case $\rho\mathbf{v}$, and this quantity is neither created nor destroyed. The temporal derivative of the density (at a fixed point in space) added to the divergence of the flux must then vanish. Of course, not all physical quantities have to be conserved. If there were sources or sinks of mass, then these would be added to the right hand side of Eq. (11.3).

The next quantity to treat in this manner is momentum. If we just consider the mechanical momentum, its density is $\rho\mathbf{v}$, and it is a vector field. The momentum $d\mathbf{p}$ crossing a small element of area $d\mathbf{\Sigma}$, from the back side of $d\mathbf{\Sigma}$ to the front (in the "positive sense"; cf. Fig. 1.12b) during unit time is given by $d\mathbf{p} = (\rho\mathbf{v}\cdot d\mathbf{\Sigma})\mathbf{v}$. This is also a vector and it is a linear function of the element of area $d\mathbf{\Sigma}$. This then allows us to define a second rank tensor, the *momentum flux*, by the equivalent relations

$$d\mathbf{p} = (\rho\mathbf{v}\cdot d\mathbf{\Sigma})\mathbf{v} = \mathbf{T}_m(\_, d\mathbf{\Sigma})\ , \quad \mathbf{T}_m = \rho\mathbf{v}\otimes\mathbf{v}. \tag{11.4}$$

This tensor is manifestly symmetric. We are now in a position to write down a conservation law for momentum by direct analogy with Eq. (11.3), namely

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T}_m = \mathbf{f}, \tag{11.5}$$

where $\mathbf{f}$ is the net rate of increase of momentum in a unit volume, i.e. the force per unit volume acting on the material.

We have used the symbol $\mathbf{T}_m$ for the mechanical momentum flux because, as well as being the momentum crossing unit area in unit time, it is also a piece of the stress tensor (force per unit area): the force that acts across a unit area is just the rate at which momentum crosses that area; cf. the relativistic discussion in Sec. 1.12

Now consider what happens in an elastic medium. When the medium is deformed, elastic forces produce on it a force per unit volume $\mathbf{f} = -\boldsymbol{\nabla} \cdot \mathbf{T}_{el}$ [cf. Eqs. (10.31) and (10.35)], where the subscript el labels the elastic contribution to the stress [Eq. (10.34)]. We can therefore combine this with $\mathbf{T}_m$ to obtain a combined conservation law with no extraneous forces (sources or sinks of momentum),

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} = 0, \quad \text{where} \quad \mathbf{T} = \mathbf{T}_m + \mathbf{T}_{el} = \rho \mathbf{v} \otimes \mathbf{v} - K\Theta \mathbf{g} - 2\mu \boldsymbol{\Sigma}\ . \tag{11.6}$$

Here $\mathbf{T} = \mathbf{T}_m + \mathbf{T}_{el}$ is the total stress tensor. Mechanical plus elastic momentum is therefore conserved. (A conservation law for energy can also be derived, but we will not need it in our treatment of elasticity and will defer its introduction to the following chapter on fluid dynamics where it will be crucial.)

We conclude this section with two remarks. The first is that in going from Newtonian physics (this chapter) to special relativity (Chap. 1), mass and energy get combined (added) to form a conserved mass-energy or total energy; that total energy and the momentum are the temporal and spatial parts of a spacetime 4-vector, the 4-momentum; and correspondingly, the conservation laws for mass [Eq. (11.3)], nonrelativistic energy [Chap. 12], and momentum [Eq. (11.6)] get unified into a single conservation law for 4-momentum, which is expressed as the vanishing 4-dimensional, spacetime divergence of the 4-dimensional stress-energy tensor (Sec. 1.12). The second remark is that there may seem something tautological about our procedure. We have argued that the mechanical momentum will not be conserved in the presence of forces such as elastic forces. Then we argued that we can actually associate a momentum flux, or more properly a stress tensor, with the strained elastic medium, so that the combined momentum is conserved. It is almost as if we regard conservation of momentum as a principle to be preserved at all costs and so every time there appears to be a momentum deficit, we simply define it as a bit of the momentum flux. (An analogous accusation could be made about the conservation of energy.) This, however, is not the whole story. What is important about our formulation of the theory of elasticity is that the elastic force density can be expressed as the divergence of a stress tensor. An erroneous formulation would not necessarily have this property and there would not be a differential conservation law. So the fact that we *can* create elastostatic, thermodynamic, viscous, electromagnetic, gravitational *etc* contributions to some grand stress tensor (that go to zero outside the regions occupied by the relevant matter or fields), as we shall see in the coming chapters, *is* significant and affirms that our physical model is complete at the level of approximation to which we are working.

11.3 Basic Equations of Elastodynamics

In this section we shall use the laws of mass and momentum conservation to derive a vectorial equation that governs the dynamical displacement $\boldsymbol{\xi}(\mathbf{x}, t)$, and shall show how that equation can be converted into two wave equations, one for "longitudinal" waves and the other for "transverse" waves. Our method of analysis is a special case of a very general approach to deriving wave equations in continuum mechanics. That general approach is sketched in Box 11.1. We shall follow that approach not only here, for elastic waves, but also in Part IV for waves in fluids, Part VI for waves in plasmas and Part V for general relativistic gravitational waves.

11.3.1 Equation of Motion

We derive the elastodynamic equation of motion by multiplying the law of mass conservation [Eq. (11.3)] by $\mathbf{v}$ and subtracting from the law of momentum conservation [Eq. (11.6)] to obtain

$$\rho \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \right] \mathbf{v} = -\boldsymbol{\nabla} \cdot \mathbf{T}_{el}. \tag{11.7}$$

The operator appearing on the left hand side of (11.7),

$$\frac{\partial}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \equiv \frac{d}{dt} , \tag{11.8}$$

is known as the *convective derivative* and crops up often in continuum mechanics. Its physical interpretation is very simple. Consider first the partial derivative $(\partial/\partial t)_{\mathbf{x}}$. This is the rate of change of some quantity [the velocity $\mathbf{v}$ in Eq. (11.7)] at a fixed point in space in some reference frame. In other words, if there is motion, $\partial/\partial t$ compares this quantity at the same point in space for two different points in the solid. By contrast, the convective derivative operator, (d/dt) follows the motion and takes the difference in the value of the quantity at successive instants at the same point in the moving matter. It therefore measures the rate of change of the physical quantity following the material rather than at a fixed point in space. This convective derivative is the Newtonian limit of relativity's proper time derivative along the world line of a bit of matter, $d/d\tau = u^\alpha \partial/\partial x^\alpha = (dx^\alpha/d\tau)\partial/\partial x^\alpha$ [Secs. 1.4 and 1.6]. In Eq. (11.7), $d\mathbf{v}/dt \equiv [\partial/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})]\mathbf{v}$ is the acceleration of the solid and therefore, Eq. (11.7) can be interpreted as Newton's second law for a fixed mass of the solid. Note that the velocity itself is the convective derivative of the displacement

$$\mathbf{v} = \frac{d\boldsymbol{\xi}}{dt} . \tag{11.9}$$

Now, in elastodynamics, it is more convenient to work with the displacement $\boldsymbol{\xi}$ than the velocity $\mathbf{v}$. If the strains in the solid are much smaller than unity, and we have already

Box 11.1
Wave Equations in Continuum Mechanics

In this box, we make an investment for future chapters by considering wave equations in some generality.

Most wave equations arise as approximations to the full set of equations that govern a dynamical physical system. It is usually possible to arrange those full equations as a set of first order partial differential equations that describe the dynamical evolution of a set of n physical quantities, V_A, with $A = 1, 2, ..., n$: i.e.

$$\frac{\partial V_A}{\partial t} + F_A(V_B) = 0 \ . \qquad (1)$$

[For elastodynamics there are $n = 7$ quantities V_A: $\{\rho, \rho v_x, \rho v_y, \rho v_z, \xi_x, \xi_y, \xi_z\}$; and the 7 equations (1) include mass conservation, momentum conservation, and $\partial\xi_j/\partial t = \xi_j$.]

Now most dynamical systems are intrinsically nonlinear (Maxwell's equations *in vacuo* being a conspicuous exception) and it is usually quite hard to find nonlinear solutions. However, it is generally possible to make a perturbation expansion in some small physical quantity about an equilibrium and just retain terms that are linear in this quantity. We then have a set of n linear partial differential equations that are much easier to solve than the nonlinear ones—and that usually turn out to have the character of wave equations (i.e., to be "hyperbolic"). Of course the solutions will only be a good approximation for small amplitude waves. [In elastodynamics, we justify linearization by requiring that the strains be below the elastic limit, we linearize in the strain or displacement, and the resulting linear wave equation is $\rho\partial^2\boldsymbol{\xi}/\partial t^2 = (K + \frac{1}{3}\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi}$.]

Boundary Conditions

In some problems, e.g. determining the normal modes of vibration of a building during an earthquake, or analyzing the sound from a violin or the vibrations of a finite-length rod, the boundary conditions are intricate and have to be incorporated as well as possible, to have any hope of modeling the problem. The situation is rather similar to that familiar from elementary quantum mechanics. The waves are often localised within some region of space, like bound states, in such a way that the eigenfrequencies are discrete, for example, standing wave modes of a plucked string. In other problems the volume in which the wave propagates is essentially infinite as happens with unbound states (e.g. waves on the surface of the ocean or seismic waves propagating through the earth). The only boundary condition is essentially that the wave amplitude remain finite at large distances. In this case, the wave spectrum is usually continuous.

