Chapter 13

Vorticity

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13.1 Overview

In the last chapter, we introduced an important quantity called *vorticity* which is the subject of the present chapter. Although the most mathematically simple flows are "potential flows", with velocity of the form $\mathbf{v} = \boldsymbol{\nabla}\psi$ for some ψ so the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ vanishes, the majority of naturally occuring flows are vortical. We shall find that studying vorticity allows us to develop an intuitive understanding of how flows evolve. Furthermore computing the vorticity can provide an important step along the path to determining the full velocity field of a flow.

We all think we know how to recognise a vortex. The most hackneyed example is water disappearing down a drainhole in a bathtub. Here what happens is that water at large distances has a small angular velocity about the drain, which increases as the water flows towards the drain in order to conserve angular momentum. This in turn means that the product of the circular velocity v_ϕ and the radius r is independent of radius, which, in turn, implies that $\boldsymbol{\nabla} \times \mathbf{v} \sim 0$. So this is a vortex without much vorticity! (except, as we shall see, a delta-function spike of vorticity right at the drainhole's center). Vorticity is a precise physical quantity defined by $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$, not any vaguely circulatory motion.[1]

In Sec. 13.2 we shall introduce several tools for analyzing and utilizing vorticity: Vorticity is a vector field and therefore has integral curves obtained by solving $d\mathbf{x}/d\lambda = \boldsymbol{\omega}$ for some parameter λ. These are called *vortex lines* and they are quite analogous to magnetic field lines. (We shall also introduce an integral quantity called the *circulation* analogous to the magnetic flux and show how this is also helpful for understanding flows.) In fact, the analogy with magnetic fields turns out to be extremely useful. Vorticity, like a magnetic field, has vanishing divergence which means that the vortex lines are continuous, just like magnetic

[1] Incidentally, in a bathtub the magnitude of the Coriolis force resulting from the earth's rotation with angular velocity Ω is a fraction $\sim \Omega(r/g)^{1/2} \sim 3 \times 10^{-6}$ of the typical centrifugal force in the vortex (where g is the acceleration of gravity and $r \sim 2$ cm is the radius of the vortex at the point where the water's surface achieves an appreciable downward slope). Thus, only under the most controlled of conditions will the hemisphere in which the drainhole is located influence the direction of rotation.

field lines. Vorticity, again like a magnetic field, is an axial vector and thus can be written as the curl of a polar vector potential, the velocity $\mathbf{v}$. Vorticity has the interesting property that it evolves in a perfect fluid in such a manner that the flow carries the vortex lines along with it. Furthermore, when viscous stresses are important, vortex lines diffuse through the moving fluid with a diffusion coefficient that is equal to the kinematic viscosity.

In Sec. 13.3 we study a classical problem that illustrates both the action and the propagation of vorticity: the *creeping* flow of a low Reynolds' number fluid around a sphere. (Low Reynolds' number flow arises when the magnitude of the viscous stress in the equation of motion exceeds the magnitude of the inertial acceleration.) The solution to this problem finds contemporary application in computing the sedimentation rates of soot particles in the atmosphere.

In Sec. 13.4, we turn to high Reynolds' number flows, in which the viscous stress is quantitatively weak over most of the fluid. Here, the action of vorticity can be concentrated in relatively thin *boundary layers* in which the vorticity, created at the wall, diffuses away into the main body of the flow. Boundary layers arise because in real fluids, intermolecular attraction requires that the component of the fluid velocity parallel to the boundary (not just the normal component) vanish. It is the vanishing of both components of velocity that distinguishes real fluid flow at high Reynolds' number (i.e. small viscosity) from the solutions obtained assuming vanishing vorticity. Nevertheless, it is often a good approximation to adopt a solution to the equations of fluid dynamics in which vortex-free fluid slips freely past the solid and then match it onto the solid using a boundary-layer solution.

All the above issues will be discussed in some detail in this chapter. As a final issue, in Sec. 13.5 we shall consider a simple vortex sheet ignoring viscous stresses. We shall find that this type of flow is generically unstable. This will provide a good introduction to the principal topic of the next chapter, turbulence.

13.2 Vorticity and Circulation

We have already defined the vorticity as the curl of the velocity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$, analogous to defining the magnetic field as the curl of a vector potential.

