Chapter 12

Foundations of Fluid Dynamics

12.1 Overview

Having studied elasticity theory, we now turn to a second branch of continuum mechanics: *fluid dynamics*. Three of the four states of matter (gases, liquids and plasmas) can be regarded as fluids and so it is not surprising that interesting fluid phenomena surround us in our everyday lives. Fluid mechanics is an experimental discipline and much of what has been learned has come in response to laboratory investigations. Fluid dynamics finds experimental and observational applications in engineering (aeronautics, civil engineering, hydrology, petrology, ...), and in physics, biophysics, chemistry, agriculture and other fields. The observational sciences of oceanography, meteorology, astrophysics, and geophysics, in which experiments are harder to perform, are also heavily reliant upon fluid dynamics, and the fundamental principles are common to all of these subjects. Many of these fields have enhanced our appreciation of fluid dynamics by presenting flows under conditions that are inaccessible to laboratory study.

The equations of fluid dynamics describe the motions of fluids when the length and time scales associated with the flow are long compared with the corresponding microscopic scales, so the continuum approximation can be invoked. In this chapter, we will derive and discuss these fundamental equations. These equations are, in some respects, simpler than the corresponding laws of elastodynamics. However, as with particle dynamics, simplicity in the equations does not imply that the solutions are simple, and indeed they are not! One reason is that there is no restriction that fluid displacements be small (by constrast with elastodynamics where the elastic limit keeps them small), so most fluid phenomena are immediately non-linear.

Relatively few problems in fluid dynamics admit complete, closed-form, analytic solutions, so progress in describing fluid flows has usually come from the introduction of clever physical "models" and the use of judicious mathematical approximations. In more recent years numerical fluid dynamics has come of age and in many areas of fluid mechanics, finite difference simulations have begun to complement laboratory experiments and measurements.

Fluid dynamics is a subject where considerable insight accrues from being able to visualize the flow. This is true of fluid experiments where much technical skill is devoted to marking the fluid so it can be photographed, and numerical simulations where frequently more time is devoted to computer graphics than to solving the underlying partial differential equations. We shall pay some attention to flow visualization. The reader should be warned that obtaining an analytic solution to the equations of fluid dynamics is not the same as understanding the flow; it is always a good idea to be able to sketch the flow pattern at the very least.

We shall begin this chapter in Sec. 12.2 with a discussion of hydrostatics that parallels our discussion of elastostatics. Following a discussion of atmospheres, stars and planets, we shall explain the microphysical basis of Archimedes principle. The stress tensor associated with most fluids can be decomposed into an isotropic pressure and a viscous term linear in the rate of shear or velocity gradient. Under many conditions the viscous stress can be neglected over most of the flow and the fluid is then called *ideal* or *inviscid*. We shall study the laws governing ideal flows in Sec. 12.3. After deriving the relevant conservation laws and equation of motion, we shall derive and discuss the Bernoulli principle (which relies on ideality) and show how it can simplify the description of many flows. In flows for which the speed neither approaches the speed of sound, nor the gravitational escape velocity, the fractional changes in fluid density are relatively small. It can then be a good approximation to treat the fluid as *incompressible* and this leads to considerable simplification, which we also study in Sec. 12.3. As we shall see, incompressibility can be a good approximation not just for liquids which tend to have large bulk moduli, but also, more surprisingly, for gases. In Sec. 12.4 we augment our basic equations with terms describing the action of the viscous stresses. This allows us to derive the famous Navier-Stokes equation and to illustrate its use by analyzing pipe flow.

12.2 Hydrostatics

Just as we began our discussion of elasticity with a treatment of elastostatics, so we will introduce fluid dynamics by discussing hydrostatic equilibrium. Once again, we shall emphasize the roles of tensors and conservation laws.

The elementary laws of fluid dynamics follow from two simple observations. The first is that in most flows, the continuum approximation is valid and we can define a mean local velocity of the fluid's molecules, which varies smoothly both spatially and temporally. The same is true for other quantities like density which characterize the fluid. Now this need not always be the case. For example, in a shock front, which we shall study in Chapter 16, the flow varies rapidly over a length of order the collision mean free path of the molecules. In this case, the continuum hypothesis is only piecewise valid and we must perform a matching at the boundary. A second example is a turbulent flow where, it might be thought, the flow will vary rapidly on whatever length scale we choose to study. As we shall see in Chapter 14, this is not the case and in general turbulent flows there is a length scale within which the flow is smoothly varying.

The second observation is that fluids do not oppose a steady shear strain. This is easy to understand on microscopic grounds as there is no lattice to deform and the molecular distribution remains isotropic. By kinetic theory considerations (Chapter 2), we therefore expect that the stress tensor will be isotropic. This allows us to write $\mathbf{T} = P\mathbf{g}$ where P the pressure. Now suppose that we have a fluid with pressure P and density ρ and it undergoes

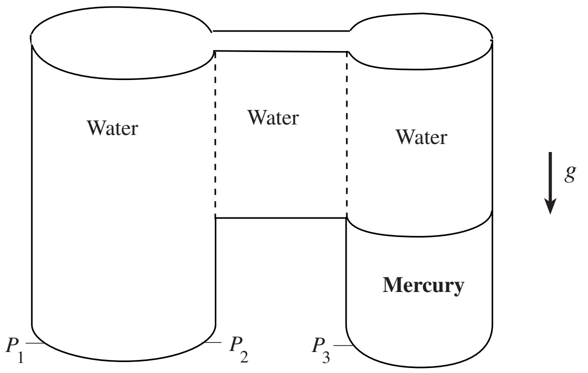

Fig. 12.1: Elementary demonstration of the principle of hydrostatic equilibrium. Water and mercury, two immisicible fluids of different density, are introduced into a container with two chambers as shown. The pressure at each point on the bottom of the container is equal to the weight per unit area of the overlying fluids. The pressures P_1 and P_2 at the bottom of the left chamber are equal, but because of the density difference between mercury and water, they differ from the pressure P_3 at the bottom of the right chamber.

a small isotropic expansion with $\Theta = -\delta\rho/\rho$ [cf. Eq. (11.9)]. The change in the stress tensor will be

$$\delta\mathbf{T} = -K\Theta\mathbf{g} = \delta P\mathbf{g} \tag{12.1}$$

For small expansions, we can again linearize and use the fact that the bulk modulus has the physical dimensions of pressure to write $\delta P = \Gamma P\delta\rho/\rho$, where Γ is a pure number so that $K = \Gamma P$. More formally, we define

$$\Gamma = \left(\frac{\partial \ln P}{\partial \ln \rho}\right) . \tag{12.2}$$

Now Γ is written as a partial derivative which implies that the variation is constrained in some manner. This must be specified. For example, the physical conditions may be such that the fluid changes so that its temperature does not change. Alternatively, and much more commonly, its entropy might remain constant because no significant heat can flow in or out of a fluid element in the time for the variation to take place. In this case it can be shown, using the laws of thermodynamics (Chapter 4), that $\Gamma = \gamma = C_P/C_V$, where C_P, C_V are the specific heats at constant pressure and volume. These matters will occupy us in the future. For the moment, though, we shall just assume that we have a knowledge of Γ and have a prescription for relating changes in the density to corresponding changes in the pressure. (See Box 12.1 for further discussion of thermodynamic aspects of fluid dynamics.)

We have already derived the equation of hydrostatic equilibrium for a fluid at rest in a gravitational field. It is just the same as the equation of elastostatic equilibrium with a vanishing shear stress, Eqs. (10.31) and (10.32):

$$\boldsymbol{\nabla}\cdot\mathbf{T} = \boldsymbol{\nabla}P = \rho\mathbf{g}_e = -\rho\boldsymbol{\nabla}\Phi . \tag{12.3}$$

Here $\mathbf{g}_e$ is the acceleration of gravity (with the subscript e to distinguish it from the metric tensor $\mathbf{g}$), and Φ is Newton's gravitational potential. (Note our sign convention: Φ is negative near a gravitating body.)

Box 12.1
Thermodynamic Considerations

One feature of fluid dynamics, especially gas dynamics, that distinguishes it from elastodynamics, is that the thermodynamic properties of the fluid are often very important and we must treat energy conservation explicitly. In this box we review, from Chapter 4, some of the necessary thermodynamic concepts; see also Reif (1959). We shall have no need for partition functions, ensembles and other statistical aspects of thermodynamics. Instead, we shall only need elementary thermodynamics.

We begin with the nonrelativistic first law of thermodynamics (4.11) for a sample of fluid with energy E, entropy S, volume V, number N_I of molecules of species I, temperature T, pressure P, and chemical potential μ_I for species I:

$$dE = TdS - PdV + \sum_I \mu_I dN_I \ . \tag{1}$$

Almost everywhere in our treatment of fluid mechanics (and throughout this chapter), we shall assume that the term $\sum_I \mu_I dN_I$ vanishes. Physically this happens because all relevant nuclear reactions are frozen (occur on timescles $\tau_{\rm react}$ far longer than the dynamical timescales $\tau_{\rm dyn}$ of interest to us), so $dN_I = 0$; and each chemical reaction is either frozen, or goes so rapidly ($\tau_{\rm react} \ll \tau_{\rm dyn}$) that it and its inverse are in local thermodynamic equilibrium (LTE): $\sum_I \mu_I dN_I = 0$ for those species involved in the reactions. In the intermediate situation, where some relevant reaction has $\tau_{\rm react} \sim \tau_{\rm dyn}$, we would have to carefully keep track of the relative abundances of the chemical or nuclear species and their chemical potentials.

Consider a small fluid element with mass Δm, energy per unit mass u, entropy per unit mass s, and volume per unit mass $1/\rho$. Then inserting $E = u\Delta m$, $S = s\Delta m$ and $V = \Delta m/\rho$ into the first law $dE = TdS - pdV$, we obtain the form of the first law that we shall use in almost all of our fluid dynamics studies:

There are some immediate and important inferences that can be drawn. Take the curl of Eq. (12.3):

$$\boldsymbol{\nabla}\Phi \times \boldsymbol{\nabla}\rho = 0 \tag{12.4}$$

This tells us that, in hydrostatic equilibrium, the contours of constant density, coincide with the equipotential surfaces, i.e. $\rho = \rho(\Phi) = -dP/d\Phi$. This, in turn, implies that the difference in pressure between two equipotential surfaces,

$$\Delta P = -\int \rho(\Phi) d\Phi, \tag{12.5}$$

just depends on the difference in potential between these two surfaces and, as $\nabla P \propto \nabla\Phi$, the surfaces of constant pressure, the *isobars*, coincide with the gravitational equipotentials. The gravitational force is constant to high accuracy in most non-astrophysical applications of fluid dynamics, for example on the surface of the earth. In this case, the pressure at a

Box 12.1, Continued

$$du = Tds - Pd\left(\frac{1}{\rho}\right) . \tag{2}$$

The internal energy (per unit mass) u comprises the random translational energy of the molecules that make up the fluid, together with the energy associated with their internal degrees of freedom (rotation, vibration etc.) and with their intermolecular forces. The term Tds represents some amount of heat (per unit mass) that may get injected into a fluid element, e.g. by viscous heating (last section of this chapter), or may get removed, e.g. by radiative cooling.