Box 11.1, Continued

Geometric Optics Limit and Dispersion Relations

The solutions to the wave equation will reflect the properties of the medium through which the wave is propagating, as well as its boundaries. If the medium and boundaries have a finite number of discontinuities but are otherwise smoothly varying, there is a simple limiting case: waves of short enough wavelength and high enough frequency that they can be analyzed in the *geometric optics approximation* (Chap. 6).

The key to geometric optics is the *dispersion relation*, which acts as a Hamiltonian for the propagation. Recall from Chap. 6 that although the medium may actually be inhomogeneous and might even be changing with time, when deriving the dispersion relation we can approximate it as precisely homogeneous and time-independent.

In such a situation we can resolve the waves into plane-wave modes, i.e. modes in which the perturbations vary $\propto \exp i(\mathbf{k}\cdot\mathbf{x}-\omega t)$. Here $\mathbf{k}$ is the *wave vector* and ω is the *angular frequency*. This allows us to remove all the temporal and spatial derivatives and converts our set of partial differential equations into a set of homogeneous, linear algebraic equations. When we do this, we say that our normal modes are *local*. If, instead, we were to go to the trouble of solving the partial differential equation with its attendant boundary conditions, the modes would be referred to as *global*.

The linear algebraic equations for a local problem can be written in the form $M_{AB}V_B = 0$, where V_A is the vector of dependent variables and the elements M_{AB} of the matrix $||M_{AB}||$ depend on $\mathbf{k}$ and ω as well as on parameters p_α that describe the local conditions of the medium. This set of equations can be solved in the usual manner by requiring that the determinant of $||M_{AB}||$ vanish. Carrying through this procedure yields a polynomial, usually of n'th order, for $\omega(\mathbf{k}, p_\alpha)$. This polynomial is the dispersion relation. It can be solved (analytically in simple cases and numerically in general) to yield a number of complex solutions for ω, with $\mathbf{k}$ regarded as real. (Of course, we might just as well treat the wave vector as a complex number, but for the moment we will regard it as real.) Armed with these solutions, we can solve for the associated eigenvectors. The eigenfrequencies fully characterize the solution of the local problem, and can be used to solve for the temporal evolution from some given initial conditions in the usual manner. (As we shall see several times, especially when we discuss Landau damping in Chap. 21, there are some subtleties that can arise.)

What does a complex value of the angular frequency ω mean? We have posited that all small quantities vary $\propto \exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$. If ω has a positive imaginary part, then the small quantities will grow exponentially with time. Conversely, if it has a negative imaginary part, it will decay. Now polynomial equations with real coefficients have complex conjugate solutions. Therefore if there is a decaying mode there must also be a growing mode. Growing modes correspond to *instability*, a topic that we shall encounter often.

established that they are typically $\lesssim 10^{-3}$ for elastic deformations, then the density can be treated as effectively constant. More specifically, the equation of mass conservation (11.3) can be integrated with respect to time to obtain for the fractional change of density due to an elastic strain

$$\frac{\delta\rho}{\rho} \simeq -\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = -\Theta \lesssim 10^{-3} \text{ in magnitude,} \tag{11.10}$$

where we have used Eq. (10.9); so for most purposes in elasticity theory we can set

$$\rho = \text{constant.} \tag{11.11}$$

We can make a similar approximation to the equation of motion (11.7). If the displacement $\boldsymbol{\xi}$ varies over some characteristic timescale T and some characteristic lengthscale L, then the ratio of the second term $\rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$ to the first term $\rho\partial\mathbf{v}/\partial t$ is $O(\xi/L)$, which is of order the strain. Again, for a conventionally elastic material, this is a small number, $\lesssim 10^{-3}$ in magnitude, so we need only retain the first term. Substituting Eq. (10.32) for the elastic force density, we then obtain

$$\begin{aligned}\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} &= -\boldsymbol{\nabla}\cdot\mathbf{T}_{el} \\ &= (K+\frac{1}{3}\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi},\end{aligned} \tag{11.12}$$

which is a linear wave equation for the elastic displacement $\boldsymbol{\xi}$. Remember that we have assumed that the elastic moduli are constant. If their variation is significant, then extra terms will appear. Equation (11.12) is the first of many wave equations we shall encounter in elastodynamics, fluid mechanics, and plasma physics.

11.3.2 Elastodynamic Waves

Continuing to follow our general procedure for deriving and analyzing wave equations as outlined in Box 11.3, we next derive dispersion relations for the two types of waves (longitudinal and transverse) that are jointly incorporated into the general elastodynamic wave equation (11.12).

Recall from Chap. 6 that, although a dispersion relation can be used as a Hamiltonian for computing wave propagation through an *inhomogeneous medium*, one can derive the dispersion relation most easily by specializing to monochromatic plane waves propagating through a medium that is precisely homogeneous. Therefore, we seek seek a plane-wave solution, i.e. a solution of the form

$$\boldsymbol{\xi}(\mathbf{x},t) \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\ , \tag{11.13}$$

to the wave equation (11.12) with ρ, K and μ regarded as homogeneous (constant). (To deal with more complicated perturbations of a homogeneous medium, we can think of this wave as being an individual Fourier component and linearly superpose many such waves as a Fourier integral.) Since our wave is planar and monochromatic, we can remove the derivatives in Eq. (11.12) by making the substitutions $\nabla^2 \to -k^2$ and $\partial/\partial t \to -i\omega$. (More

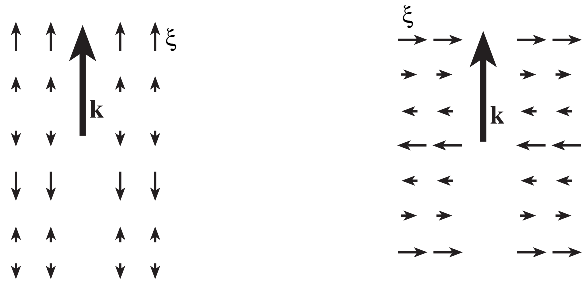

Fig. 11.1: Displacements in an isotropic, elastic solid, perturbed by a) a longitudinal mode, b) a transverse mode.

generally we can also use the substitutions $\boldsymbol{\nabla} \to i\mathbf{k}$, $\boldsymbol{\nabla}\cdot \to i\mathbf{k}\cdot$, $\boldsymbol{\nabla}\times \to i\mathbf{k}\times$.) We thereby reduce the partial differential equation (11.12) to a set of algebraic equations:

$$\rho\omega^2\boldsymbol{\xi} = \mu k^2\boldsymbol{\xi} + (K + \frac{1}{3}\mu)\mathbf{k}(\mathbf{k}\cdot\boldsymbol{\xi}) \ . \tag{11.14}$$

(This reduction is only possible because the medium is uniform or in the geometric optics limit of near uniformity; otherwise, we must solve the second order partial differential equation (11.12) using standard techniques.)

11.3.3 Longitudinal sound waves

How do we solve this equation? The sure way is to write it as a 3×3 matrix equation for the vector $\boldsymbol{\xi}$ and set the determinant to zero (Box 11.3). This is not hard for small or sparse matrices. However, some wave equations are more complicated and it repays us to think about the waves in a geometric, coordinate-independent way before resorting to brute force.

The quantity that oscillates in the elastodynamic waves (11.14) is the vector field $\boldsymbol{\xi}$. The nature of its oscillations is influenced by some scalar constants (ρ, μ, K, ω) and by just one quantity that has directionality: the constant vector $\mathbf{k}$. It seems reasonable to expect the description (11.14) of the oscillations to simplify, then, if we resolve the oscillations into a "longitudinal" component along $\mathbf{k}$ and a "transverse" component perpendicular to $\mathbf{k}$:

$$\boldsymbol{\xi} = \boldsymbol{\xi}_L + \boldsymbol{\xi}_T \ , \quad \boldsymbol{\xi}_L = \xi_L\hat{\mathbf{k}} \ , \quad \boldsymbol{\xi}_T\cdot\hat{\mathbf{k}} = 0 \ . \tag{11.15}$$

Here $\hat{\mathbf{k}} \equiv \mathbf{k}/k$ is the unit vector along the propagation direction. It is easy to see that the longitudinal mode has nonzero expansion $\Theta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\xi} \neq 0$ but vanishing rotation $\boldsymbol{\phi} = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\xi} = 0$, and can therefore be written as the gradient of a scalar potential, $\xi_L = \boldsymbol{\nabla}\psi$. By contrast, the transverse mode has zero expansion but nonzero rotation and can thus be written as the curl of a vector potential, $\boldsymbol{\xi}_T = \boldsymbol{\nabla}\times\mathbf{A}$; cf. Exercise 11.1.

For the longitudinal mode the algebraic wave equation (11.14) reduces to the following simple relation [as one can easily see by inserting $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_L = \xi_L\hat{\mathbf{k}}$ into Eq. (11.14) , or, alternatively, by taking the divergence of (11.14), which is equivalent to taking the scalar product with $\mathbf{k}$]:

$$\omega^2 = \frac{K + \frac{4}{3}\mu}{\rho}k^2 \ . \tag{11.16}$$

This relation between ω and $\mathbf{k}$ is the longitudinal mode's dispersion relation.

As we discussed in great detail in Sec. 6.2, associated with any wave mode is its phase velocity, $\mathbf{V}_{\rm ph} = (\omega/k)\hat{\mathbf{k}}$ and its phase speed $V_{\rm ph} = \omega/k$. The dispersion relation (11.16) implies that for longitudinal elastodynamic modes, the phase speed is

$$c_L = \frac{\omega}{k} = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{1/2} . \tag{11.17}$$

As this does not depend on the frequency ω, the mode is *non-dispersive*. As the wave properties do not depend on the direction of propagation through the medium, the phase speed is also isotropic, naturally enough, and the group velocity is equal to the phase velocity, $\mathbf{V}_g = \mathbf{V}_{\rm ph} = c_L\hat{\mathbf{k}}$.