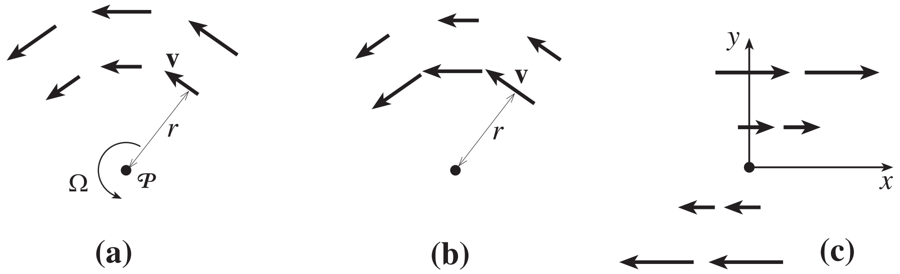

Fig. 13.1: Illustration of vorticity in three two-dimensional flows. a) Constant angular velocity Ω. If we measure radius r from the center $\mathcal{P}$, the circular velocity satisfies $v = \Omega r$. This flow has vorticity $\omega = 2\boldsymbol{\Omega}$. b) Constant angular momentum per unit mass j, with $v = j/r$. This flow has zero vorticity except at its center, $\omega = (j/2\pi)\delta(\mathbf{x})$. c) Shear flow in a laminar boundary layer, $v_x = \omega y$. In this flow the vorticity is $\omega = v_x/y$.

We can illustrate vorticity by considering the three simple 2-dimensional flows shown in Fig. 13.1:

Drawing (a) shows uniform rotation with angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_z$. The velocity field is $\mathbf{v} = \mathbf{\Omega} \times \mathbf{x}$, where $\mathbf{x}$ is measured from the rotation axis. Taking its curl, we discover that $\boldsymbol{\omega} = 2\mathbf{\Omega}$.

Drawing (b) shows a flow in which the angular momentum per unit mass $\mathbf{j} = j\mathbf{e}_z$ is constant, i.e. $\mathbf{v} = \mathbf{j} \times \mathbf{x}/r^2$ (where $r = |\mathbf{x}|$). This is the kind of flow that occurs around a bathtub vortex, and around a tornado. In this case, the vorticity is $\boldsymbol{\omega} = (\mathbf{j}/2\pi)\delta(\mathbf{x})$, .i.e. it vanishes everywhere except at the center, $\mathbf{x} = 0$. What is different in this case is that the fluid rotates differentially and although two neighboring fluid elements, separated tangentially, rotate about each other with an angular velocity $\mathbf{j}/r^2$, when the two elements are separated radially, their angular velocity is $-\mathbf{j}/r^2$. The average of these two angular velocities vanishes, and the vorticity vanishes.

The vanishing vorticity in this case is an illustration of a simple geometrical description of vorticity in any two dimensional flow: If we orient the $\mathbf{e}_z$ axis of a Cartesian coordinate system normal to the velocity field, then

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \, . \tag{13.1}$$

From this expression, it is apparent that the vorticity at a point is the sum of the angular velocities of any pair of mutually perpendicular, infinitesimal lines passing through that point and moving with the fluid. If we float a little vane with orthogonal fins in the flow, the vane will rotate with an angular velocity that is the average of the flow's angular velocities at its fins, which is half the vorticity. Equivalently, the vorticity is twice the rotation rate of the vane. In the case of constant angular momentum flow in Fig. 13.1(b), the average of the two angular velocities is zero, the vane doesn't rotate, and the vorticity vanishes.

Diagram 13.1(c) shows the flow in a plane-parallel shear layer. In this case, a line in the flow along the x direction does not rotate, while a line along the y direction rotates with angular velocity ω. The sum of these two angular velocities, $0 + \omega = \omega$ is the vorticity. Evidently, curved streamlines are not a necessary condition for vorticity.