In fluid mechanics it is useful to introduce the enthalpy $H = E + PV$ of a fluid element (cf. Ex. 4.3) and the corresponding enthalpy per unit mass $h = u + p/\rho$. Inserting $u = h - P/\rho$ into the left side of the first law (1), we obtain the first law in the "enthalpy representation":

$$dh = Tds + \frac{dP}{\rho}. \tag{3}$$

Because all reactions are frozen or are in LTE, the relative abundances of the various nuclear and chemical species are fully determined by a fluid element's density ρ and temperature T (or by any two other variables in the set ρ, T, s, and P). Correspondingly, the thermodynamic state of a fluid element is completely determined by any two of these variables. In order to calculate all features of that state from two variables, we must know the relevant *equations of state*, such as $P(\rho, T)$ and $s(\rho, T)$, or the fluid's fundamental thermodynamic potential (Table 4.1) from which follow the equations of state.

We shall often deal with *ideal gases* (in which intermolecular forces and the volume occupied by the molecules are both negligible). For any ideal gas, the equation of state $P(\rho, T)$ is [cf. Eq. (3.64)]

$$P = \frac{\rho k T}{\mu m_p} \tag{4}$$

where μ is the *mean molecular weight* and m_p is the proton mass. The mean molecular weight μ is the mean mass per gas molecule in units of the proton mass, and should not be confused with the chemical potential of species I, μ_I (which will rarely if ever be used in our fluid mechanics analyses).

point in a fluid is, from Eq. (12.5), equal to the total weight of fluid per unit area above the point,

$$P = g_e \int \rho dx \ , \tag{12.6}$$

where the integral is performed by integrating upward in the gravitational field; cf. Fig. 12.1). For example, the Marianas trench in the Pacific Ocean is just over 10km deep. Adopting a density $\sim 10^3$kg m^{-3}for water and a value $\sim$ 10 m s^{-2} for g, we obtain a pressure of $\sim 10^8$N m^{-2} or $\sim 10^3$ atmospheres. This is comparable with the yield stress of the strongest

Box 12.1, Continued

An idealisation that is often accurate in fluid dynamics is that the fluid is *adiabatic*; that is to say there is no heating resulting from dissipative processes, such as viscosity, thermal conductivity or the emission and absorption of radiation. When this is a good approximation, the entropy per unit mass s of a fluid element is constant following a volume element with the flow, i.e.

$$\frac{ds}{dt} = 0. \tag{5}$$

In an adiabatic flow, there is only one thermodynamic degree of freedom and so we can write $P = P(\rho, s) = P(\rho)$. Of course, this function will be different for fluid elements that have different s. In the case of an ideal gas, a standard thermodynamic argument [Ex. 12.7] shows that the pressure in an adiabatically expanding or contracting fluid element varies with density as

$$P \propto \rho^\gamma \ , \tag{6}$$

where γ, the *adiabatic index*, is equal to the ratio of specific heats

$$\gamma = C_P/C_V \ . \tag{7}$$

[Our specific heats, like the energy, entropy and enthalpy, are defined on a per unit mass basis, so $C_P = T(\partial s/\partial T)_P$ is the amount of heat that must be added to a unit mass of the fluid to increase its temperature by one unit, and similarly for $C_V = T(\partial s/\partial T)_\rho$.] A special case of adiabatic flow is *isentropic* flow. In this case, the entropy is constant everywhere, not just along individual streamlines.

Whenever the pressure can be regarded as a function of the density alone (the same function everywhere), the fluid is called *barotropic*. A particular type of barytrope is the *polytrope* in which $P \propto \rho^{1+1/n}$ for some constant n (the *prolytropic index*. Another is a liquid of infinite bulk modulus for which ρ =constant, everywhere. Note that barytropes are not necessarily isentropes; for example, in a fluid of sufficiently high thermal conductivity, the temperature will be constant everywhere, thereby causing both P and s to be unique functions of ρ.

materials. It should therefore come as no surprize to discover that the deepest dive ever recorded by a submersible was made by the *Trieste* when it achieved a depth of 10.9 km in 1960.

12.2.1 Archimedes' Law

The Law of Archimedes, states that when a solid body is totally or partially immersed in a liquid, the total buoyant upward force of the liquid on the body is equal to the weight of the displaced liquid. A formal proof can be made as follows; see Fig. 12.2. The fluid,

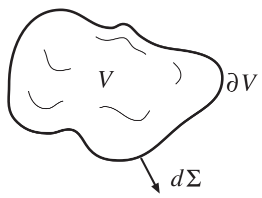

Fig. 12.2: Derivation of Archimedes Law.

pressing inward on the body across a small element of the body's surface $d\mathbf{\Sigma}$, exerts a force $d\mathbf{F}^{\rm buoy} = \mathbf{T}(\_, -d\mathbf{\Sigma})$, where the minus sign is because, by convention, $d\mathbf{\Sigma}$ points out of the body rather than into it. Converting to index notation and integrating over the body's surface ∂V, we obtain for the net buoyant force

$$F_i^{\rm buoy} = \int_{\partial V} T_{ij} d\Sigma_j \ . \tag{12.7}$$

Now, imagine removing the body and replacing it by fluid that has the same pressure $P(z)$ and density $\rho(z)$, at each height z, as the surrounding fluid; this is the fluid that was originally displaced by the body. Since the fluid stress on ∂V has not changed, the buoyant force will be unchanged. Use Gauss's law to convert the surface integral (12.7) into a volume integral over the interior fluid (the originally displaced fluid)

$$F_i^{\rm buoy} = \int_V T_{ij;j} dV \ . \tag{12.8}$$

The displaced fluid obviously is in hydrostatic equilibrium with the surrounding fluid, and its equation of hydrostatic equilibrium (12.3), when inserted into Eq. (12.8), implies that

$$F_i^{\rm buoy} = -\mathbf{g}_e \int_V \rho dV = -M\mathbf{g}_e \ , \tag{12.9}$$

where M is the mass of the displaced fluid. Thus, the upward buoyant force on the original body is equal in magnitude to the weight Mg_e of the displaced fluid. Clearly, if the body has a higher density than the fluid, then the downward gravitational force on it (its weight) will exceed the weight of the displaced fluid and thus exceed the buoyant force it feels, and the body will fall; but if the body's density is less than that of the fluid, the buoyant force will exceed its weight and it will be pushed upward.

A key piece of physics underlying Archimedes law is the fact that the intermolecular forces acting in a fluid, like those in a solid (cf. Sec. 10.4), are of short range. If, instead, the forces were of long range, Archimedes' law could fail. For example, consider a fluid that is electrically conducting with currents flowing through it, so there are long-range electromagnetic forces. If we then substitute an insulating solid for some region V of the fluid, the force that acts upon the solid will be different from the force that acted upon the displaced fluid.

12.2.2 Stars and Planets

Stars and planets—if we ignore their rotation—are self-gravitating spheres, part fluid and part solid. We can model the structure of a non-rotating spherical, fluid body by combining the equation (12.3) of hydrostatic equilibrium in spherical polar coordinates,

$$\frac{dP}{dr} = -\rho\frac{d\Phi}{dr} , \tag{12.10}$$

with Poisson's equation,

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right) = 4\pi G\rho , \tag{12.11}$$

to obtain

$$\frac{1}{r^2}\frac{d}{dr}\frac{r^2}{\rho}\frac{dP}{dr} = -4\pi G\rho. \tag{12.12}$$

This can be integrated once radially with the aid of the boundary condition $dP/dr = 0$ at $r = 0$ (pressure cannot have a cusp-like singularity) to obtain

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} , \tag{12.13}$$

where

$$m = \int_0^r 4\pi\rho r^2 dr \tag{12.14}$$

is the total mass inside radius r. Eq. (12.13) is an alternative form of the equation of hydrostatic equilibrium at radius r inside the body: Gm/r^2 is the gravitational acceleration g at r, $\rho Gm/r^2$ is the downward gravitational force per unit volume on the body's fluid, and dP/dr is the upward buoyant force per unit volume.

Equations (12.10)—(12.14) are a good approximation for solid planets, as well as for stars and liquid planets, because, at the enormous stresses encountered in the interior of a solid planet, the strains are so large that plastic flow will occur. In other words, the limiting shear stresses are much smaller than the isotropic part of the stress tensor.

Let us make an order of magnitude estimate of the interior pressure in a star of mass M and radius R. We use the equation of hydrostatic equilibrium (12.3) or (12.13), approximating m by M, the density ρ by M/R^3 and the gravitational acceleration by GM/R^2, so that

$$P \sim \frac{GM^2}{R^4} . \tag{12.15}$$

In order to improve upon this estimate, we must solve Eq. (12.12). We therefore need a prescription for relating the pressure to the density. A common idealization is the polytropic relation, namely that

$$P \propto \rho^{1+1/n} \tag{12.16}$$

where n is called the polytropic index (cf. last part of Box 12.1). This finesses the issue of the thermal balance of stellar interiors. Low mass white dwarf stars are well approximated as $n = 1.5$ polytropes, and red giant stars are somewhat similar in structure to $n = 3$

polytropes. The giant planets, Jupiter and Saturn mainly comprise a H-He fluid which is well approximated by an $n = 1$ polytrope, and the density of a small planet like Mercury is very roughly constant ($n = 0$). We also need boundary conditions to solve Eq. (12.12); the appropriate choices are the vanishing of the gravity at the center, $r = 0$, and the pressure at the surface, $r = R$. For details of polytropic stellar models see, e.g., Chandrasekhar (1939).