Elastodynamic longitudinal modes are similar to sound waves in a fluid. However, in a fluid, as we shall see in subsequent chapters, the sound waves travel with phase speed $V_{\rm ph} = (K/\rho)^{1/2}$ [the limit of Eq. (11.17) when the shear modulus vanishes]. This fluid sound speed is lower than the c_L of a solid with the same bulk modulus because the longitudinal displacement necessarily entails shear (note that in the left half of Fig. 11.1 the motions are not an isotropic expansion), and in a solid there is a restoring shear stress (proportional to μ) that is absent in a fluid.

Because the longitudinal phase velocity is independent of frequency, we can write down general planar longitudinal-wave solutions to the elastodynamic wave equation (11.12) in the following form:

$$\boldsymbol{\xi} = \xi_L\hat{\mathbf{k}} = [A(\hat{\mathbf{k}}\cdot\mathbf{x} - c_L t) + B(\hat{\mathbf{k}}\cdot\mathbf{x} + c_L t)]\hat{\mathbf{k}} , \tag{11.18}$$

where A, B are arbitrary functions. The two terms in Eq. (11.18) describe waves propagating in opposite directions, with arbitrary profiles determined by the functions A and B.

11.3.4 Transverse shear waves

To derive the dispersion relation for a transverse wave we can simply make use of the transversality condition $\mathbf{k}\cdot\boldsymbol{\xi}_T = 0$ in Eq. (11.14); or, equally well, we can take the curl of Eq. (11.14) (multiply it by $i\mathbf{k}\times$), thereby projecting out the transverse piece, since the longitudinal part of $\boldsymbol{\xi}$ has vanishing curl. The result is

$$\omega^2 = \frac{\mu}{\rho}k^2 . \tag{11.19}$$

This dispersion relation implies that the transverse waves propagate with a speed c_T given by

$$c_T = \left(\frac{\mu}{\rho}\right)^{1/2} . \tag{11.20}$$

As $K > 0$, the shear wave speed is always less than the speed of longitudinal waves.

These transverse modes are known as *shear waves* because they are driven by the shear stress; cf. Fig. 11.1. There is no expansion and therefore no change in volume associated with shear waves. They do not exist in fluids, but they are close analogs of the transverse vibrations of a string.

Longitudinal waves can be thought of as scalar waves, since they are fully describable by a single component ξ_L of the displacement $\boldsymbol{\xi}$: that along $\hat{\mathbf{k}}$. Shear waves, by contrast, are inherently vectorial. Their displacement $\boldsymbol{\xi}_T$ can point in any direction orthogonal to $\mathbf{k}$. Since the directions orthogonal to $\mathbf{k}$ form a two-dimensional space, once $\mathbf{k}$ has been chosen, there are two independent states of polarization for the shear wave. These two polarization states, together with the single one for the scalar, longitudinal wave, make up the three independent degrees of freedom in the displacement $\boldsymbol{\xi}$.

11.3.5 Energy of Elastodynamic Waves

Elastodynamic waves transport energy, just like waves on a string. The energy density is the sum of the kinetic energy density and the elastic energy density computed using Eq. (10.39). For the longitudinal mode this becomes

$$U_L = \left\langle \frac{1}{2}\rho\dot{\xi}^2 + \frac{1}{2}\left(K + \frac{4}{3}\mu\right)\Theta^2 \right\rangle , \tag{11.21}$$

where $\langle ... \rangle$ denotes an average over one period or wavelength of the wave. If we now substitute $\nabla \cdot \boldsymbol{\xi}$ for Θ, and the dispersion relation (11.16) into Eq. (11.21), then we find that the two terms are equal and so there is equipartition of energy between the kinetic and the elastic energy density just as in a simple harmonic oscillator. Therefore,

$$U_L = \rho\omega^2\langle\xi_L^2\rangle . \tag{11.22}$$

Similar formulae holds for the shear-wave energy density and the total energy density,

$$U_T = \rho\omega^2\langle\boldsymbol{\xi}_T \cdot \boldsymbol{\xi}_T\rangle , \quad U = U_L + U_T = \rho\omega^2\langle\boldsymbol{\xi}\cdot\boldsymbol{\xi}\rangle . \tag{11.23}$$

How fast is the energy transported? Just as electromagnetic waves transport electromagnetic energy, via the Poynting flux $F_{EM} = U_{EM}c$, at the same speed as the waves propagate (the speed of light c), so elastodynamic waves have an associated energy flux of

$$F_{L,T} = Uc_{L,T}, \tag{11.24}$$

corresponding to a propagation speed equal to that of the wave itself, $c_{L,T}$. One derivation of this was given, quite generally, for any dispersionless wave, in Ex. 6.7; another derivation will be given when we study fluid sound waves in Chap. 15.

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EXERCISES

Exercise 11.1 *Example: Scalar and Vector Potentials for Elastic Waves in a Homogeneous Solid*

Just as in electromagnetic theory, it is sometimes useful to write the displacement $\boldsymbol{\xi}$ in terms of scalar and vector potentials,

$$\boldsymbol{\xi} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\times\mathbf{A} . \tag{11.25}$$

(The vector potential $\mathbf{A}$ is, as usual, only defined up to a gauge transformation, $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\varphi$, where φ is an arbitrary scalar field.) Show that the scalar and vector potentials satisfy the following wave equations in a homogeneous solid:

$$\frac{\partial^2 \psi}{\partial t^2} = c_L^2 \nabla^2 \psi \ , \quad \frac{\partial^2 \mathbf{A}}{\partial t^2} = c_T^2 \nabla^2 \mathbf{A} \ . \tag{11.26}$$

Thus, the scalar potential ψ generates longitudinal waves, while the vector potential $\mathbf{A}$ generates transverse waves.

Exercise 11.2 *Problem: Influence of gravity on wave speed*
Modify the wave equation (11.12) to include the effect of gravity. Assume that the gravitational field is constant. By comparing the orders of magnitude of the terms involving small perturbations to the displacement, stress and density verify that the gravitational terms can be ignored for short wavelength elastodynamic modes.

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11.4 Waves in Rods, Strings and Beams

Let us now illustrate some of these ideas using the type of waves that can arise in some practical applications. In particular we discuss how the waves can be modified when the medium through which they propagate is not uniform but instead is bounded. Despite this situation being formally “global” in character (cf. Box 11.3), elementary considerations enable us to derive the relevant dispersion relations without much effort.

11.4.1 Compression waves

First, consider a longitudinal wave propagating along a light, thin, unstressed rod. Introduce a Cartesian coordinate system with the z-axis parallel to the rod. When there is a small displacement ξ_z independent of x and y, the restoring stress is given by $T_{zz} = -E\partial\xi_z/\partial z$, where E is Young’s modulus (cf. end of Sec. 10.4). Hence the restoring force density can be evaluated from the divergence of the stress tensor as $f_z = E\partial^2\xi_z/\partial z^2$. The wave equation then becomes

$$\frac{\partial^2 \xi_z}{\partial t^2} = \left(\frac{E}{\rho}\right) \frac{\partial^2 \xi_z}{\partial z^2} \ , \tag{11.27}$$

and so the sound speed for compression waves in a long straight rod is

$$c_C = \left(\frac{E}{\rho}\right)^{\frac{1}{2}} \ . \tag{11.28}$$

Referring to Table 10.1 (on page 10-19), we see that a typical value of Young’s modulus in a solid is ~ 100 GPa. If we adopt a typical density $\sim 3 \times 10^3$ kg m^{-3}, then we estimate the longitudinal sound speed to be ~ 5 km s^{-1}. This is roughly 15 times the sound speed in air.

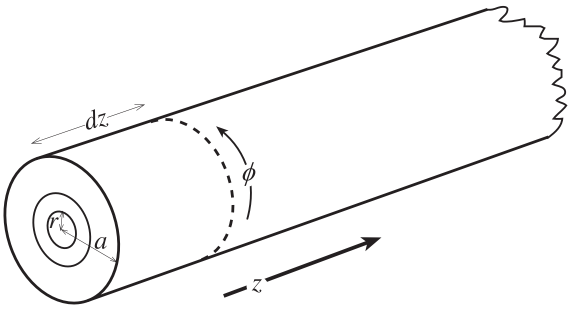

Fig. 11.2: When a wire of circular cross section is twisted, there will be a restoring torque.

11.4.2 Torsion Waves

Next consider a wire with circular cross section of radius a subjected to a twisting force. Let us introduce an angular displacement ϕ that depends on z. We can calculate the total torque by integating over a circular cross section. For small twists, there will be no expansion and the only components of the shear tensor are

$$\Sigma_{\phi z} = \Sigma_{z\phi} = \frac{r}{2}\frac{\partial \phi}{\partial z} \; . \tag{11.29}$$

The torque contributed by an annular ring of radius r and thickness dr is $r \cdot T_{\phi z} \cdot 2\pi r dr$ and we substitute $T_{\phi z} = -2\mu \Sigma_{\phi z}$ to obtain the total torque

$$G = \int_0^a 2\pi \mu r^3 dr \frac{\partial \phi}{\partial z} \; . \tag{11.30}$$

Now the moment of inertia per unit length is

$$I = \frac{\pi}{2}\rho a^4 \; , \tag{11.31}$$

so equating the net torque per unit length to the rate of change of angular momentum, also per unit length, we obtain

$$\frac{\partial G}{\partial z} = I \frac{\partial^2 \phi}{\partial t^2} \; , \tag{11.32}$$

or

$$\frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\rho}{\mu}\right) \frac{\partial^2 \phi}{\partial t^2} \; . \tag{11.33}$$

The speed of torsional waves is thus

$$c_T = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \; . \tag{11.34}$$

Note that this is same speed as that of shear waves in a uniform medium. This might have been anticipated as there is also no change in volume in a torsional oscillation and so only the shear stress acts to produce a restoring force.