13.2.1 Vorticity Transport

By analogy with magnetic field lines, we define a flow's *vortex lines* to be parallel to the vorticity vector $\boldsymbol{\omega}$ and to have a line density proportional to $\omega = |\boldsymbol{\omega}|$. These vortex lines will always be continuous throughout the fluid because the vorticity field, like the magnetic field, is a curl and therefore is necessarily solenoidal ($\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$). However, vortex lines can begin and end on solid surfaces, as the equations of fluid dynamics no longer apply there. Vorticity depends on the velocity field at a particular instant, and will evolve with time as the velocity field evolves. We can determine how by manipulating the Navier-Stokes equation

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi + \nu \nabla^2 \mathbf{v} \tag{13.2}$$

[Eq. (12.43)]. (Here and throughout this chapter we keep matters simple by assuming that the bulk viscosity is ignorable and the coefficient of shear viscosity is constant.) We take the curl of (13.2) and use the vector identity $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \boldsymbol{\nabla}(v^2)/2 - \mathbf{v}\times\boldsymbol{\omega}$ (easily derivable using the Levi-Civita tensor and index notation) to obtain

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega}) - \frac{\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\rho}{\rho^2} + \nu\nabla^2\boldsymbol{\omega}. \tag{13.3}$$

It is convenient to rewrite this vorticity evolution law with the aid of the relation

$$\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega}) = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v} + \mathbf{v}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}) - \boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\mathbf{v}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}\ . \tag{13.4}$$

Inserting this into Eq. (13.3), using $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$ and introducing a new type of time derivative

$$\begin{aligned}\frac{D\boldsymbol{\omega}}{Dt} &= \frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}\\ &= \frac{d\boldsymbol{\omega}}{dt} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}\ ,\end{aligned} \tag{13.5}$$

we bring Eq. (13.3) into the following form:

$$\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega}\boldsymbol{\nabla}\cdot\mathbf{v} - \frac{\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\rho}{\rho^2} + \nu\nabla^2\boldsymbol{\omega}. \tag{13.6}$$

This is our favorite form for the vorticity evolution law. In the remainder of this section we shall explore its predictions.

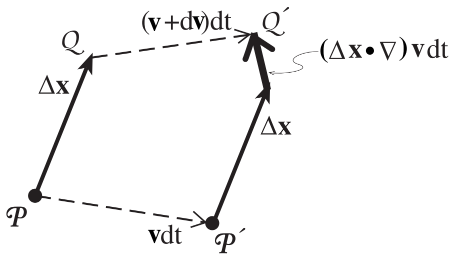

Fig. 13.2: Equation of motion for an infinitesimal vector $\Delta\mathbf{x}$ connecting two fluid elements. As the fluid elements at $\mathcal{P}$, and $\mathcal{Q}$ move to $\mathcal{P}'$ and $\mathcal{Q}'$ in a time interval dt, the vector changes by $(\Delta\mathbf{x}\cdot\nabla)\mathbf{v}dt$.

The operator D/Dt (defined by Eq. (13.4) when acting on a vector and by $D/Dt = d/dt$ when acting on a scalar) is called the *fluid derivative*. (The reader should be warned that the notation D/Dt is used in some older texts for the convective derivative d/dt.) The geometrical meaning of the fluid derivative can be understood from Fig. 13.2. Denote by $\Delta\mathbf{x}(t)$ the vector connecting two points $\mathcal{P}$ and $\mathcal{Q}$ that are moving with the fluid. Then the convective derivative $d\Delta\mathbf{x}/dt$ must equal the relative velocity of these two points, namely $(\Delta\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{v}$. In other words, the fluid derivative of $\Delta\mathbf{x}$ vanishes

$$\frac{D\Delta\mathbf{x}}{Dt} = 0\ . \tag{13.7}$$

More generally, the fluid derivative of any vector can be understood as its rate of change relative to a vector moving with the fluid.

In order to understand the vorticity evolution law (13.6) physically, let us consider a barotropic $[P = P(\rho)]$, inviscid ($\nu = 0$) fluid flow. (This is the kind of flow that usually occurs in the Earth's atmosphere and oceans, well away from solid boundaries.) Then the last two terms in Eq. (13.6) will vanish, leaving

$$\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega}\boldsymbol{\nabla}\cdot\mathbf{v} \ . \tag{13.8}$$

Equation (13.8) says that the vorticity has a fluid derivative parallel to itself. In this sense we can speak of the vortex lines as being *frozen* into the moving fluid.