We can easily solve the equation of hydrostatic equilibrium (12.13) for a constant density star to obtain

$$P = P_0\left(1 - \frac{r^2}{R^2}\right) , \tag{12.17}$$

where the central pressure is

$$P_0 = \left(\frac{3}{8\pi}\right)\frac{GM^2}{R^4} , \tag{12.18}$$

consistent with our order of magnitude estimate.

12.2.3 Rotating Fluids

Newton's laws apply in an inertial frame. In a body that rotates with uniform angular velocity $\boldsymbol{\Omega}$, it is more convenient to work in a co-rotating frame. In such a frame, the equation of hydrostatic equilibrium (12.3) must be augmented by the centrifugal force:

$$\boldsymbol{\nabla} P = \rho(\mathbf{g} + \mathbf{g}_{\rm cen}) = -\rho\boldsymbol{\nabla}(\Phi + \Phi_{\rm cen}) . \tag{12.19}$$

Here

$$\mathbf{g}_{\rm cen} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\boldsymbol{\nabla}\Phi_{\rm cen} , \tag{12.20}$$

is the centrifugal acceleration, and

$$\Phi_{\rm cen} = -\frac{1}{2}(\boldsymbol{\Omega} \times \mathbf{r})^2. \tag{12.21}$$

is a *centrifugal potential* whose gradient is equal to the centrifugal acceleration in our situation of constant $\boldsymbol{\Omega}$. The centrifugal potential can be regarded as an augmentation of the gravitational potential Φ; indeed, in the presence of uniform rotation, all hydrostatic theorems [e.g., Eqs. (12.4) and (12.5)] remain valid with Φ replaced by $\Phi + \Phi_{\rm cen}$.

We can illustrate this by considering the shape of a spinning fluid planet. Let us suppose that almost all the mass of the planet is concentrated in its core so that the gravitational potential $\Phi = -GM/r$ is unaffected by the rotation. (Here M is the planet mass and r is the radius.) Now the surface of the planet must be an equipotential of $\Phi + \Phi_{\rm cen}$ (coinciding with the zero pressure isobar); cf. Eq. (12.4) with $\Phi \to \Phi + \Phi_{\rm cen}$. The contribution of the centrifugal potential at the equator is $-\Omega^2 R_e^2/2$ and at the pole zero. The difference in the gravitational potential between the equator and the pole is $\sim g_e(R_e - R_p)$ where R_e, R_p are the equatorial and polar radii respectively. Therefore, adopting this mass model, we estimate the difference between the polar and equatorial radii to be

$$R_e - R_p \simeq \frac{\Omega^2 R^2}{2g_e} \tag{12.22}$$

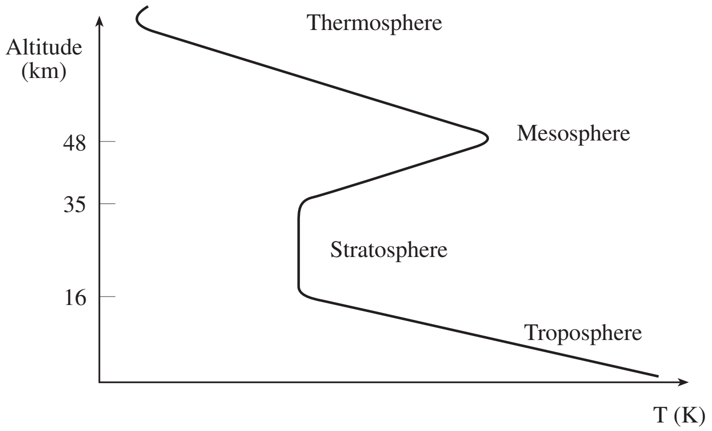

Fig. 12.3: Actual temperature variation in the Earth's mean atmosphere at temperate latitudes.

As was discussed in Chapter 10, the earth, although not a fluid, is unable to withstand large shear stresses and its surface will not deviate by more than the maximum height of a mountain from its equipotential. If we substitute $g_e \sim 10\text{m s}^{-2}$, $R \sim 6 \times 10^6\text{m}$ and $\Omega \sim 7 \times 10^{-5}\text{rad s}^{-1}$, we obtain $R_e - R_p \sim 10\text{km}$, about half the correct value of 21km. The reason for this discrepancy lies in our assumption that all the mass lies in the center. In fact it is distributed fairly uniformly in radius and, in particular, some mass is found in the equatorial bulge. This deforms the gravitational equipotential surfaces from spheres to ellipsoids, which accentuates the flattening. If, following Newton, we assume that the earth has uniform density then the flattening estimate is 2.5 times larger. Naturally, this is an overestimate as the true density variation lies between these two limits.

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EXERCISES

Exercise 12.1 *Practice: Weight in Vacuum*
How much more would you weigh *in vacuo*?

Exercise 12.2 *Example: Earth's Atmosphere*
As mountaineers know, it gets cooler as you climb. However, the rate at which the temperature falls with altitude depends upon the assumed thermal properties of air. Consider two limiting cases.

(i) In the lower stratosphere, the air is isothermal. Use the equation of hydrostatic equilibrium (12.3) to show that the pressure decreases exponentially with height z

$$P \propto \exp(-z/H),$$

where the scale height H is given by

$$H = \frac{kT}{\mu m_p g_e}$$

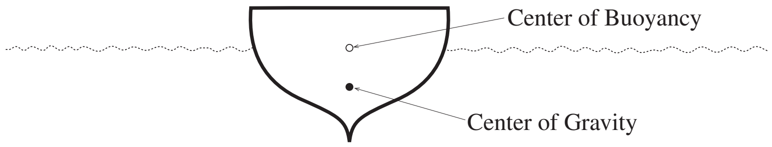

Fig. 12.4: Stability of a Boat. We can understand the stability of a boat to small rolling motions by defining both a center of gravity for weight of the boat and also a center of buoyancy for the upthrust exerted by the water.

and μ is the mean molecular weight of air and m_p is the proton mass. Estimate your local isothermal scale height.

(ii) Suppose that the air is isentropic so that $P \propto \rho^\gamma$, where γ is the specific heat ratio. (For diatomic gases like nitrogen and oxygen, $\gamma \sim 1.4$.) Show that the temperature gradient satisfies

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma}\frac{g_e \mu m_p}{k}.$$

Note that the temperature gradient vanishes when $\gamma \to 1$. Evaluate the temperature gradient, otherwise known as the *lapse rate*. At low altitude, the average lapse rate is measured to be $\sim$ 6K km^{-1}, Show that this is intermediate between the two limiting cases (Figure 12.3).

Exercise 12.3 *Problem: Stability of Boats*
Use Archimedes Principle to explain qualitatively the conditions under which a boat floating in still water will be stable to small rolling motions from side to side. (Hint, you might want to introduce a *center of buoyancy* inside the boat, as in Figure 12.4.

Exercise 12.4 *Problem: Jupiter and Saturn*
The text described how to compute the central pressure of a non-rotating, constant density planet. Repeat this exercise for the polytropic relation $P = K\rho^2$, appropriate to Jupiter and Saturn. Use the information that $M_J = 2 \times 10^{27}$kg, $M_S = 6 \times 10^{26}$kg, $R_J = 7 \times 10^4$km to estimate the radius of Saturn. Hence, compute the central pressures, gravitational binding energy and polar moments of inertia of both planets.

Exercise 12.5 *Example: Shape of a constant density, spinning planet*

(i) Show that the spatially variable part of the gravitational potential for a uniform density, non-rotating planet can be written as $\Phi = 2\pi G\rho r^2/3$, where ρ is the density.

(ii) Hence argue that the gravitational potential for a slowly spinning planet can be written in the form

$$\Phi = \frac{2\pi G\rho r^2}{3} + Ar^2 P_2(\mu)$$

where A is a constant and P_2 is a Legendre polynomial of $\mu = \sin(\text{latitude})$. What happens to the P_1 term?

(iii) Give an equivalent expansion for the potential outside the planet.

(iv) Now transform into a frame spinning with the planet and add the centrifugal potential to give a total potential.

(v) By equating the potential and its gradient at the planet's surface, show that the difference between the polar and the equatorial radii is given by

$$R_e - R_p \simeq \frac{5\Omega^2 R^2}{2g_e},$$

where g_e is the surface gravity. The true result lies in between this estimate and that given in Sec. 12.2

Exercise 12.6 *Problem: Von Zeipel's Theorem*
Consider a differentially rotating star with a *barotropic* equation of state $P = P(\rho)$. Write down the equations of hydrostatic equilibrium including the centrifugal force in cylindrical polar coordinates. Hence show that the angular velocity must be constant on surfaces of constant cylindrical radius. (As an application, Jupiter is differentially rotating and therefore might be expected to have similar rotation periods at the same latitude in the north and the south. This is only roughly true, suggesting that the equation of state is not completely barotropic.)

Exercise 12.7 *Derivation: Adiabatic Index*
Show that for an ideal gas (one with equation of state $P = (k/\mu m_p)\rho T)$, the specific heats are related by $C_P = C_V + k/(\mu m_p)$, and the adiabatic index is $\gamma = C_P/C_V$. [The solution is given in most thermodynamics textbooks.]

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12.3 Conservation Laws for an Ideal Fluid

We now turn from hydrostatic situations to fully dynamical fluids. We shall derive the fundamental equations of fluid dynamics in several stages. In this section, we will confine our attention to *ideal* fluids, i.e., flows for which it is safe to ignore dissipative processes (viscosity and thermal conductivity), and for which, therefore, the entropy of a fluid element remains constant with time. In the next section we will introduce the effects of viscosity, and in Chap. 17 we will introduce heat conductivity. At each stage, we will derive the fundamental fluid equations from the even-more-fundamental conservation laws for mass, momentum, and energy. Our derivations will be analogous to those given in Chap. 11 for elastodynamics.

12.3.1 Mass Conservation

We have already derived the first conservation law, that of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0, \tag{12.23}$$

in our treatment of elasticity, Eq. (11.3). Equation (12.23) can be re-written as

$$\frac{d \ln \rho}{dt} = -\rho\theta \equiv -\rho \boldsymbol{\nabla} \cdot \mathbf{v} \ , \tag{12.24}$$

where d/dt is the convective derivative introduced in Eq. (11.8):

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \ , \tag{12.25}$$

and $\theta \equiv \boldsymbol{\nabla} \cdot \mathbf{v}$ is the rate of expansion of the fluid (time derivative of the expansion $\Theta = \boldsymbol{\nabla} \cdot \xi$ used in elastodynamics).