11.4.3 Strings

This example is surely all too familiar. When a string under tension T is plucked there will be a restoring force proportional to the curvature of the string. If $\xi_x \equiv \eta$ is the transverse displacement, then the wave equation will be

$$T\frac{\partial^2\eta}{\partial z^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} , \tag{11.35}$$

where Λ is the mass per unit length. The wave speed is thus

$$c_S = \left(\frac{T}{\Lambda}\right)^{1/2} \tag{11.36}$$

11.4.4 Flexural Waves on a Beam

Now consider the small amplitude displacement of a rod or beam that can be flexed. In Sec. 1.6 we showed that such a flexural displacement produces a net elastic restoring force per unit length given by $D\partial^4\eta/\partial z^4$, and we considered a situation where that force was balanced by the beam's weight per unit length, $W = \Lambda g_e$ [cf. Eq. (10.69)]. Here D is the flexural rigidity [Eq. (10.66)], $\eta = \xi_x$ is the transverse displacement of the neutral surface from the horizontal, Λ is the mass per unit length, and g_e is the earth's acceleration of gravity. The solution of the resulting force-balance equation $-D\partial^4\eta\partial z^4 = W$, was the quartic (10.70), which defined the equilibrium beam shape. When gravity is absent, the flexural restoring force can be equated to the beam's mass per unit length times its acceleration to obtain the wave equation

$$-D\frac{\partial^4\eta}{\partial z^4} = \Lambda\frac{\partial^2\eta}{\partial t^2} . \tag{11.37}$$

(This an elementary illustration of the Principle of Equivalence that we shall discuss in Part VI.)

The wave equations we have encountered so far in this chapter have all described non-dispersive waves, for which the wave speed is independent of the frequency. Flexural waves, by contrast, are dispersive. We can see this by assuming that $\eta \propto \exp[i(kz-\omega t)]$ and thereby deducing from Eq. (11.37) the dispersion relation

$$\omega^2 = \frac{D}{\Lambda}k^4 . \tag{11.38}$$

Before considering the implications of this dispersion, we shall complicate the equilibrium a little. Let us suppose that, in addition to the net shearing force per unit length, the beam is also held under a tension force T as well. We can then combine the two wave equations (11.35), (11.37) to obtain

$$-D\frac{\partial^4\eta}{\partial z^4} + T\frac{\partial^2\eta}{\partial z^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} , \tag{11.39}$$

for which the dispersion relation is

$$\omega^2 = c_S^2 k^2 \left(1 + \frac{k^2}{k_c^2}\right) , \tag{11.40}$$

where

$$k_c^2 = T/D \tag{11.41}$$

defines a critical wave vector. If the average strain induced by the tension is ϵ, then to order of magnitude $k_c \sim \epsilon^{1/2} t^{-1}$. where t is the thickness of the beam. For short wavelengths, $k \gg k_c$, the shearing force dominates and the beam behaves like a tension-free beam; for long wavelengths $k \ll k_c$ it behaves like a string.

A consequence of dispersion is that waves with different wave numbers k propagate with different speeds, and correspondingly the group velocity $V_g = d\omega/dk$ with which wave packets propagate differs from the phase velocity with which a wave's crests and troughs move (see Sec. 6.2). For the dispersion relation (11.40), the phase and group velocities are

$$\begin{aligned} V_\phi &\equiv \omega/k = c_S(1 + k^2/k_c^2)^{1/2} , \\ V_g &\equiv d\omega/dk = c_S(1 + 2k^2/k_c^2)(1 + k^2/k_c^2)^{-1/2} . \end{aligned} \tag{11.42}$$

As we discussed in detail in Sec. 6.2 and Ex. 6.2, for dispersive waves such as this one, the the fact that different Fourier components in the wave packet propagate with different speeds causes the packet to gradually spread; we explore this quantitatively for longitudinal waves on a beam in Ex. 11.3.

11.4.5 Buckling (once more)

We conclude this discussion by returning to the problem of buckling, which we introduced in the last chapter. The problem we discussed there was a playing card compressed until an equilibrium was no longer possible. As promised, we can treat this as a dynamical problem by treating the tension T as negative. We can solve Eq. (11.39) by separation of variables and find that the spatial variation satisfies

$$\eta = A \sin(kz) \tag{11.43}$$

where $k = \pi/l$ for the ground state excitation. From the dispersion relation (11.40) we see that ω^2 becomes negative for $T < -\pi^2 D/l^2$ as we derived by a different argument in Sec. 10.8. When this happens, there are two normal-mode solutions, one decaying exponentially with time, $\eta \propto \exp(-|\omega|t)$ and the other increasing exponentially with time, $\eta \propto \exp(+|\omega|t)$. It is the second solution that signifies *instability*.

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EXERCISES

Exercise 11.3 *Derivation: Dispersion of Flexural Waves*
Verify Eqs. (11.40) and (11.42). Sketch the evolution of a Gaussian wave packet as it propagates along a stretched beam.

Exercise 11.4 *Problem: Speeds of Elastic Waves*

Show that the sound speeds for the following types of elastic waves in an isotropic material are in the ratio $1 : (1-\nu^2)^{-1/2} : \left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)^{1/2} : [2(1+\nu)]^{1/2} : [2(1+\nu)]^{1/2}$. Longitudinal waves along a rod, longitudinal waves along a sheet, longitudinal waves along a rod embedded in an incompressible medium, shear waves in an extended solid, torsional waves along a rod.

Exercise 11.5 *Problem: Xylophones*
Consider a beam of length l, whose weight is neglible in the elasticity equations, supported freely at both ends (so the slope of the beam is unconstrained at the ends). Show that the frequencies of standing flexural waves satisfy

$$\omega = \left(\frac{n\pi}{l}\right)^2 \left(\frac{D}{\rho A}\right)^{1/2} ,$$

where A is the cross-sectional area and n is an integer. Now repeat the exercise when the ends are clamped. Hence explain why xylophones don't have clamped ends.

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11.5 Body Waves and Surface Waves–Seismology

In Sec. 11.3 we derived the dispersion relations $\omega = c_L k$ and $\omega = c_T k$ for elastodynamic waves in uniform media. We now consider how the waves are modified in an inhomogeneous, finite body, the earth. The earth is well approximated as a sphere of radius $R_e \sim 6000$ km and mean density $\bar{\rho} \sim 6000$ kg m^{-3}. The outer crust comprising rocks of high tensile strength rests on a denser but more malleable mantle, the two regions being separated by the famous Moho discontinuity. Underlying the mantle is an outer core mainly comprised of liquid iron, which itself surrounds a denser, solid inner core; see Table 11.1 and Fig.11.4 below.

The pressure in the Earth's interior is much larger than atmospheric and the rocks are therefore quite compressed. Their atomic structure cannot be regarded as a small perturbation from their structure *in vacuo*. Nevertheless, we can still use linear elasticity theory to discuss small perturbations about this equilibrium. This is because the crystal lattice has had plenty of time to re-establish a new equilibrium with a much smaller lattice spacing (Figure 11.3). The density of lattice defects and dislocations will probably not differ appreciably from the density on the earth's surface. Hooke's law should still apply below the elastic limit, though the elastic moduli are much greater than those measured at atmospheric pressure.

We can estimate the magnitude of the pressure P in the Earth's interior by idealizing the earth as a fluid with a stress tensor $\mathbf{T} = P\mathbf{g}$ (where $\mathbf{g}$ is the metric tensor), for which the equation of static equilibrium (10.29) becomes

$$\frac{dP}{dr} = -g\rho \, , \qquad (11.44)$$

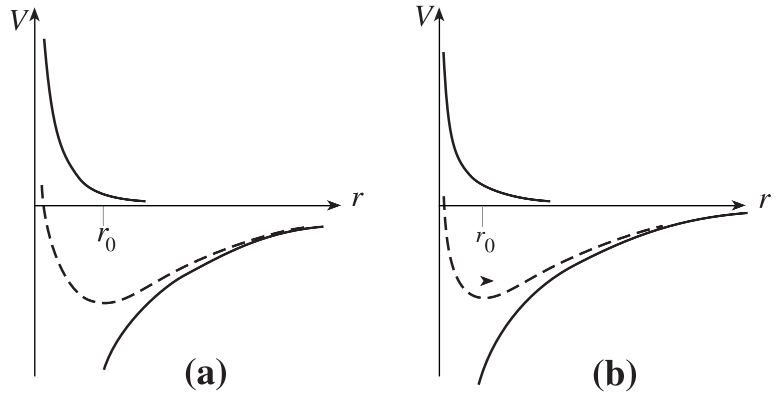

Fig. 11.3: Potential energy curves (dashed) for nearest neighbors in a crystal lattice. (a) At atmospheric (effectively zero) pressure, the equilibrium spacing is set by the minimum in the potential energy which is a combination of hard electrostatic repulsion by the nearest neighbors (upper solid curve) and a softer overall attraction associated with all the nearby ions (lower solid curve). (b) At much higher pressure, the softer, attractive component is moved inward and the equilibrium spacing is greatly reduced. The bulk modulus is given by the curvature of the potential energy curve and is considerably increased.

which can be approximated by

$$P \sim \bar{\rho} g_e R_e \sim 300\text{GPa} \sim 3 \times 10^6 \text{atmospheres} . \tag{11.45}$$

This agrees well with the accurate value of 360GPa. The bulk modulus produces the isotropic pressure $P = -K\Theta$ [Eq. (11.34)]; and since $\Theta = -\delta\rho/\rho$ [Eq. (11.10)], the bulk modulus can be expressed as

$$K = \frac{dP}{d\ln\rho} . \tag{11.46}$$

[Strictly speaking, we should distinguish between adiabatic and isothermal variations in Eq. (11.46), but the distinction is small for solids; cf. Sec. 10.6b. It is significant for gases.] Typically, the bulk modulus inside the earth is 4-5 times the pressure and the shear modulus in the crust and mantle is about half the bulk modulus.