We can actually make the fluid derivative vanish by substituting $\boldsymbol{\nabla}\cdot\mathbf{v} = -\rho^{-1}d\rho/dt$ (the equation of mass conservation) into Eq. (13.8); the result is

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = 0 \ . \tag{13.9}$$

Therefore, the quantity $\boldsymbol{\omega}/\rho$ evolves according to the same equation as the separation $\Delta\mathbf{x}$ of two points in the fluid. To see what this implies, consider a small cylindrical fluid element whose symmetry axis is parallel to $\boldsymbol{\omega}$ (Fig. 13.3). Denote its vectorial length by $\Delta\mathbf{x}$ and its vectorial cross sectional area by $\Delta\boldsymbol{\Sigma}$. Then since $\boldsymbol{\omega}/\rho$ points along $\Delta\mathbf{x}$ and both are frozen into the fluid, it must be that $\boldsymbol{\omega}/\rho = \text{constant}\times\Delta\mathbf{x}$. Therefore, the fluid element's conserved mass is $\Delta M = \rho\Delta\mathbf{x}\cdot\Delta\boldsymbol{\Sigma} = \text{constant}\times\boldsymbol{\omega}\cdot\Delta\boldsymbol{\Sigma}$, so $\boldsymbol{\omega}\cdot\Delta\boldsymbol{\Sigma}$ is conserved as the cylindrical fluid element moves and deforms. We thereby conclude that the fluid's vortex lines, with number per unit area directly proportional to $|\boldsymbol{\omega}|$, are convected by our barotropic, inviscid fluid, without having to be created or destroyed.

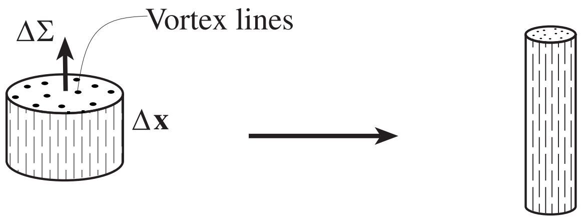

Fig. 13.3: Simple demonstration of the kinematics of vorticity propagation in a barotropic, inviscid flow. A short, thick cylindrical fluid element with generators parallel to the local vorticity gets deformed, by the flow, into a long, slender cylinder. By virtue of Eq. (13.9), we can think of the vortex lines as being convected with the fluid, with no creation of new lines or destruction of old ones, so that the number of vortex lines passing through the cylinder (through its end surface $\Delta\boldsymbol{\Sigma}$) remains constant.

If the flow is not only barotropic and inviscid, but also incompressible (as it usually is to high accuracy in the oceans and atmosphere), then Eqs. (13.8) and (13.9) say that $D\boldsymbol{\omega}/Dt = 0$. Suppose, in addition, that the flow is 2-dimensional (as it commonly is to moderate accuracy averaged over scales large compared to the thickness of the atmosphere and oceans), so $\mathbf{v}$ is in the x and y directions and independent of z. This means that $\boldsymbol{\omega} = \omega\mathbf{e}_z$ and we can regard the vorticity as the scalar ω. Then Eq. (13.5) with $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v} = 0$ implies that the vorticity obeys the simple propagation law

$$\frac{d\omega}{dt} = 0 \ . \tag{13.10}$$

Thus, in a 2-dimensional, incompressible, baroytropic, inviscid flow, the vorticity is convected conservatively, just like entropy in an adiabatic fluid.

13.2.2 Tornadoes

A particular graphic illustration of the behavior of vorticity is provided by a tornado. Tornadoes in North America are most commonly formed at a front where cold, dry air from the north meets warm, moist air from the south, and huge, cumulo-nimbus thunderclouds form. Air is set into counter-clockwise circulatory motion just below the clouds by Coriolis forces. A low pressure vortical core is created at the center of this spinning fluid and there will be an upflow of air which will cause the spinning region to lengthen. Now, consider this in the context of vorticity propagation. As the air, to first approximation, is incompressible, a lengthening of the vortex lines corresponds to a reduction in the cross section and a strengthening of the vorticity. This, in turn, corresponds to an increase in the circulatory speeds found in a tornado. (Speeds in excess of 300 mph have been reported.)

If and when the tornado touches down to the ground and its very-low-pressure core passes over the walls and roof of a building, the far larger, normal atmospheric pressure inside the building can cause the building to explode.