We define a fluid to be *incompressible* when $d\rho/dt = 0$. *Note:* incompressibility does *not* mean that the fluid cannot be compressed; rather, it merely means that in the situation being studied, the density of each fluid element remains constant as time passes. From Eq. (12.25), we see that incompressibility implies that the velocity field has vanishing divergence (i.e. it is *solenoidal*, i.e. expressible as the curl of some potential). The condition that the fluid be incompressible is a weaker condition than that the density be constant everywhere; for example, the density varies substantially from the earth's center to its surface, but if the material inside the earth were moving more or less on surfaces of constant radius, the flow would be incompressible. As we shall shortly see, approximating a flow as incompressible is a good approximation when the flow speed is much less than the speed of sound and the fluid does not move through too great gravitational potential differences.

12.3.2 Momentum Conservation

The momentum conservation law for a fluid in a gravitational field can be obtained by adding the gravitational force to the right-hand side of Eq. (11.6):

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \mathsf{T} = \rho \mathbf{g}_e \ . \tag{12.26}$$

Here the stress or momentum flux tensor is given by

$$\mathsf{T} = \rho \mathbf{v} \otimes \mathbf{v} + P\mathbf{g}. \tag{12.27}$$

[cf. Eqs. (11.4), (11.6) and (12.3)]. The first term, $\rho\mathbf{v}\otimes\mathbf{v}$, is the momentum flux produced by the motion of the fluid (also called the *kinetic* stress), and the second, $P\mathbf{g}$, is that associated with the fluid's pressure.

In most of our applications, the gravitational field $\mathbf{g}_e$ will be externally imposed, i.e., it will be produced by some object such as the Earth that is different from the fluid we are studying. However, the law of momentum conservation remains the same, Eq. (12.26), independently of what produces gravity, the fluid or an external body or both.

The most fundamental formulation of the law of momentum conservation is one in which *all* forces, including gravity, are expressed as the divergence of a stress, so, e.g., Eq. (12.26) becomes

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T}_{\rm total} = 0\ , \tag{12.28}$$

where

$$\mathbf{T}_{\rm total} = \rho\mathbf{v}\otimes\mathbf{v} + P\mathbf{g} + \mathbf{T}_g \tag{12.29}$$

is the total stress tensor, including a gravitational contribution

$$\mathbf{T}_g = \frac{\mathbf{g}_e\otimes\mathbf{g}_e - \frac{1}{2}g_e^2\mathbf{g}}{4\pi G}\ . \tag{12.30}$$

(We discuss this gravitational stress tensor in Box 12.2 at the end of this section. Although it is very fundamental, we shall typically not use it but instead shall employ the less fundamental version (12.26) of momentum conservation.)

Equation (12.28) is the most fundamental version of momentum conservation in two senses: *First*, Eq. (12.28) has the standard form for a conservation law: $(\partial/\partial t)$(density of conserved quantity) + $\boldsymbol{\nabla}\cdot$(flux of conserved quantity) = 0 — a form that can be converted immediately into a global conservation equation by integrating over the interior of some region V of space and then applying Gauss's law to get (d/dt)(amount of the conserved quantity in region V) = (rate that conserved quantity flows into V through its boundary ∂V). In the case of momentum, this global conservation law reads

$$\frac{d}{dt}\int_V \rho\mathbf{v}dV = -\int_{\partial\mathrm{V}}\mathbf{T}\cdot d\boldsymbol{\Sigma}\ . \tag{12.31}$$

Second, Eq. (12.28) is fundamental in that it is the Newtonian limit of the relativistic law of momentum conservation,

$$0 = T^{j\alpha}{}_{;\alpha} = \frac{\partial T^{j0}}{\partial t} + T^{jk}{}_{;k} \tag{12.32}$$

[Eq. (1.134) in special relativity where gravity is absent; Part VI of this book in general relativity where gravity is present]. For an ideal fluid, in the Newtonian limit and in an orthonormal basis, the relativistic momentum density becomes $T^{j0} = \rho v_j$, and the relativistic momentum flux becomes $T^{jk} = T^{\rm total}_{jk}$ [Eq. (12.29)].

The Newtonian law of momentum conservation, $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla}\cdot\mathbf{T}_{\rm total} = 0$, will be a foundation for our studies of dynamics throughout the remainder of the Newtonian part of this book (ideal fluids, viscous fluids, magnetized and conducting fluids, and a variety of models of plasmas). An essential principle in applying this law is that *you get out what you put in.* We shall invoke this input-output principle time and again.

For an ideal fluid, we put into the momentum flux $\mathbf{T}_{\rm total}$ (i) the fluid's kinetic stress (i.e., its momentum of motion) $\rho\mathbf{v}\otimes\mathbf{v}$, (ii) the fluid's isotropic pressure $P\mathbf{g}$, and (iii) the gravitational field's stress $\mathbf{T}_g$. Consequently, from $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla}\cdot\mathbf{T}_{\rm total} = 0$ we get out (in the next subsection) an equation of motion that includes inertial forces (coming from $\rho\mathbf{v}\otimes\mathbf{v}$), pressure forces (coming from $P\mathbf{g}$) and gravitational forces (coming from $\mathbf{T}_g$). In Sec. 12.4 below we shall put one more piece of physics into $\mathbf{T}^{\rm tot}$, viscous stresses, and correspondingly we shall get out of $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla}\cdot\mathbf{T}^{\rm tot} = 0$ an equation of motion that includes viscous forces in additional to inertial, pressure, and gravitational forces.

12.3.3 Euler Equation

The "Euler equation" is the equation of motion that one gets out of the momentum conservation law $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla}\cdot\mathbf{T}_{\rm total} = 0$, when one puts into $\mathbf{T}_{\rm total}$ the kinetic stress, the pressure, and the gravitational stress (12.29). Specifically, a straightforward evaluation of momentum conservation followed by invoking mass conservation (12.23) yields:

$$\frac{d\mathbf{v}}{dt} = -\frac{\boldsymbol{\nabla}P}{\rho} + \mathbf{g}_e \; . \tag{12.33}$$

This Euler equation was first derived in 1785 by the Swiss mathematician and physicist Leonhard Euler.

In Cartesian coordinates, the Euler equation (12.33) and mass conservation (12.23) comprise four equations in five unknowns, ρ, P, v_x, v_y, v_z. In order to close this system of equations, we must relate P to ρ. For an ideal fluid, we use the fact that the entropy of each fluid element is conserved (because there is no mechanism for dissipation),

$$\frac{ds}{dt} = 0 \; , \tag{12.34}$$

together with an equation of state for the pressure in terms of the density and the entropy, $P = P(\rho, s)$. In practice, the equation of state is often well approximated by incompressibility, $\rho =$ constant, or by a polytropic relation, $K = K(s)\rho^{1+1/n}$ [Eq. (12.16)].

12.3.4 Bernoulli's Principle

Bernoulli's principle is well known. Less well appreciated are the conditions under which it is true. In order to deduce these, we must first introduce a kinematic quantity known as the *vorticity*,

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}. \tag{12.35}$$

The attentive reader may have noticed that there is a parallel between elasticity and fluid dynamics. In elasticity, we are concerned with the gradient of the displacement vector field $\boldsymbol{\xi}$ and we decompose it into expansion, rotation and shear. In fluid dynamics, we are interested in the gradient of the velocity field $\mathbf{v} = d\boldsymbol{\xi}/dt$ and we make an analogous decomposition. The fluid analogue of expansion $\Theta = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ is its time derivative $\theta \equiv \boldsymbol{\nabla}\cdot\mathbf{v} = d\boldsymbol{\Theta}/dt$, which we call the *rate of expansion*. This has already appeared in the equation of mass conservation. Rotation $\boldsymbol{\phi} = \frac{1}{2}\boldsymbol{\nabla}\times\xi$ is uninteresting in elastostatics because it causes no stress. Vorticity $\boldsymbol{\omega} \equiv \boldsymbol{\nabla}\times\mathbf{v} = 2d\boldsymbol{\phi}/dt$ is its fluid counterpart, and although primarily a kinematic quantity, it plays a vital role in fluid dynamics; we shall discuss it in more detail in the following chapter. Shear $\boldsymbol{\Sigma}$ is responsible for the shear stress in elasticity. We shall meet its counterpart, the *rate of shear* tensor $\boldsymbol{\sigma} = d\boldsymbol{\Sigma}/dt$ below when we introduce the viscous stress tensor.

To derive the Bernoulli principle, we begin with the Euler equation $d\mathbf{v}/dt = -(1/\rho)\boldsymbol{\nabla}P + \mathbf{g}_e$; we express $\mathbf{g}_e$ as $\boldsymbol{\nabla}\Phi$; we convert the convective derivative of velocity (i.e. the acceleration) into its two parts $d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$; and we rewrite $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ using the vector identity

$$\mathbf{v}\times\boldsymbol{\omega} \equiv \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \frac{1}{2}\boldsymbol{\nabla}v^2 - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\ . \tag{12.36}$$

The result is

$$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\frac{1}{2}v^2 + \Phi) + \frac{\boldsymbol{\nabla}P}{\rho} - \mathbf{v}\times\boldsymbol{\omega} = 0. \tag{12.37}$$

This is just the Euler equation written in a new form, but it is also the most general version of the Bernoulli principle. Two special cases are of interest:

(i) *Steady flow of an ideal fluid.* A steady flow is one in which $\partial(\text{everything})/\partial t = 0$, and an ideal fluid is one in which dissipation (due to viscosity and heat flow) can be ignored. Ideality implies that the entropy is constant following the flow, i.e. $ds/dt = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = 0$. From the thermodynamic identity, $dh = Tds + dP/\rho$ [Eq. (3) of Box 12.1] we obtain

$$(\mathbf{v}\cdot\boldsymbol{\nabla})P = \rho(\mathbf{v}\cdot\boldsymbol{\nabla})h. \tag{12.38}$$

(Remember that the flow is steady so there are no time derivatives.) Now, define the *Bernoulli constant*, B, by

$$B \equiv \frac{1}{2}v^2 + h + \Phi\ . \tag{12.39}$$

This allows us to take the scalar product of the gradient of Eq. (12.39) with the velocity $\mathbf{v}$ to rewrite Eq. (12.37) in the form

$$\frac{dB}{dt} = (\mathbf{v}\cdot\boldsymbol{\nabla})B = 0, \tag{12.40}$$

This says that the Bernoulli constant, like the entropy, does not change with time in a fluid element. Let us define *streamlines*, analogous to lines of force of a magnetic field, by the differential equations