Zone	R	ρ	K	μ	V_P	V_S
	10^3km	10^3kg m^{-3}	GPa	GPa	km s^{-1}	km s^{-1}
Inner Core	1.2	13	1400	160	11	2
Outer Core	3.5	10-12	600-1300	-	8-10	-
Mantle	6.35	3-5	100-600	70-250	8-14	5-7
Crust	6.37	3	50	30	6-7	3-4
Ocean	6.37	1	2	-	1.5	-

Table 11.1: Typical outer radii (R), densities (ρ), bulk moduli (K), shear moduli(μ), P-wave speeds and S-wave speeds within different zones of the earth. Note the absence of shear waves in the fluid regions. (Adapted from Stacey 1977.)

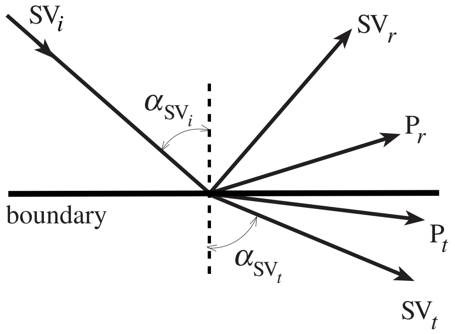

Fig. 11.4: An incident shear wave polarized in the vertical direction (SV$_i$), incident from above on a boundary, produces both a longitudinal (P) wave and a SV wave in reflection and in transmission. If the wave speeds increase across the boundary (the case shown), then the transmitted waves, SV$_t$, P$_t$, will be refracted away from the vertical. A shear mode, SV$_r$, will be reflected at the same angle as the incident wave. However, the reflected P mode, P$_r$, will be reflected at a greater angle to the vertical as it is has a greater speed.

11.5.1 Body Waves

Virtually all our direct information about the internal structure of the earth comes from measurements of the propagation times of elastic waves generated by earthquakes. There are two fundamental kinds of body waves: the longitudinal and shear modes of Sec. 11.3. These are known in seismology as P-modes and S-modes respectively. The two polarizations of the shear waves are designated SH and SV, where H and V stand for "horizontal" and "vertical" displacements, i.e., displacements orthogonal to $\mathbf{k}$ that are fully horizontal, or that are obtained by projecting the vertical direction $\mathbf{e}_z$ orthogonal to $\hat{\mathbf{k}}$.

We shall first be concerned with what seismologists call high-frequency (of order 1Hz) modes. This leads to three related simplifications. As typical wave speeds lie in the range 3–14 km s^{-1}, the wavelengths lie in the range $\sim 1-10$ km which is generally small compared with the distance over which gravity causes the pressure to change significantly – the *pressure scale height*. It turns out that we then can ignore the effects of gravity on the propagation of small perturbations. In addition, we can regard the medium as effectively homogeneous and infinite and use the local dispersion relations $\omega = c_{L,T}k$, Finally, as the wavelengths are short we can trace rays through the earth using geometrical optics (Sec. 6.3).

Despite these simplifications, the earth is quite inhomogeneous and the sound speeds vary significantly with radius. Two types of variation can be distinguished, the abrupt and the gradual. To a fair approximation, the earth is horizontally stratified below the outer crust. However, there are several abrupt changes in composition in the crust and mantle (including the Moho discontinuity) where the density, pressure and elastic constants apparently change over distances short compared with a wavelength. Seismic waves incident on these discontinuities behave like light incident on the surface of a glass plate and can be reflected and refracted. In addition, as there are now two different waves with different phase speeds, it is possible to generate SV waves from pure P waves and *vice versa* at a discontinuity (Fig. 11.4). However, this wave-wave mixing is confined to SV and P; the SH waves do not mix with SV or P.

The junction conditions that control this wave mixing and all other details of the waves' behavior at a discontinuity are: (i) the displacement $\boldsymbol{\xi}$ must be continuous across the boundary (otherwise there would be infinite strain and infinite stress there); and (ii) the net force acting on an element of surface must be zero (otherwise the surface, having no mass, would have infinite acceleration), so the force per unit area acting from the front face of the boundary to the back must be balanced by that acting from the back to the front. If we take the unit normal to the horizontal boundary to be $\mathbf{e}_z$, then these boundary conditions become

$$[\boldsymbol{\xi}_j] = [\mathbf{T}_{jz}] = 0, \qquad (11.47)$$

where the notation $[X]$ signifies the difference in X across the boundary and the j is a vector index. (For an alternative, more formal derivation of $[T_{jz}] = 0$, see Ex. 11.6.)

One consequence of these boundary conditions is Snell's law for the directions of propagation of the waves: Since these continuity conditions must be satisfied all along the boundary and at all times, the phase $\phi = \mathbf{k}\cdot\mathbf{x} - \omega t$ of the wave must be continuous across the boundary at all locations $\mathbf{x}$ on it and all times, which means that the phase ϕ must be the same on the boundary for all transmitted waves and all reflected waves as for the incident waves. This is possible only if the horizontal phase speed $c_H = \omega/k_H = \omega/(k \sin\alpha)$, is the same for all the waves. (Here $k_H = k \sin\alpha$ is the magnitude of the horizontal component of a wave's propagation vector and α is the angle between its propagation direction and the vertical; cf. Fig. 11.4.) Thus, we arrive at Snell's law: For every reflected or transmitted wave J, the horizontal phase speed must be the same as for the incident wave:

$$\frac{c_J}{\sin\alpha_J} = c_H \text{ is the same for all } J. \qquad (11.48)$$

In Fig. 11.5, we show schematically the propagation of a seismic wave through the earth. At each discontinuity in the earth's material, Snell's law governs the directions of the reflected and transmitted waves. As an example, note from Eq. (11.48) that an SV mode incident on a boundary cannot generate any P mode when its angle of incidence exceeds $\sin^{-1}(c_{Ti}/c_{Lt})$. (Here we use the standard notation c_T for the phase speed of an S wave and c_L for that of a P wave.) As a result, the SV mode undergoes a total reflection, which is analogous to total internal reflection for light in a crystal. Such total reflection happens, in particular, at the earth's surface, but also, for sufficiently large angles of incidence, at other discontinuities inside the earth.

It is straightforward though tedious to compute the reflection and transmission coefficients (e.g. the strength of transmitted P-wave produced by an incident SV wave) for the general case using the boundary conditions (11.47); see, e.g., Eringen and Suhubi (1975). The analysis is straightforward but algebraically complex.

Gradual variation in the wave speeds, due to gradual variations of the elastic moduli and density inside the earth, can be handled using geometrical optics:

In the regions between the discontinuities, the pressures and consequently the elastic moduli increase steadily, over many wavelengths, with depth. The elastic moduli generally increase more rapidly than the density so the wave speeds generally also increase with depth, i.e. $dc/dr < 0$. This radial variation in c causes the rays along which the waves propagate

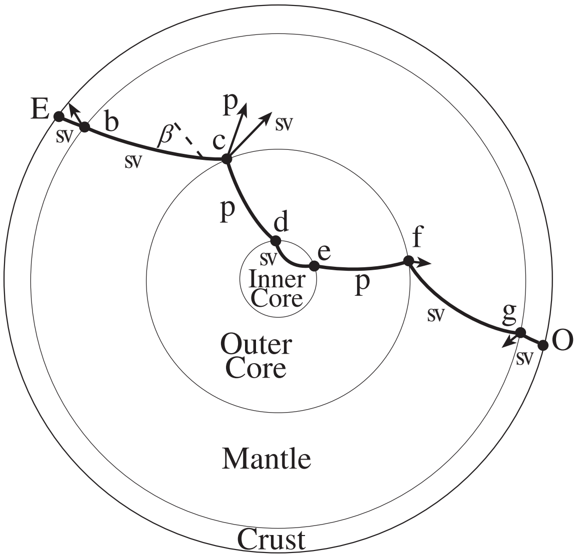

Fig. 11.5: Seismic wave propagation in a schematic earth model. A SV wave made by an earthquake, E, is partially reflected and refracted at the crust-mantle boundary b. The transmitted shear wave then is incident upon the liquid outer core at c. Only a P wave can be transmitted. This is incident upon the solid inner core at d where a shear wave can be reformed. The reverse transformations occur at e, f and g so that the arrival time of a P wave packet at the seismological observatory, S, can be measured.

to bend. The details of this bending are governed by Hamilton's equations, with the Hamiltonian $\Omega(\mathbf{x}, \mathbf{k})$ determined by the simple nondispersive dispersion relation $\Omega = c(\mathbf{x})k$ (Sec. 6.3). Hamilton's equations in this case reduce to the simple ray equation (6.44), which (since the index of refraction is $\propto 1/c$) can be rewritten as

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\mathbf{x}}{ds}\right) = \boldsymbol{\nabla}\left(\frac{1}{c}\right) . \tag{11.49}$$

Here s is distance along the ray, so $d\mathbf{x}/ds = \mathbf{n}$ is the unit vector tangent to the ray. This ray equation can be reexpressed in the following form:

$$d\mathbf{n}/ds = -(\boldsymbol{\nabla} \ln c)_\perp , \tag{11.50}$$

where the subscript $\perp$ means "projected perpendicular to the ray;" and this in turn means that the ray bends *away* from the direction in which c increases (i.e., it bends upward inside the earth since c increases downward) with the radius of curvature of the bend given by

$$\mathcal{R} = \frac{1}{|(\boldsymbol{\nabla} \ln c)_\perp|} = \frac{1}{|(d \ln c/dr) \sin\alpha|} . \tag{11.51}$$

Here α is the angle between the ray and the radial direction; see the bending rays in Fig. 11.5.