13.2.3 Kelvin's Theorem

Intimately related to vorticity is a quantity called the *circulation* Γ; it is defined as the line integral of the velocity around a closed contour ∂S lying in the fluid

$$\Gamma \equiv \int_{\partial S} \mathbf{v} \cdot d\mathbf{x} \ , \tag{13.11}$$

and it can be regarded as a property of the closed contour ∂S. We can invoke Stokes' theorem to convert this circulation into a surface integral of the vorticity passing through a surface S bounded by the same contour:

$$\Gamma = \int_S \boldsymbol{\omega} \cdot d\boldsymbol{\Sigma} \ . \tag{13.12}$$

[Note, though, that Eq. (13.12) is only valid if the area bounded by the contour is simply connected; in particular, if the area enclosed contains a solid body, Eq. (13.12) may fail.] Equation (13.12) says that the circulation Γ is the flux of vorticity through S, or equivalently the number of vortex lines passing through S. Circulation is thus the fluid counterpart of magnetic flux.

Kelvin's theorem tells us the rate of change of the circulation associated with a particular contour ∂S that is attached to the moving fluid. Let us evaluate this directly using the convective derivative of Γ. We do this by differentiating the two vector quantities inside the integral (13.11):

$$
\begin{aligned}
\frac{d\Gamma}{dt} &= \int_{\partial S} \frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} + \int_{\partial S} \mathbf{v} \cdot d\left(\frac{d\mathbf{x}}{dt}\right) \\
&= -\int_{\partial S} \frac{\boldsymbol{\nabla} P}{\rho} \cdot d\mathbf{x} - \int_{\partial S} \boldsymbol{\nabla}\Phi \cdot d\mathbf{x} + \nu \int_{\partial S} (\nabla^2\, \mathbf{v}) \cdot d\mathbf{x} + \int_{\partial S} d\frac{1}{2}v^2, \qquad (13.13)
\end{aligned}
$$

where we have used the Navier-Stokes equation (13.2) with $\nu =$ constant. The second and fourth terms on the right hand side of Eq. (13.13) vanish around a closed curve and the first can be rewritten in different notation to give

$$\frac{d\Gamma}{dt} = -\int_{\partial S} \frac{dP}{\rho} + \nu \int_{\partial S} (\nabla^2\, \mathbf{v}) \cdot d\mathbf{x}\ . \qquad (13.14)$$

This is Kelvin's theorem for the evolution of circulation. In a rotating reference frame it must be augmented by the integral of the Coriolis acceleration $-2\boldsymbol{\Omega} \times \mathbf{v}$ around the closed curve ∂S, and if the fluid is electrically conducting and possesses a magnetic field it must be augmented by the integral of the Lorentz force per unit mass $\mathbf{J} \times \mathbf{B}/\rho$ around ∂S.

If the fluid is barotropic, $P = P(\rho)$, and the effects of viscosity are negligible (and the coordinates are inertial and there is no magnetic field and electric current), then the right hand side of Eq. (13.14) vanishes, and Kelvin's theorem takes the simple form

$$\frac{d\Gamma}{dt} = 0 \quad \text{for barotropic, inviscid flow.} \qquad (13.15)$$

This is just the global version of our result that the circulation $\boldsymbol{\omega} \cdot \Delta\boldsymbol{\Sigma}$ of an infinitesimal fluid element is conserved.

The qualitative content of Kelvin's theorem is that vorticity in a fluid is long-lived. Once a fluid develops some circulation, this circulation will persist unless and until the fluid can develop some shear stress, either directly though viscous action, or indirectly through the $\boldsymbol{\nabla} P \times \boldsymbol{\nabla}\rho$ term or a Coriolis or Lorentz force term in Eq. (13.3).

13.2.4 Diffusion of Vortex Lines

Next, consider the action of viscous stresses on an existing vorticity field. For a barotropic fluid with nonneglible viscosity, the vorticity evolution law (13.6) says

$$\frac{D\boldsymbol{\omega}}{Dt} = \nu\nabla^2\boldsymbol{\omega}. \qquad (13.16)$$

This is a "convective" diffusion equation: the viscous term $\nu\nabla^2\boldsymbol{\omega}$ causes the vortex lines to diffuse through the moving fluid. When viscosity is negligible, the vortex lines are frozen into the flow. When the shear viscuous stress is large, vorticity will spread away from its source to occupy an area $\sim \nu t$ in the moving fluid after a time t. The kinematic viscosity, therefore, not only has the dimensions of a diffusion coefficient, it actually controls the diffusion of vortex lines relative to the moving fluid.