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \tag{12.41}$$

In the language of Sec. 1.5, these are just the integral curves of the (steady) velocity field; they are also the spatial world lines of the fluid elements. Equation (12.40) says that *the Bernoulli constant is constant along streamlines in a steady, ideal flow.*

(ii) *Irrotational flow of an isentropic fluid.* An even more specialized type of flow is one where the vorticity vanishes and the entropy is constant *everywhere*. A flow in which $\boldsymbol{\omega} = 0$ is called an *irrotational* flow. (Later we shall learn that, if an incompressible flow initially is irrotational and it encounters no walls and experiences no significant viscous stresses, then it remains always irrotational.) Now, as the curl of the velocity field vanishes, we can follow the electrostatic precedent and introduce a *velocity potential* $\psi(\mathbf{x}, t)$ so that at any time,

$$\mathbf{v} = \boldsymbol{\nabla}\psi. \tag{12.42}$$

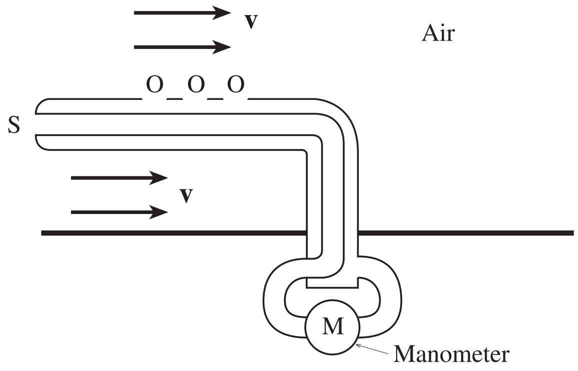

Fig. 12.5: Schematic illustration of a Pitot tube used to measure airspeed. The tube points into the flow well away from the boundary layer. A manometer measures the pressure difference between the stagnation points S, where the external velocity is very small, and several orfices O in the side of the tube where the pressure is almost equal to that in the free air flow. The air speed can then be inferred by application of the Bernoulli principle.

A flow in which the entropy is constant everywhere is called *isentropic* (Box 12.1). In an isentropic flow, $\boldsymbol{\nabla} P = \rho \boldsymbol{\nabla} h$. Imposing these conditions on Eq. (12.37), we obtain, for an isentropic, irrotational flow:

$$\boldsymbol{\nabla}\left[\frac{\partial \psi}{\partial t} + B\right] = 0. \tag{12.43}$$

Thus, the quantity $\partial\psi/\partial t + B$ will be constant everywhere in the flow, not just along streamlines. Of course, if the flow is steady so $\partial/\partial t(\text{everything}) = 0$, then B itself is constant. Note the important restriction that the vorticity in the flow vanish.

The most immediate consequence of Bernoulli's theorem in a steady, ideal flow (constancy of $B = \frac{1}{2}v^2 + h + \Phi$ along flow lines) is that the enthalpy falls when the speed increases. For a perfect gas the enthalpy is simply $h = c^2/(\gamma - 1)$, where c is the speed of sound. For an incompressible liquid, it is P/ρ. Microscopically, what is happening is that we can decompose the motion of the constituent molecules into a bulk motion and a random motion. The total kinetic energy should be constant after allowing for variation in the gravitational potential. As the bulk kinetic energy increases, the random or thermal kinetic energy must decrease, leading to a reduction in pressure.

A simple, though important application of the Bernoulli principle is to the Pitot tube which is used to measure air speed in an aircraft (Figure 12.5). A Pitot tube extends out from the side of the aircraft and points into the flow. There is one small orfice at the end where the speed of the gas relative to the tube is small and several apertures along the tube, where the gas moves with approximately the air speed. The pressure difference between the end of the tube and the sides is measured using an instrument called a manometer and is then converted into an airspeed using the formula $v = (2\Delta P/\rho)^{1/2}$. For $v \sim 100$m s^{-1}, $\rho \sim 1$kg m^{-3}, $\Delta P \sim 5000$N m^{-3} ~ 0.05atmospheres. Note that the density of the air ρ will vary with height.

12.3.5 Conservation of Energy

As well as imposing conservation of mass and momentum, we must also address energy conservation. So far, in our treatment of fluid dynamics, we have finessed this issue by simply postulating some relationship between the pressure P and the density ρ. In the case of ideal fluids, this is derived by requiring that the entropy be constant following the flow. In this case, we are not required to consider the energy to derive the flow. However, understanding how energy is conserved is often very useful for gaining physical insight. Furthermore, it is imperative when dissipative processes operate.

The most fundamental formulation of the law of energy conservation is

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F} = 0 \,, \tag{12.44}$$

where U is the total energy density and $\mathbf{F}$ is the total energy flux. This has the standard conservation-law form, so it is readily converted into a global conservation law by integrating over a volume V and applying Gauss's law. Moreover, it is the Newtonian limit of the relativistic equation of energy conservation $T^{0\alpha}{}_{;\alpha} = \partial T^{00}/\partial t + T^{0j}{}_{;j} = 0$, with mass conservation removed so the rest-mass contribution to the energy density is gone: $T^{00} - \rho c^2$ is the energy density U and $T^{0j} - \rho v^j$ is the energy flux F_j.

As with momentum conservation, *you get out what you put in.* For an ideal fluid (no dissipation), one puts into U and $\mathbf{F}$ kinetic energy, internal energy, and gravitational energy; and one gets out the ideal-fluid law of energy balance.

More specifically: when (for simplicity) the fluid is in an externally produced gravitational field Φ, its energy density is obviously

$$U = \rho \left(\frac{1}{2} v^2 + u + \Phi \right) \,, \tag{12.45}$$

where the three terms are kinetic, internal, and gravitational. When the fluid participates in producing gravity and one includes the energy of the gravitational field itself, the energy density is a bit more subtle; see Box 12.2. In an external field one might expect the energy flux to be $\mathbf{F} = U\mathbf{v}$, but this is not quite correct. Consider a bit of surface area dA orthogonal to the direction in which the fluid is moving, i.e., orthogonal to $\mathbf{v}$. The fluid element that crosses dA during time dt moves through a distance $dl = vdt$, and as it moves, the fluid behind this element exerts a force PdA on it. That force, acting through the distance dl, feeds an energy $dE = (PdA)dl = PvdAdt$ across dA; the corresponding energy flux across dA has magnitude $dE/dAdt = Pv$ and obviously points in the $\mathbf{v}$ direction, so it contributes $P\mathbf{v}$ to the energy flux $\mathbf{F}$. This contribution is missing from our initial guess $\mathbf{F} = U\mathbf{v}$. We shall explore its importance at the end of this subsection. When it is added to our guess, we obtain for the total energy flux in our ideal fluid with external gravity,

$$\mathbf{F} = \rho\mathbf{v} \left(\frac{1}{2} v^2 + h + \Phi \right) \,, \tag{12.46}$$

where $h = u + P/\rho$ is the enthalpy per unit mass. Inserting Eqs. (12.45) and (12.46) into the law of energy conservation (12.44), we get out the following ideal-fluid equation of energy

Quantity	**Density**	**Flux**
Mass	ρ	$\rho\mathbf{v}$
Momentum	$\rho\mathbf{v}$	$P\mathbf{g}_e + \rho\mathbf{v}\otimes\mathbf{v}$
Energy	$U = \frac{1}{2}\rho v^2 + \rho\rho\Phi$	$\mathbf{F} = (\frac{1}{2}v^2 + h + \Phi)\rho\mathbf{v}$

Table 12.1: Densities and Fluxes of mass, momentum, and energy for an ideal fluid in an externally produced gravitational field.

balance:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right)\right] = 0\ . \tag{12.47}$$

If the fluid is not ideal because heat is being injected into it by viscous heating, or being injected or removed by diffusive heat flow or by radiative cooling or by some other agent, then that rate of heat change per unit volume will be $\rho T ds/dt$, where s is the entropy per unit mass; and correspondingly, in this non-ideal case, the equation of energy balance will be changed from (12.47) to

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right)\right] = \rho T\frac{ds}{dt}. \tag{12.48}$$

It is instructive and builds confidence to derive this law of energy balance from other laws, so we shall do so: We begin with the laws of mass and momentum conservation in the forms (12.23) and (12.26). We multiply Eq. (12.23) by $v^2/2$ and add it to the scalar product of Eq. (12.26) with $\rho\mathbf{v}$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho v^2\mathbf{v}\right) = -(\mathbf{v}\cdot\boldsymbol{\nabla})P - \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\Phi\ . \tag{12.49}$$

Assuming for simplicity that the fluid's own gravity is negligible and that the external gravitational acceleration $\mathbf{g}_e - \boldsymbol{\nabla}\Phi$ is constant (see Box 12.2 for more general gravitational fields), we rewrite this as

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + \Phi\right)\right] = -(\mathbf{v}\cdot\boldsymbol{\nabla})P\ . \tag{12.50}$$

We can now use thermodynamic identities to transform the right-hand side:

$$\begin{aligned}(\mathbf{v}\cdot\boldsymbol{\nabla})P &= \rho(\mathbf{v}\cdot\boldsymbol{\nabla})h - \rho T(\mathbf{v}\cdot\boldsymbol{\nabla})s \\ &= \boldsymbol{\nabla}\cdot(\rho\mathbf{v}h) - \rho T\frac{ds}{dt} + \rho\frac{\partial u}{\partial t} + \left(h - \frac{P}{\rho}\right)\frac{\partial\rho}{\partial t} \\ &= \boldsymbol{\nabla}\cdot(\rho\mathbf{v}h) - \rho T\frac{ds}{dt} + \frac{\partial(\rho u)}{\partial t}\ ,\end{aligned} \tag{12.51}$$

where we have used mass conservation (12.23), the first law of thermodynamics [Eq. (1) of Box 12.1] and the definition of enthalpy $h = u + P/\rho$ [Box 12.1]. Combining Eq. (12.50) with Eq. (12.51), we obtain the expected law of energy balance (12.48).