11.5.2 Edge waves

One phenomenon that is important in seismology but is absent for many other types of wave motion is the existence of "edge waves", i.e., waves that propagate along a discontinuity

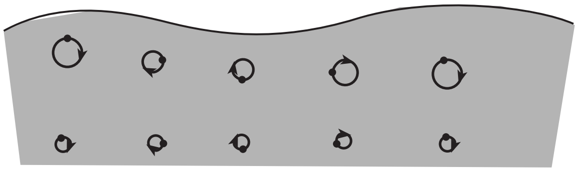

Fig. 11.6: Rayleigh waves in a semi-infinite elastic medium.

in the elastic medium. An important example is surface waves, which propagate along the surface of a medium (e.g., the earth) and that decay exponentially with depth. Waves with such exponential decay are sometimes called *evanescent*.

The simplest type of surface wave is called a *Rayleigh* wave. We shall now analyze Rayleigh waves for the idealisation of a plane semi-infinite solid. This discussion must be modified to allow for both the density stratification and the surface curvature when it is applied to the earth. However, the qualitative character of the mode is unchanged.

Rayleigh waves are an intertwined mixture of P and SV waves; and, in analyzing them, it is useful to resolve their displacement vector $\boldsymbol{\xi}$ into a sum of a (longitudinal) P-wave component, $\boldsymbol{\xi}^L$, and a (transverse) S-wave component, $\boldsymbol{\xi}^T$.

Consider a semi-infinite elastic medium and introduce a local Cartesian coordinate system with $\mathbf{e}_z$ normal to the surface, with $\mathbf{e}_x$ lying in the surface, and with the propagation vector $\mathbf{k}$ in the $\mathbf{e}_z$-$\mathbf{e}_x$ plane. The propagation vector will have a real component along the horizontal, $\mathbf{e}_x$ direction, corresponding to true propagation, and an imaginary component along the $\mathbf{e}_z$ direction, corresponding to an exponential decay of the amplitude as one goes downward into the medium. In order for the longitudinal (P-wave) and transverse (SV-wave) parts of the wave to remain in phase with each other as they propagate along the boundary, they must have the same values of the frequency ω and horizontal wave number k_x. However, there is no reason why their vertical e-folding lengths should be the same, i.e. why their imaginary k_z's should be the same. We therefore shall denote their imaginary k_z's by $-iq_L$ for the longitudinal (P-wave) component and $-iq_T$ for the transverse (S-wave) component, and we shall denote k_x by k.

Focus attention, first, on the longitudinal part of the wave. Its displacement must have the form

$$\boldsymbol{\xi}_L = \mathbf{A}e^{q_L z + i(kx - \omega t)}, \quad z \leq 0\,. \tag{11.52}$$

Substituting into the general dispersion relation $\omega^2 = c_L^2 \mathbf{k}^2$ for longitudinal waves, we obtain

$$q_L = (k^2 - \omega^2/c_L^2)^{1/2}\,. \tag{11.53}$$

Now, the longitudinal displacement field is irrotational (curl-free), so we can write

$$\xi^L_{x,z} = \xi^L_{z,x} \tag{11.54}$$

or

$$\xi^L_z = \frac{iq_L \xi^L_x}{k} \tag{11.55}$$

As the transverse component is solenoidal (divergence-free), the expansion of the combined P-T wave is produced entirely by the P component:

$$\Theta = \boldsymbol{\nabla} \cdot \boldsymbol{\xi}^L = ik\left(1 - \frac{q_L^2}{k^2}\right) A \, . \tag{11.56}$$

Now turn to the transverse (SV-wave) component. We write

$$\boldsymbol{\xi}_T = \mathbf{B} \exp^{q_T z + i(kx - \omega t)}, \quad z \leq 0 \, , \tag{11.57}$$

where (by virtue of the transverse dispersion relation)

$$q_T = \left(k^2 - \frac{\omega^2}{c_T^2}\right)^{1/2} \, . \tag{11.58}$$

As the transverse mode is solenoidal, we obtain

$$\xi_z^T = \frac{ik\xi_1^T}{q^T} \tag{11.59}$$

and for the rotation

$$\boldsymbol{\phi}_2 = \frac{1}{2}\mathbf{e}_2 \cdot \boldsymbol{\nabla} \times \boldsymbol{\xi}^T = -\frac{1}{2} q_T \left(1 - \frac{k^2}{q_T^2}\right) B \, . \tag{11.60}$$

We must next impose boundary conditions at the surface. Now, as the surface is free, there will be no force acting upon it, so,

$$\mathbf{T} \cdot \mathbf{e}_z |_{z=0} = 0, \tag{11.61}$$

which is a special case of the general boundary condition (11.47). (Note that we can evaluate the stress at the unperturbed surface location rather than at the displaced surface as we are only working to linear order.) The normal stress is

$$-T_{zz} = K\Theta + 2\mu(\xi_{z,z} - \frac{1}{3}\Theta) = 0 \, , \tag{11.62}$$

and the tangential stress is

$$-T_{xz} = 2\mu(\xi_{z,x} + \xi_{x,z}) = 0 \, . \tag{11.63}$$

Combining Eqs. (11.56), (11.60), (11.62) and (11.63), we obtain

$$(k^2 + q_T^2)^2 = 4 q_L q_T k^2 \, . \tag{11.64}$$

Next we substitute for q_L, q_T from (11.53) and (11.58) to obtain the dispersion relation

$$\zeta^3 - 8\zeta^2 + 8\left(\frac{2-\nu}{1-\nu}\right)\zeta - \frac{8}{(1-\nu)} = 0 \, , \tag{11.65}$$

where

$$\zeta = \left(\frac{\omega}{c_T k}\right)^2 \, . \tag{11.66}$$

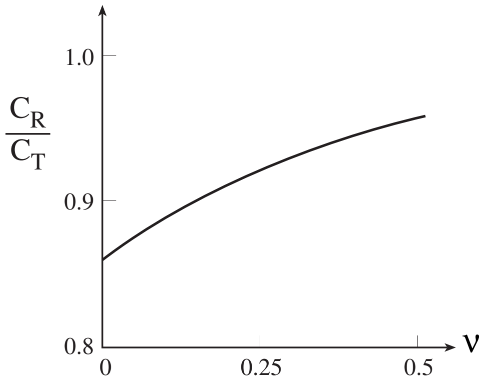

Fig. 11.7: Solution of the dispersion relation (11.65) for different values of Poisson's ratio, ν.

The dispersion relation (11.65) is a third order polynomial in ω^2 with generally just one positive real root. From Eqs. (11.65) and (11.66), we see that for a Poisson ratio characteristic of rocks, $0.2 \lesssim \nu \lesssim 0.3$, the phase speed of a Rayleigh wave is roughly 0.9 times the speed of a pure shear wave; cf. Fig. 11.7.

Rayleigh waves propagate around the surface of the earth rather than penetrate the interior. However, our treatment is inadequate because their wavelengths, typically 1–10 km if generated by an earthquake, are not necessarily small compared with the pressure scale heights in the outer crust. Our wave equation has to be modified to include these vertical gradients.

This vertical stratification has an important additional consequence. If, ignoring these gradients, we attempt to find an orthogonal surface mode just involving SH waves, we find that we cannot simultaneously satisfy the surface boundary conditions on displacement and stress with a single evanescent wave. We need two modes to do this. However, when we allow for stratification, the strong refraction allows an SH surface wave to propagate. This is known as a *Love wave*. The reason for its practical importance is that seismic waves are also created by underground nuclear explosions and it is necessary to be able to distinguish explosion-generated waves from earthquake waves. Now, an earthquake is usually caused by the transverse slippage of two blocks of crust across a fault line. It is therefore an efficient generator of shear modes including Love waves. By contrast, explosions involve radial motions away from the point of explosion and are inefficient emitters of Love waves. This allows these two sources of seismic disturbance to be distinguished.