As a simple example of the spreading of vortex lines, consider an infinite plate moving parallel to itself relative to a fluid at rest. Let us transform to the frame of the plate

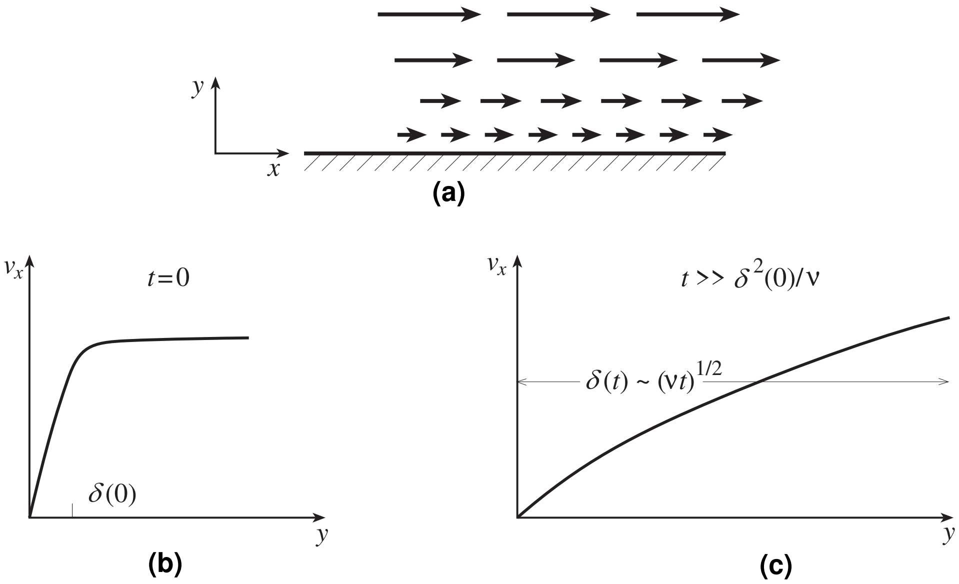

Fig. 13.4: A simple shear layer in which the vortex line freezing term, $\nabla \times (\mathbf{v} \times \boldsymbol{\omega})$ vanishes. Vorticity will diffuse away from the midplane under the action of viscous torques in much the same way that heat diffuses away from a heated surface.

[Fig. 13.4(a)] so the fluid moves past it. Suppose that at time $t = 0$ the velocity has only a component v_x parallel to the plate, which depends solely on the distance y from the plate, and suppose further that v_x is constant away from the plate, but in a thin boundary layer along the plate it decreases to 0 at $y = 0$ (as it must, because of the plate's "no-slip" boundary condition); cf. Fig. 13.4(b). As the flow is a function only of y (and t), and $\mathbf{v}$ and $\boldsymbol{\omega}$ point in directions orthogonal to $\mathbf{e}_y$, the fluid derivative (13.5) in our situation reduces to a partial derivative and the convective diffusion equation (13.16) becomes an ordinary diffusion equation, $\partial\boldsymbol{\omega}/\partial t = \nu\nabla^2\boldsymbol{\omega}$. Let the initial thickness of the boundary layer be $\delta(0)$. Then this diffusion equation says that the viscosity will diffuse through the fluid, under the action of viscous stress, and as a result, the boundary-layer thickness will increase with time as

$$\delta(t) \sim (\nu t)^{\frac{1}{2}} \quad \text{for} \quad t \gtrsim \delta(0)^2/\nu \ . \tag{13.17}$$

13.2.5 Sources of Vorticity.

Having discussed how vorticity is conserved in simple inviscid flows and how it diffuses away under the action of viscosity, we must now consider its sources. The most important source is a solid surface. When fluid suddenly encounters a solid surface like the leading edge of an airplane wing, intermolecular forces act to decelerate the fluid very rapidly in a thin boundary layer along the surface. This introduces circulation and consequently vorticity into the flow, where none existed before; and that vorticity then diffuses into the bulk flow, thickening the boundary layer (Sec. 13.4 below).