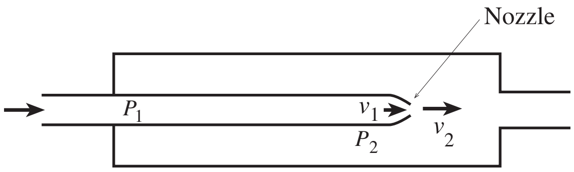

Fig. 12.6: Joule-Kelvin cooling of a gas. Gas flows steadily through a nozzle from a chamber at high pressure to one at low pressure. The flow proceeds at constant enthalpy. Work done against the intermolecular forces leads to cooling. The efficiency of cooling is enhanced by exchanging heat between the two chambers. Gases can also be liquefied in this manner as shown here.

Let us return to the contribution $P\mathbf{v}$ to the energy flux. A good illustration of the necessity for this term is provided by the Joule-Kelvin method commonly used to cool gases (Fig. 12.6). In this method, gas is driven under pressure through a nozzle or porous plug into a chamber where it can expand and cool. Microscopically, what is happening is that the molecules in a gas are not completely free but attract one another through intermolecular forces. When the gas expands, work is done against these forces and the gas therefore cools. Now let us consider a steady flow of gas from a high pressure chamber to a low pressure chamber. The flow is invariably so slow (and gravity so weak!) that the kinetic and gravitational potential energy contributions can be ignored. Now as the mass flux $\rho\mathbf{v}$ is also constant the enthalpy per unit mass, h must be the same in both chambers. The actual temperature drop is given by

$$\Delta T = \int_{P_1}^{P_2} \mu_{JK} dP, \tag{12.52}$$

where $\mu_{JK} = (\partial T/\partial P)_h$ is the Joule-Kelvin coefficient. A straighforward thermodynamic calculation yields the identity

$$\mu_{JK} = -\frac{1}{\rho^2 C_p}\left(\frac{\partial(\rho T)}{\partial T}\right)_P \tag{12.53}$$

The Joule-Kelvin coefficient of a perfect gas obviously vanishes.

12.3.6 Incompressible Flows

A common assumption that is made when discussing the fluid dynamics of highly subsonic flows is that the density is constant, i.e., that the fluid is *incompressible.* This is a natural approximation to make when dealing with a liquid like water which has a very large bulk modulus. It is a bit of a surprise that it is also useful for flows of gases, which are far more compressible under static conditions.

To see its validity, suppose that we have a flow in which the characteristic length L over which the fluid variables P, ρ, v etc. vary is related to the characteristic timescale T over which they vary by $L \lesssim vT$—and in which gravity is not important. In this case, we can compare the magnitude of the various terms in the Euler equation (12.33) to obtain an estimate of the magnitude of the pressure variation $\delta P \sim \rho\delta(v^2)$. (We could just as easily have used the Bernoulli constant.) Now the variation in pressure will be related to the variation in density by $\delta P \sim c^2 \delta\rho$, where c is the *sound* speed (not light speed) and we drop constants of order unity in making these estimates. Combining these two estimates, we obtain the estimate for the relative density fluctuation

$$\frac{\delta\rho}{\rho} = \frac{\delta(v^2)}{c^2} \tag{12.54}$$

Therefore, provided that the fluid speeds are highly subsonic ($v \ll c$), then we can ignore the density variation along a streamline in solving for the velocity field. Using the equation of continuity, written as in Eq. (12.24), we can make the approximation

$$\boldsymbol{\nabla} \cdot \mathbf{v} \simeq 0. \tag{12.55}$$

This argument breaks down when we are dealing with sound waves for which $L \sim cT$

It should be emphasized, though, that "incompressibility", which is an approximation made in deriving the velocity field does not imply that the density variation can be neglected in other contexts. A particularly good example of this is provided by convection flows which are driven by buoyancy as we shall discuss in Chapter 17.

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EXERCISES

Exercise 12.8 *Problem: A Hole in My Bucket*
There's a hole in my bucket. How long will it take to empty? (Try an experiment and if the time does not agree with the estimate suggest why this is so.)

Exercise 12.9 *Problem: Crocco's Theorem*
Consider steady flow of an adiabatic fluid. The Bernoulli constant is conserved along stream lines. Show that the variation of B across streamlines is given by

$$\boldsymbol{\nabla} B = T\boldsymbol{\nabla} s + \mathbf{v} \times \boldsymbol{\omega}$$

Exercise 12.10 *Derivation: Joule-Kelvin Coefficient*
Verify Eq. (12.53)

Exercise 12.11 *Problem: Cavitation*
A hydrofoil moves with velocity $\mathbf{V}$ at a depth $h = 3$m below the surface of a lake. (See Figure 12.7.) How fast must the hydrofoil move to make the water next to it boil? (Boiling results from the pressure P trying to go negative.)

Box 12.2
Self Gravity

In the text, we mostly treat the gravitational field as externally imposed and independent of the behavior of the fluid. This is usually a good approximation. However, it is inadequate for discussing the properties of planets and stars. It is easiest to discuss the necessary modifications required by self-gravitational effects by amending the conservation laws.

As long as we work within the domain of Newtonian physics, the mass conservation equation (12.23) is unaffected. However, we included the gravitational force per unit volume $\rho\mathbf{g}_e$ as a source of momentum in the momentum conservation law. It would fit much more neatly into our formalism if we could express it as the divergence of a gravitational stress tensor $\mathbf{T}_g$. To see that this is indeed possible, use Poisson's equation $\boldsymbol{\nabla}\cdot\mathbf{g}_e = 4\pi G\rho$ to write

$$\boldsymbol{\nabla}\cdot\mathbf{T}_g = -\rho\mathbf{g}_e = \frac{(\boldsymbol{\nabla}\cdot\mathbf{g}_e)\mathbf{g}_e}{4\pi G} = \frac{\boldsymbol{\nabla}\cdot[\mathbf{g}_e\otimes\mathbf{g}_e - \frac{1}{2}g_e^2\mathbf{g}]}{4\pi G}\ ,$$

so

$$\mathbf{T}_g = \frac{\mathbf{g}_e\otimes\mathbf{g}_e - \frac{1}{2}g_e^2\mathbf{g}}{4\pi G}\ . \tag{1}$$

Readers familiar with classical electromagnetic theory will notice an obvious and understandable similarity to the Maxwell stress tensor whose divergence equals the Lorentz force density.

What of the gravitational momentum density? We expect that this can be related to the gravitational energy density using a Lorentz transformation. That is to say it is $O(v/c^2)$ times the gravitational energy density, where v is some characteristic speed. However, in the Newtonian approximation, the speed of light, c, is regarded as infinite and so we should expect the gravitational momentum density to be identically zero in Newtonian theory—and indeed it is. We therefore can write the full equation of motion (12.26), including gravity, as a conservation law

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T}_{\rm total} = 0$$

where $\mathbf{T}_{\rm total}$ includes $\mathbf{T}_g$.

Exercise 12.12 *Example: Collapse of a bubble*
Suppose that a spherical bubble has just been created in the water above the hydrofoil in the previous question. We will analyze its collapse, i.e. the decrease of its radius $R(t)$ from its value R_o at creation. First show that the assumption of incompressibility implies that the radial velocity of the fluid at any radial location r can be written in the form $v = F(t)/r^2$.

Box 12.2, Continued

Turn to energy conservation: We have seen in the text that, in a constant, external gravitational field, the fluid's total energy density U and flux $\mathbf{F}$ are given by Eqs. (12.45) and (12.46). In a general situation, we must add to these some field energy density and flux. On dimensional grounds, these must be $U_{\rm field} \propto \mathbf{g}_e^2/G$ and $\mathbf{F}_{\rm field} \propto \Phi_{,t}\mathbf{g}_e/G$ (where $g_e = -\boldsymbol{\nabla}\Phi$). The proportionality constants can be deduced by demanding that [as in the derivation (12.49)–(12.51) of Eq. (12.48)] the laws of mass and momentum conservation imply energy conservation. The result [Ex. 12.13] is

$$U = \rho(\frac{1}{2}v^2 + u + \Phi) + \frac{g_e^2}{8\pi G}\;, \tag{2}$$

$$\mathbf{F} = \rho\mathbf{v}(\frac{1}{2}v^2 + h + \Phi) + \frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\mathbf{g}_e\;. \tag{3}$$

Actually, there is an ambiguity in how the gravitational energy is localized. This ambiguity arises physically from the fact that one can transform away the gravitational acceleration $\mathbf{g}_e$, at any point in space, by transforming to a reference frame that falls freely there. Correspondingly, it turns out, one can transform away the gravitational energy density at any desired point in space. This possibility is embodied mathematically in the possibility to add to the energy flux $\mathbf{F}$ the time derivative of $\alpha\Phi\boldsymbol{\nabla}\Phi/4\pi G$ and add to the energy density U minus the divergence of this quantity (where α is an arbitrary constant), while preserving energy conservation $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$. Thus, the following choice of energy density and flux is just as good as Eqs. (2) and (3); both satisfy energy conservation:

$$U = \rho(\frac{1}{2}v^2 + u + \Phi) + \frac{g_e^2}{8\pi G} - \alpha\boldsymbol{\nabla}\cdot\left(\frac{\Phi\boldsymbol{\nabla}\Phi}{4\pi G}\right) = \rho[\frac{1}{2}v^2 + u + (1-\alpha)\Phi] + (1-2\alpha)\frac{g_e^2}{8\pi G}\;, \tag{4}$$

$$\begin{aligned}\mathbf{F} &= \rho\mathbf{v}(\frac{1}{2}v^2 + h + \Phi) + \frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\mathbf{g}_e + \alpha\frac{\partial}{\partial t}\left(\frac{\Phi\boldsymbol{\nabla}\Phi}{4\pi G}\right)\\ &= \rho\mathbf{v}(\frac{1}{2}v^2 + h + \Phi) + (1-\alpha)\frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\mathbf{g}_e + \frac{\alpha}{4\pi G}\Phi\frac{\partial\mathbf{g}_e}{\partial t}\;.\end{aligned} \tag{5}$$

[Here we have used the gravitational field equation $\nabla^2\Phi = 4\pi G\rho$ and $\mathbf{g}_e = -\boldsymbol{\nabla}\Phi$.] Note that the choice $\alpha = 1$ puts all of the energy density into the $\rho\Phi$ term, while the choice $\alpha = 1/2$ puts all of the energy density into the field term $\mathbf{g}_e^2$. In Ex. 12.10 it is shown that the total gravitational energy of an isolated system is independent of the arbitrary parameter α, as it must be on physical grounds.

A full understanding of the nature and limitations of the concept of gravitational energy requires the general theory of relativity (Part VI). The relativistic analog of the arbitrariness of Newtonian energy localization is an arbitrariness in the gravitational "stress-energy pseudotensor".