11.5.3 Green's Function for a Homogeneous Half Space

To get insight into the combination of waves generated by a localized source, such as an explosion or earthquake, it is useful to examine the *Green's function* for excitations in a homogeneous half space. Physicists define the Green's function $G_{jk}(\mathbf{x}, t; \mathbf{x}', t')$ to be the displacement response $\xi_j(\mathbf{x}, t)$ to a unit delta-function force at location $\mathbf{x}'$ and time t', $F_k = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$. Geophysicists sometimes find it useful to work, instead, with the "Heaviside Green's function," $G^H_{jk}(\mathbf{x}, t; \mathbf{x}', t')$, which is the displacement response $\xi_j(\mathbf{x}, t)$ to a unit step-function force (one that turns on to unit strength and remains forever constant afterwards)

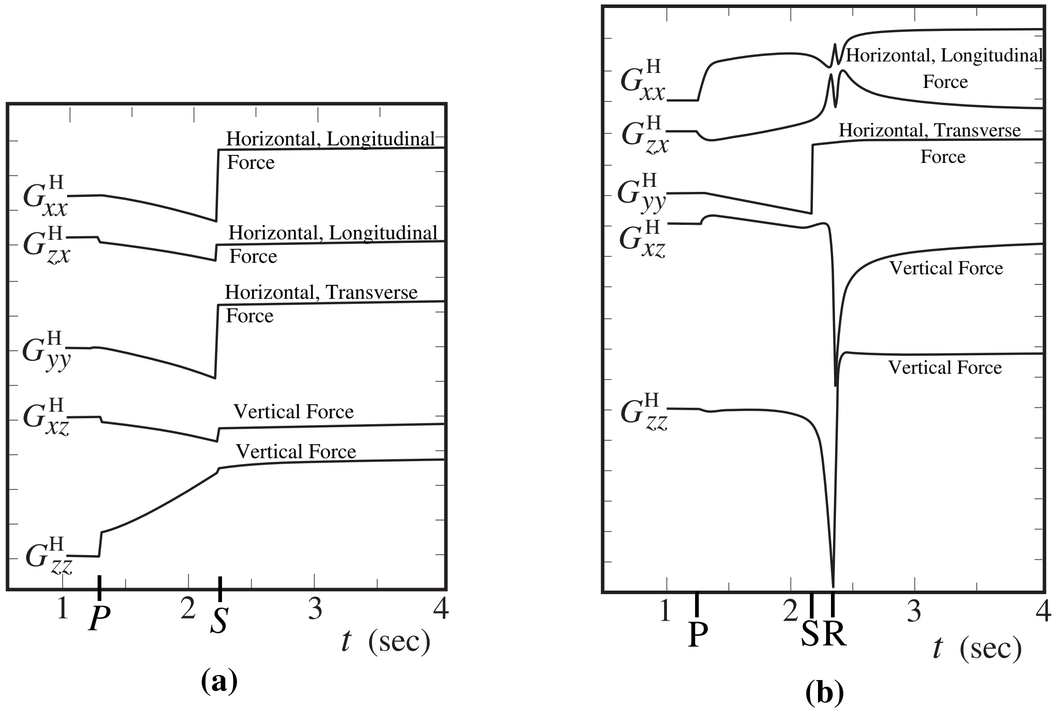

Fig. 11.8: The Heaviside Green's function (displacement response to a step-function force) in a homogeneous half space; adapted from Figs. 2 and 4 of Johnson (1974). The observer is at the surface. The force is applied at a point in the $x-z$ plane, with a direction given by the second index of G^H; the displacement direction is given by the first index of G^H. In (a), the source is nearly directly beneath the observer so the waves propagate nearly vertically upward; more specifically, the source is at 10 km depth and 2 km distance along the horizontal x direction. In (b), the source is close to the surface and the waves propagate nearly horizontally, in the x direction; more specifically, the source is at 2 km depth and is 10 km distance along the horizontal x direction. The longitudinal and transverse speeds are $c_H = 8$ km/s and $c_S = 4.62$ km/s, and the density is 3.30 g/cm^3. For a force of 1 dyne, a division on the vertical scale is 10^{-19} cm. The moments of arrival of the P-wave, S-wave and Rayleigh wave from the moment of force turnon are indicated on the horizontal axis.

at $\mathbf{x}'$ and t': $F_k = \delta(\mathbf{x}-\mathbf{x}')H(t-t')$. Because $\delta(t-t')$ is the time derivative of the Heaviside step function $H(t-t')$, the Heaviside Green's function is the time integral of the physicists' Green's function.

It is a rather complicated task to compute the Heaviside Green's function, and geophysicists have devoted much effort to doing so. We shall not give details of such computations, but merely show the Green's function graphically in Fig. 11.8 for an instructive situation: the displacement produced by a step-function force in a homogeneous half space with the observer at the surface and the force at two different locations: (a) a point nearly beneath the observer, and (b) a point close to the surface and some distance away in the x direction.

Several features of this Green's function deserve note: (i) For the source nearly beneath the observer [graphs (a)], there is no sign of any Rayleigh wave, whereas for the source close to the surface, the Rayleigh wave is the strongest feature in the x and z (longitudinal and

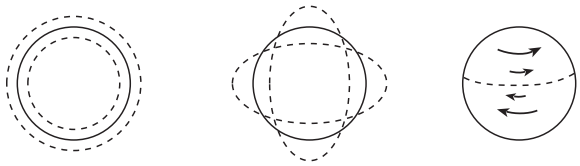

Fig. 11.9: Surface displacements associated with three simple classes of free oscillation. a)Radial modes. b) l=2 spheroidal mode. c) Torsional mode.

vertical) displacements but is absent from the y (transverse) displacement. (ii) The y (transverse) component of force produces a transverse displacement that is strongly concentrated in the S-wave. (iii) The x and z (longitudinal and vertical) components of force produce x and z displacements that include P-waves, S-waves, and (for the source near the surface) Rayleigh waves. (iv) The gradually changing displacements that occur between the arrival of the turn-on P-wave and turn-on S-wave are due to P-waves that hit the surface some distance from the observer, and from there diffract to the observer as a mixture of P- and S-waves, and similarly for gradual changes of displacement after the turn-on S-wave.

The complexity of seismic waves arises in part from the richness of features in this homogeneous-half-space Green's function, in part from the influences of the earth's inhomogeneities, and in part from the complexity of an earthquake's or explosion's forces.

11.5.4 Free Oscillations of Solid Bodies

In computing the dispersion relations for body (P- and S-wave) and surface (Rayleigh-wave) modes, we have assumed that the wavelength is small compared with the earth's radius and therefore can have a continuous frequency spectrum. However, it is also possible to excite global wave modes in which the whole earth "rings". If we regard the earth as spherically symmetric, then we can isolate three fundamental types of oscillation, *radial*, *spheroidal* and *torsional*.

If we introduce spherical polar coordinates for the displacement, then it is possible to separate and solve the equations of elastodynamics to find the normal modes just like solving the Schrodinger equation for a central potential. Each of the three types of modes has a displacement vector $\boldsymbol{\xi}$ characterized by its own type of spherical harmonic.

The spheroidal modes have radial displacements proportional to $Y_l^m(\theta,\phi)\mathbf{e}_r$ (where θ, ϕ are spherical coordinates, Y_l^m is the scalar spherical harmonic of order l and m, and $\mathbf{e}_r$ is the unit radial vector; and they have nonradial components proportional to $\boldsymbol{\nabla} Y_l^m$). These modes are called "spheroidal" because (when one ignores the tiny nonsphericity of the earth and ignores Coriolis and centrifugal forces due to the earth's rotation), their eigenfrequencies are independent of m, and thus can be studied by specializing to $m = 0$, in which case the displacements become

$$\xi_r \propto P_l(\cos\theta)\ , \quad \xi_\theta \propto \sin\theta P_l'(\cos\theta). \tag{11.67}$$

These displacements deform the earth in a spheroidal manner for the special case $l = 2$. [In Eq. (11.67) P_l is the Legendre polynomial and P_l' is the derivative of P_l with respect to its argument.] The radial modes are the special case $l = 0$ of these spheroidal modes. It is often mistakenly asserted that there are no $l = 1$ modes because of conservation of momentum. In fact, $l = 1$ modes do exist: for example, the central regions of the earth can move up, while the outer regions move down. The $l = 2$ spheroidal mode has a period of 53 min. and can ring for about 1000 periods. (We say that its quality factor is $Q \sim 1000$.) This is typical for solid planets.

Toroidal modes have vanishing radial displacements, and their nonradial displacements are proportional to the vector spherical harmonic $\mathbf{e}_r \times \boldsymbol{\nabla} Y_l^m$. As for spheroidal modes, spherical symmetry of the unperturbed earth guarantees that the eigenfrequencies will be independent of the azimuthal quantum number m, so $m = 0$ is representative. For $m = 0$ the only nonzero component of the vector spherical harmonic $\mathbf{e}_r \times \boldsymbol{\nabla} Y_l^m$ is in the ϕ direction, and it gives

$$\xi_\phi \propto \sin\theta P_l'(\cos\theta) \ . \tag{11.68}$$

In these modes alternate zones of different latitude oscillate in opposite directions (clockwise or counterclockwise at some chosen moment of time), in such a way as to conserve total angular momentum.

When one writes the displacement vector $\boldsymbol{\xi}$ for a general vibration of the earth as a sum over these various types of normal modes, and inserts that sum into the wave equation (11.12) (augmented, for greater realism, by gravitational forces), spherical symmetry of the unperturbed earth guarantees that the various modes will separate from each other, and for each mode the wave equation will give a radial wave equation, analogous to that for a hydrogen atom in quantum mechanics. The boundary condition $\mathbf{T} \cdot \mathbf{e}_r = 0$ at the earth's surface constrains the solutions of the radial wave equation, for each mode, to be a discrete set, which one can label by the number n of radial nodes that they possess (just as for the hydrogen atom). The frequencies of the modes increase with both n and l.

For small values of the quantum numbers, the modes are quite sensitive to the model assumed for the earth's structure. For example, they are sensitive to whether one correctly includes the gravitational restoring force in the wave equation. However, for large l and n, the spheroidal and toroidal modes become standing combinations of P waves, SV waves, SH waves, Rayleigh and Love waves, and therefore they are rather insensitive to one's ignoring the effects of gravity.

11.5.5 Seismic Tomography

Observations of all of these types of seismic waves clearly code much information about the earth's structure and inverting the measurements to infer this structure has become a highly sophisticated and numerically intensive branch of geophysics. The travel times of the P and S body waves can be measured at various points over the earth's surface and essentially allow c_L and c_T and hence K/ρ and μ/ρ to be determined as functions of radius inside the earth. Travel times are $\lesssim 1$ hour. Using this type of analysis, seismologists can infer the presence of hot and cold regions within the mantle and then show how the rocks are circulating under the crust.