Box 12.3
Flow Visualization

There are different methods for visualizing fluid flows. We have already met the *streamlines* which are the integral curves of the velocity field $\mathbf{v}$ at a given time. They are the analog of magnetic lines of force. They will coincide with the *paths* of individual fluid elements if the flow is stationary. However, when the flow is time-dependent, the paths will not be the same as the streamlines. In general, the paths will be the solutions of the equations

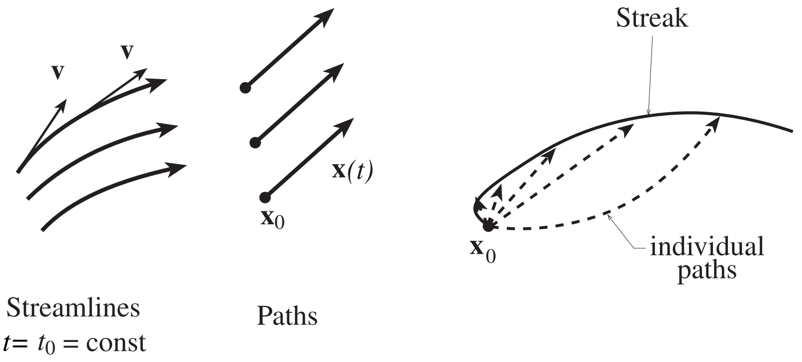
$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t). \tag{1}$$

These paths are the analog of particle trajectories in mechanics.

Yet another type of flow line is a *streak*. This is a common way of visualizing a flow experimentally. Streaks are usually produced by introducing some colored or fluorescent tracer into the flow continuously at some fixed point, say $\mathbf{x}_0$, and observing the locus of the tracer at some fixed time, say t_0. Now, if $\mathbf{x}(t; \mathbf{x}_0, t_0)$ is the expression for the location of a particle released at time t at $\mathbf{x}_0$ and observed at time t_0, then the equation for the streak emanating from $\mathbf{x}_0$ and observed at time t_0 is the parametric relation

$$\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0, t_0)$$

Streamlines, paths and streaks are exhibited below.



Then use the radial component of the Euler equation (12.33) to show that

$$\frac{1}{r^2}\frac{dF}{dt} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0$$

and integrate this outward from the bubble surface at radius R to infinite radius to obtain

$$\frac{-1}{R}\frac{dF}{dt} + \frac{1}{2}v^2(R) = \frac{P_0}{\rho}$$

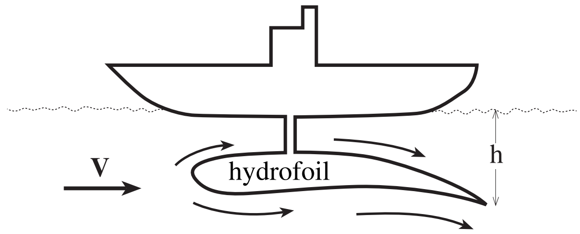


Fig. 12.7: Water flowing past a hydrofoil as seen in the hydrofoil's rest frame.

where P_0 is the ambient pressure. Hence show that the bubble surface moves with speed

$$v(R) = \left(\frac{2P_0}{3\rho}\right)^{1/2}\left[\left(\frac{R_0}{R}\right)^3 - 1\right]^{1/2}$$

Suppose that bubbles formed near the pressure minimum on the surface of the hydrofoil are swept back onto a part of the surface where the pressure is much larger. By what factor must the bubbles collapse if they are to create stresses which inflict damage on the hydrofoil?

Exercise 12.13 *Derivation: Gravitational energy density and flux*
Show that, when the fluid with density ρ produces the gravitational field via $\nabla^2\Phi = 4\pi G\rho$, then the law of mass conservation (12.23), the law of momentum conservation (12.26) and the first law of thermodynamics (Box 12.1) for an ideal fluid imply the law of energy conservation $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$, where U and $\mathbf{F}$ have the forms given in Eqs. (2) and (3) of Box 12.2.

Exercise 12.14 *Example: Gravitational Energy*
Integrate the energy density U of Eq. (4) of Box 12.2 over the interior and surroundings of an isolated gravitating system to obtain the system's total energy. Show that the gravitational contribution to this total energy (i) is independent of the arbitrariness (parameter α) in the energy's localization, and (ii) can be written in the following forms:

$$\begin{aligned} E_g &= \int dV \frac{1}{2}\rho\Phi \\ &= -\frac{1}{8\pi G}\int dV\, g_e^2 \\ &= \frac{-G}{2}\int\int dV\, dV' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \end{aligned}$$

Interpret each of these expressions physically.

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12.4 Viscous Flows - Pipe Flow

12.4.1 Decomposition of the Velocity Gradient

It is an observational fact that many fluids develop a shear stress when they flow. Pouring honey from a spoon provides a convenient example. The stresses that are developed are known as viscous stresses. Most fluids, however, appear to flow quite freely; for example, a cup of tea appears to offer little resistance to stirring other than the inertia of the water. It might then be thought that viscous effects only account for a negligible correction to the description of the flow. However, this is not the case. Despite the fact that many fluids behave in a nearly ideal fashion almost always and almost everywhere, the effects of viscosity are still of great consequence. One of the main reasons for this is that most flows that we encounter touch solid bodies at whose surfaces the velocity must vanish. This leads to the formation of boundary layers whose thickness is controlled by strength of the viscous forces. This boundary layer can then exert a controlling influence on the bulk flow. It may also lead to the development of turbulence.

We must therefore augment our equations of fluid dynamics to include viscous stress. Our formal development proceeds in parallel to that used in elasticity, with the velocity field $\mathbf{v} = d\boldsymbol{\xi}/dt$ replacing the displacement field $\boldsymbol{\xi}$. We decompose the velocity gradient tensor $\boldsymbol{\nabla}\mathbf{v}$ into a *rate of expansion*, θ, a symmetric *rate of shear* tensor $\boldsymbol{\sigma}$ and an antisymmetric *rate of rotation* tensor r, i.e.

$$\boldsymbol{\nabla}\mathbf{v} = \frac{1}{3}\theta\mathsf{g} + \boldsymbol{\sigma} + \mathsf{r}. \tag{12.56}$$

Note that we use lower case symbols to distinguish the fluid case from its elastic counterpart: $\theta = d\Theta/dt$, $\boldsymbol{\sigma} = d\boldsymbol{\Sigma}/dt$, $\mathsf{r} = d\mathsf{R}/dt$. Proceeding directly in parallel to the treatment in Chapter 2, we write

$$\begin{aligned}
\theta &= \boldsymbol{\nabla}\cdot\mathbf{v} \\
\sigma_{ij} &= \frac{1}{2}(v_{i;j} + v_{j;i}) - \frac{1}{3}\theta g_{ij} \\
r_{ij} &= \frac{1}{2}(v_{i;j} - v_{j;i}) = -\frac{1}{2}\epsilon_{ijk}\omega^k
\end{aligned} \tag{12.57}$$

where $\boldsymbol{\omega} = d\boldsymbol{\phi}/dt$ is the vorticity, which is the counterpart of the rotation vector $\boldsymbol{\phi}$.

12.4.2 Navier-Stokes Equation

Although, as we have emphasized, a fluid at rest does not exert a shear stress, and this distinguishes it from an elastic solid, a fluid in motion can resist shear in the velocity field. It has been found experimentally that in most fluids the magnitude of this shear stress is linearly related to the velocity gradient. This law, due to Hooke's contemporary, Isaac Newton, is the analogue of the linear relation between stress and strain that we used in our discussion of elasticity. Fluids that obey this law are known as *Newtonian.* (Some examples of the behavior of non-Newtonian fluids are exhibited in Figure 12.8.)

Fluids are usually isotropic. (Important exceptions include *smectic* liquid crystals.) Therefore, by analogy with the theory of elasticity, we can describe the linear relation between stress and rate of strain using two constants called the coefficients of *bulk* and *shear*

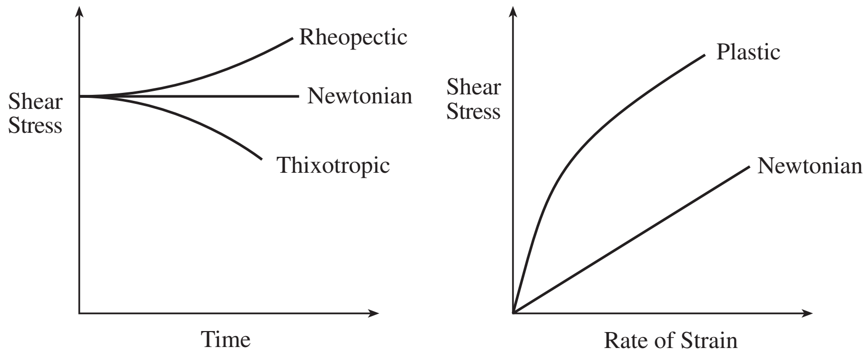

Fig. 12.8: Some examples of non-Newtonian behavior in fluids. a). In a Newtonian fluid the shear stress is proportional to the rate of shear $\boldsymbol{\sigma}$ and does not vary with time when $\boldsymbol{\sigma}$ is constant. However, some substances, such as paint, flow more freely with time and are said to be *thixotropic*. Microscopically, what happens is that the molecules become aligned with the flow which reduces the resisitance. The opposite behaviour is exhibited by *rheopectic* substances. b). An alternative type of non-Newtonian behavior is exhibited by various plastics where a threshold stress is needed before flow will commence.

viscosity and denoted ζ and η respectively. We write the viscous contribution to the stress tensor as

$$\mathbf{T}_{vis} = -\zeta\theta\mathbf{g} - 2\eta\boldsymbol{\sigma} \tag{12.58}$$

by analogy to Eq. (10.34).

If we add this viscous contribution to the stress tensor (12.29), then the law of momentum conservation $\partial(\rho\mathbf{v})/\partial t + \boldsymbol{\nabla}\cdot\mathbf{T}_{\text{total}} = 0$ gives the following modification of Euler's equation (12.33), which contains viscous forces ("your get out what you put in"):

$$\rho\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla}P + \rho\mathbf{g}_e + \boldsymbol{\nabla}(\zeta\theta) + 2\nabla\cdot(\eta\boldsymbol{\sigma}) \tag{12.59}$$

This is called the *Navier-Stokes equation*, and the last two terms are the viscous force density.