It is also possible to combine the observed travel times with the the earth's equation of elastostic equilibrium

$$\frac{dP}{dr} = \frac{K}{\rho}\frac{d\rho}{dr} = -g_e(r)\rho \ , \tag{11.69}$$

where the local gravity is given by

$$g_e = \frac{4\pi G}{r^2}\int_0^r r'^2\rho(r')dr' \ , \tag{11.70}$$

to determine the distributions of density, pressure and elastic constants. Measurements of Rayleigh and Love waves can be used to probe the surface layers. The results of this procedure are then input to obtain free oscillation frequencies which compare well with the observations. The damping rates for the free oscillations furnish information on the interior viscosity.

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EXERCISES

Exercise 11.6 *Derivation: Junction Condition at a Discontinuity*
Derive the junction condition $[\mathbf{T}_{jz}] = 0$ at a horizontal discontinuity between two media by the same method as one uses in electrodynamics to show that the normal component of the magnetic field must be continuous: Integrate the equation of motion $\rho d\mathbf{v}/dt = -\boldsymbol{\nabla}\cdot\mathbf{T}$ over the volume of an infinitesimally thin "pill box" centered on the boundary, and convert the volume integral to a surface integral via Gauss's theorem.

Exercise 11.7 *Example: Reflection and Transmission of Normal, Longitudinal Waves at a Boundary*
Consider a longitudinal elastic wave incident normally on the boundary between two media, labeled 1,2. By matching the displacement and the normal component of stress at the boundary, show that the ratio of the transmitted wave amplitude to the incident amplitude is given by

$$t = \frac{2Z_1}{Z_1 + Z_2}$$

where $Z_{1,2} = [\rho_{1,2}(K_{1,2} + 4\mu_{1,2}/3)]^{1/2}$ is known as the *acoustic impedance*. (The impedance is independent of frequency and just a characteristic of the material.) Likewise, evaluate the amplitude reflection coefficent and verify that wave energy flux is conserved.

Exercise 11.8 *Example: Earthquakes*
The magnitude M of an earthquake is a quantitative measure of the strength of the seismic waves it creates. Roughly speaking, the elastic wave energy release can be inferred semi-empirically from the magnitude using the formula

$$E = 10^{5.2+1.44M}\,\mathrm{J}$$

The largest earthquakes have magnitude ~ 8.5.

One type of earthquake is caused by slippage along a fault deep in the crust. Suppose that most of the seismic power in an earthquake with $M \sim 8.5$ is emitted at frequencies $\sim$ 1Hz and that the quake lasts for a time $T \sim$ 100s. If V is an average wave speed, then it is believed that the stress is relieved over an area of fault of length $\sim VT$ and a depth of order one wavelength. By comparing the stored elastic energy with the measured energy release make an estimate of the minimum strain prior to the earthquake. Is this reasonable? Hence estimate the typical displacement during the earthquake in the vicinity of the fault.
Make an order of magnitude estimate of the acceleration measurable by a seismometer in the next state and in the next continent. (Ignore the effects of density stratification, which are actually quite significant.)

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11.6 The relationship of Classical Waves to Quantum Mechanical Excitations.

In the previous chapter, we explored the limits of the continuum approximation and showed how we must acknowledge that solids are composed of atoms in order to account for the magnitude of the elastic constants and to explain why most solids yield under comparatively small strain. A quite different demonstration of the limits of the continuum approximation is provided by studying the normal modes of vibration of a solid lattice.

Consider a solid of finite size and solve the wave equation (11.12) for small amplitude normal modes, each with its own eigenfrequency ω_N and its eigenfunction $\boldsymbol{\xi}_N$. For simplicity, suppose that the shape of the body is a cube and that its surfaces are free so that the normal component of the stress tensor vanishes there. As described in Sec. 11.3, there are scalar longitudinal and vector transverse modes. A general, small amplitude oscillatory disturbance can be decomposed into a linear superposition of these modes,

$$\boldsymbol{\xi}(\mathbf{x}, t) = \sum_N a_N \exp\left(-i\omega_N t\right) \boldsymbol{\xi}_N(\mathbf{x}), \tag{11.71}$$

where, as usual, we implicitly take the real part and a_N is the *amplitude* of mode N. It turns out to be convenient to normalize the eigenfunctions so that

$$\int \rho |\boldsymbol{\xi}_N|^2 dV = M\ , \tag{11.72}$$

where M is the mass of the body; a_N then measures the mean physical displacement in mode N.

Classical electromagnetic waves *in vacuo*, are described by linear Maxwell equations, and, after they have been excited, will essentially propagate forever. This is not so for elastic waves, where the linear wave equation is only an approximation. Nonlinearities, and most especially impurities and defects in the homogeneous structure of the body's material, will cause the different modes to interact so that their amplitudes ***psi*** change slowly with time according to a damped simple harmonic oscillator differential equation of the form

$$\ddot{a}_N + (2/\tau_N)\dot{a}_N + \omega_N^2 a_N = F'/M, \tag{11.73}$$

where the second term on the left hand side is a damping term that will cause the mode to decay as long as $\tau_N > 0$, and F' is a fluctuating or *stochastic* force attributable to the coupling between different modes. Equation (11.73) is the *Langevin equation* that we studied in Chap. 5, and the strength and spectrum of the fluctuating forces F' is determined by the fluctuation-dissipation theorem, Eq. (5.111). If the modes are thermalized at temperature T, then the fluctuating forces maintain an average energy of kT in each one.

Now, what happens quantum mechanically? The ions and electrons in an elastic solid interact so strongly that it is very difficult to analyze them directly. A quantum mechanical treatment is much easier if one makes a canonical transformation from the coordinates and momenta of the individual ions or atoms to new, generalized coordinates x_N and momenta p_N which represent weakly interacting normal modes. These coordinates and momenta must now be viewed as Hermitian operators, and they are related to the quantum mechanical amplitude $\hat{a}_n$ by

$$\hat{x}_N = \frac{1}{2}(\hat{a}_N + \hat{a}_N^\dagger), \tag{11.74}$$

$$\hat{p}_N = \frac{M\omega_N}{2i}(\hat{a}_N - \hat{a}_N^\dagger), \tag{11.75}$$

where the dagger denotes the Hermitean adjoint. We can transform back to obtain an expression for the displacement of the i'th ion

$$\hat{\mathbf{x}}_i = \frac{1}{2}\Sigma_N[\hat{a}_N\xi_N(\mathbf{x}_i) + \hat{a}_N^\dagger\xi_N^*(\mathbf{x}_i)] \tag{11.76}$$

The Hamiltonian can be written in terms of these coordinates as

$$\hat{H} = \Sigma_N\left(\frac{\hat{p}_N^2}{2M} + \frac{1}{2}M\omega_N^2\hat{x}_N^2\right) + \hat{H}_{\rm int}\ , \tag{11.77}$$

where the first term is a sum of simple harmonic oscillator Hamiltonians for individual modes and $\hat{H}_{\rm int}$ is the perturbative interaction Hamiltonian which takes the place of the combined damping and stochastic forcing terms in the classical equation (11.73). When the various modes are thermalized, the mean energy in mode N takes on the standard Bose-Einstein form $\hbar\omega_N[1/2 + \{\exp(\hbar\omega_N/kT) - 1\}^{-1}]$ [Exercise 3.2, augmented by a "zero-point energy" of $\frac{1}{2}\hbar\omega$], which reduces to kT in the classical ($\hbar \to 0$) limit.

As the unperturbed Hamiltonian for each mode is identical to that for a particle in a harmonic oscillator potential well, it is sensible to think of each wave mode in a manner analogous to such a particle-in-well. Just as the particle-in-well can reside in any one of a series of discrete energy levels lying above the "zero point" energy of $\hbar\omega/2$, and separated by $\hbar\omega$, so each wave mode with frequency ω_N must have an energy $(n + 1/2)\hbar\omega_N$, where n is an integer. The operator which causes the energy of the mode to decrease by $\hbar\omega_N$ is the *annihilation operator* for mode n

$$\hat{\alpha}_N = \left(\frac{M\omega_N}{\hbar}\right)^{1/2}\hat{a}_N, \tag{11.78}$$

and the operator which causes an increase in the energy by $\hbar\omega_N$ is its Hermitian conjugate, the *creation operator* $\hat{\alpha}_N^\dagger$. In the case of wave modes, it is useful to think of each increase or decrease in the energy as the creation or annihilation of an individual quantum or "particle" of energy, so that when the energy in mode N is $(n+1/2)\hbar\omega_N$, there are n quanta (particles) present. These particles are called *phonons*. Phonons are not conserved, and because they can co-exist in the same state (the same mode), they are *bosons*. They have individual energies and momenta which must be conserved in their interactions with each other and with other types of particles, e.g. electrons.

The important question is now, given an elastic solid at finite temperature, do we think of its thermal agitation as a superposition of classical wave modes or do we regard it as a gas of quanta? The answer depends upon what we want to do. From a purely fundamental viewpoint, the quantum mechanical description takes precedence. However, for many problems where the number of phonons per mode $\sim kT/\hbar\omega_N \gg 1$, the classical description is amply adequate and much easier to handle. We do not need a quantum treatment to compute the normal modes of a vibrating building excited by an earthquake or trying to understand how to improve the sound quality of a violin. Here the difficulty is in accommodating the boundary conditions so as to determine the normal modes. All this was expected. What comes as more of a surprise is that often, for purely classical problems, where $\hbar$ is quantitatively irrelevant, the fastest way to procede formally is to follow the quantum route and then take the limit $\hbar \to 0$. We shall see this graphically demonstrated when we discuss nonlinear plasma physics in Chap. 22.

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