As we discuss shortly, it is often appropriate to ignore the bulk viscosity and treat the shear viscosity as constant. In this case, Eq. (12.59) simplifies to

$$\frac{d\mathbf{v}}{dt} = -\frac{\boldsymbol{\nabla}P}{\rho} + \mathbf{g}_e + \nu\nabla^2\mathbf{v} \tag{12.60}$$

where,

$$\nu = \frac{\eta}{\rho} \tag{12.61}$$

is known as the *kinematic viscosity*. This is the commonly quoted form of the Navier-Stokes equation.

The kinematic viscosity ν has dimensions $[L]^2[T]^{-1}$. This suggests that we quantify the importance of viscosity by comparing ν with the product of a characteristic velocity in the flow V and a characteristic length L. The dimensionless combination

$$R = \frac{LV}{\nu} \tag{12.62}$$

is known as the *Reynolds' number*, and is the first of many dimensionless numbers we shall encounter in our study of fluid mechanics. Flows with Reynolds number much less than unity – such as the tragic Boston molasses tank explosion in 1919 which caused one of the slowest floods in history – are dominated by viscosity. Large Reynolds' number flows can still be controlled by viscosity (as we shall see in later chapters), especially when acting near boundaries, despite the fact that the viscous stresses are negligible over most of the volume.

Quantity	**Kinematic viscosity** ν (m^2s^{-1})
Water	10^{-6}
Air	10^{-5}
Glycerine	10^{-3}
Blood	3×10^{-6}

Table 12.2: Kinematic viscosity for common fluids.

12.4.3 Energy conservation and entropy production

The viscous stress tensor represents an additional momentum flux which can do work on the fluid at a rate $\mathbf{T}_{vis} \cdot \mathbf{v}$ per unit area. There is therefore a contribution $\mathbf{T}_{vis} \cdot \mathbf{v}$ to the energy flux, just like the term $P\mathbf{v}$ appearing in Eq. (12.48). We do not expect the viscous stress to contribute to the energy density, though.

Reworking the derivation of equation (12.48) of energy conservation, we find that we must add the term

$$v_i T^{ij}_{vis,j} = (v_i T^{ij}_{vis})_{;j} - v_{i;j} T^{ij}_{vis} \tag{12.63}$$

to Eq. (12.51). The first term of Eq. (12.63) is just the viscous contribution to the total energy flux as promised. The second term remains on the right hand side of Eq. (12.48), which now reads

$$\begin{aligned} \frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] \quad &+ \quad \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right) - \zeta\theta\mathbf{v} - 2\eta\boldsymbol{\sigma}\cdot\mathbf{v}\right] \\ &= \quad \rho T\frac{ds}{dt} - \zeta\theta^2 - 2\eta\boldsymbol{\sigma}:\boldsymbol{\sigma}, \end{aligned} \tag{12.64}$$

where $\boldsymbol{\sigma}:\boldsymbol{\sigma}$ is to be interpreted as the double contraction $\sigma_{ij}\sigma^{ij}$.

This equation needs some interpretation. Once again it is in the form of a conservation law with the rate of change of the energy density plus the divergence of the energy flux (including the viscous contribution) equaling the rate at which energy is added to the fluid. Now let us suppose that the only form of dissipation is viscous dissipation and there is no external source or sink of energy such as radiation or chemical reactions or diffusive heat flow. In this case, the total energy must be conserved without sources or sinks and the right hand side of Eq. (12.64) should vanish. Therefore, *for a viscous fluid with negligible heat flow or other sources and sinks of energy, the rate of increase of entropy due to viscous dissipation is*

$$\rho T\left(\frac{ds}{dt}\right)_{vis} = \zeta\theta^2 + 2\eta\boldsymbol{\sigma}:\boldsymbol{\sigma}\ , \tag{12.65}$$

the energy density is unchanged from that of an ideal fluid, $U = \rho(\frac{1}{2}v^2 + u + \Phi)$, *the energy flux has the form*

$$\mathbf{F} = \rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right) - \zeta\theta\mathbf{v} - 2\eta\boldsymbol{\sigma}\cdot\mathbf{v}\ , \tag{12.66}$$

and the law of energy conservation including viscous dissipation has the standard fundamental form $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$.

Remarkably, we can combine the rate of viscous dissipation (12.65) with the equation of mass conservation (12.1) to obtain a conservation equation for entropy:

$$\frac{\partial(\rho s)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho s\mathbf{v}) = \frac{\zeta\theta^2 + 2\eta\boldsymbol{\sigma}:\boldsymbol{\sigma}}{T} \tag{12.67}$$

The left hand side of this equation describes the rate of change of entropy density plus the divergence of entropy flux. The right hand side is therefore the rate of production of entropy. Invoking the second law of thermodynamics, this must be positive definite. Therefore the two coefficients of viscosity, like the bulk and shear moduli, must be positive.

12.4.4 Molecular Origin of Viscosity

Microscopically, we can distinguish gases from liquids. In gases, molecules of mass m travel a distance of order their *mean free path* λ before they collide. If there is a velocity gradient, $\boldsymbol{\nabla}\mathbf{v}$ in the fluid, then they will, on average, transport a momentum $\sim m\lambda\boldsymbol{\nabla}\mathbf{v}$ with themselves. If there are n molecules per unit volume traveling with mean speeds $\bar{c}$, then the extra momentum crossing a unit area in unit time is $\sim nm\bar{c}\lambda\boldsymbol{\nabla}\mathbf{v}$, from which we may extract an estimate of the coefficient of shear stress

$$\eta = \frac{1}{3}\rho\bar{c}\lambda\ . \tag{12.68}$$

Here the numerical coefficient of 1/3 has been inserted to agree with a proper kinetic-theory calculation. (Since, in the language of Chapter 2, the viscosity coefficients are actually "transport coefficients" for momentum, a kinetic-theory calculation can be made using the techniques of Section 2.7.) Note from Eq. (12.68) that in a gas the coefficient of viscosity will increase with temperature ($\propto T^{1/2}$).

In a liquid, where the molecules are less mobile, it is the close intermolecular attraction that dominates the shear stress. The ability of molecules to slide past one another therefore increases rapidly with their thermal activation, causing typical liquid viscosity coefficients to fall dramatically with temperature.

12.4.5 Blood Flow

Let us now consider one simple example of a viscous stress at work, namely the flow of blood down an artery. Let us model the artery as a cylindrical pipe of radius R, down which the blood is forced by a pressure gradient. This is an example of what is called *pipe flow*. In the absence of external forces, and time-dependence, the divergence of the total stress tensor must vanish. Therefore,

$$\boldsymbol{\nabla}\cdot[\rho\mathbf{v}\otimes\mathbf{v} + P\mathbf{g} - 2\eta\boldsymbol{\sigma}] = 0 \tag{12.69}$$

Now, in most instances of pipe flow $\rho v^2 \ll \Delta P =$(the pressure difference between the two ends), so we can neglect the first term in Eq. (12.69). We now suppose that the flow is solely along the $z-$ direction only a function of cylindrical radius ϖ. (This is an example of *laminar* flow.) As the density is effectively constant (we satisfy the conditions for incompresible flow), and we must conserve mass, the velocity cannot change along the pipe. Therefore the only non-vanishing component of the shear tensor is the ϖz component. Reexpressing Eq. (12.69) in cylindrical coordinates, and inferring from it that the pressure is a function of z only and not of ϖ, we obtain

$$\frac{1}{\varpi}\frac{d}{d\varpi}\left(\varpi\eta\frac{dv}{d\varpi}\right) = -\frac{dP}{dz}\ , \tag{12.70}$$

where dP/dz is the pressure gradient along the pipe. This differential equation must be solved subject to the boundary conditions that the velocity gradient vanish at the center of the pipe and that the velocity vanish at its walls. The solution is

$$v(\varpi) = -\frac{dP}{dz}\frac{R^2 - \varpi^2}{4\eta} \tag{12.71}$$

We can now evaluate the total discharge or mass of fluid flowing along the pipe.

$$\frac{dm}{dt} = \int_0^R \rho v 2\pi\varpi d\varpi = -\frac{\pi\rho R^4}{8\eta}\frac{dP}{dz} \tag{12.72}$$

This relation is known as *Poiseuille's law.*

Now let us apply this to blood. Consider an artery of radius $R = 1$mm. An estimate of the pressure gradient may be obtained from the difference between the diastolic and systolic pressure measured by a doctor ($\sim$ 40mm of mercury $\sim 5\times10^3$N m^{-2} in a healthy adult) and dividing by the length of the artery, $\sim$ 1m. The kinematic viscosity is $\eta/\rho = \nu = 3\times10^{-6}$m^2 s^{-1} from Table 12.2. The rate of blood flow is then $\sim 3\times10^{-4}$kg s^{-1} or $\sim 3\times10^{-7}$m^{3}s^{-1}. Now, supposing there are ten such arteries of this size and length, the total blood flow will be $\sim 3\times10^{-6}$m^{3}s^{-1}.

Actually, the heart of a healthy adult pumps the full complement of blood $\sim$ 5litres or $\sim 5\times10^{-3}$ m^3 every minute at a mean rate of $\sim 10^{-4}$m^{3}s^{-1} about thirty times faster than this estimate. There are two reasons for this large discrepancy. The first is that we have assumed in our calculation that the walls of an artery are like those of an oil pipeline, i.e. rigid. They are not. They are quite elastic and are able to contract and expand in a wave-like manner so as to boost the blood flow considerably. The second reason for the discrepancy is that the Poiseuille formula is very sensitive to the radius of the pipe, $dm/dt \propto R^4$, so a factor two reduction in radius reduces the flow of blood by sixteen. Both hardening and thinning of the arteries will therefore strongly inhibit the flow of blood. Eat salads!

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EXERCISES

Exercise 12.15 *Problem: Mean free path*
Estimate the collision mean free path of the air molecules around you. Hence verify the estimate for the kinematic viscosity of air given in Table 12.2.

Exercise 12.16 *Example: Kinematic interpretation of Vorticity*
Consider a velocity field with non-vanishing curl. Define a locally orthonormal basis at a point in the velocity field so that one basis vector, $\mathbf{e}_x$ is parallel to the vorticity. Now imagine the remaining two basis vectors as being frozen into the fluid. Show that they will both rotate about the axis defined by $\mathbf{e}_x$ and that the vorticity will be the sum of their angular velocities (i.e. twice the average of their angular velocities).

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