If the fluid is non-barotropic, then pressure gradients can also create vorticity, as described by the second term on the right hand side of our vorticity evolution law (13.6). Physically

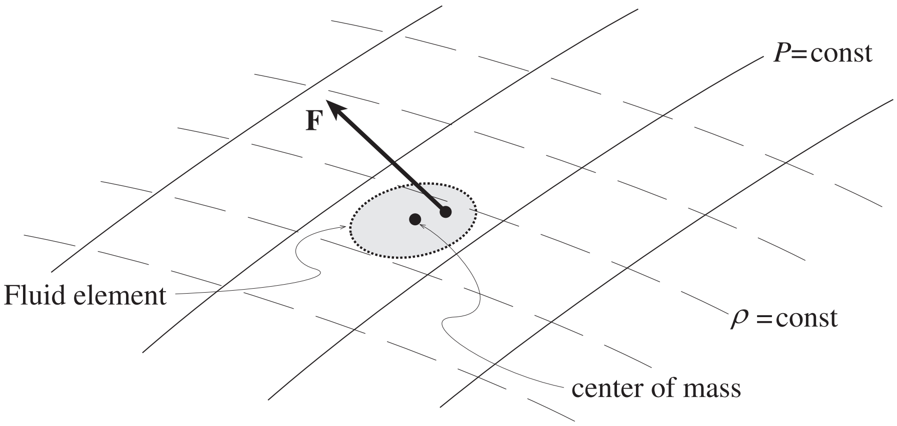

Fig. 13.5: Mechanical explanation for the creation of vorticity in a non-barytropic fluid. The net pressure gradient force **F** acting on a small fluid element is normal to the isobars (solid lines) and does not pass through the center of mass of the element; thereby a torque is produced.

what happens is that when the isobars do not coincide with the contours of constant density, known as *isochors*, the net force on a small element of fluid does not pass through its center of mass, and it therefore exerts a torque on the element, introducing some rotational motion and vorticity. (See Figure 13.5.) Non-barotropic pressure gradients can therefore create vorticity within the body of the fluid. Note that as the vortex lines must be continuous, any fresh ones that are created within the fluid must be created as loops that expand from a point or a line.

There are two other common sources of vorticity in fluid dynamics, Coriolis forces (when one's reference frame is rotating rigidly) and Lorentz forces (when the fluid is electrically conducting). We shall discuss these in Chaps. 15 and 18 respectively.

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EXERCISES

Exercise 13.1 *Practice: Vorticity and incompressibility*
Sketch the streamlines for the following stationary two dimensional flows, determine if the fluid is compressible, and evaluate its vorticity. The coordinates are Cartesian in (i) and (ii), and are circular polar with orthonormal bases $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ in (iii) and (iv).

(i) $v_x = 2xy, \quad v_y = x^2.$

(ii) $v_x = x^2, \quad v_y = -2xy$

(iii) $v_r = 0, \quad v_\phi = r$

(iv) $v_r = 0, \quad v_\phi = r^{-1}.$

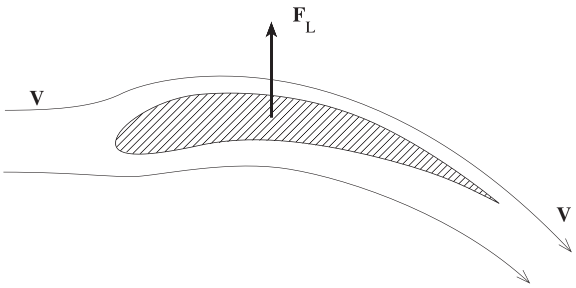

Fig. 13.6: Flow around an airfoil.

Exercise 13.2 *Example: Joukowski's Theorem*
When an appropriately curved airfoil is introduced into a steady flow of air, the air has to flow faster along the upper surface than the lower surface and this can create a lifting force (Fig. 13.6.) In this situation, compressibility and gravity are usually unimportant for the flow. Show that the pressure difference across the airfoil is given approximately by

$$\Delta P = \frac{1}{2}\rho\Delta(v^2) = \rho v \Delta v \ .$$

Hence show that the lift exerted by the air on an airfoil of length L is given approximately by

$$F_L = L\int \Delta P d\mathbf{x} = \rho v L \Gamma \ ,$$

where Γ is the circulation around the airfoil. This is known as *Joukowski's theorem.* Interpret this result in terms of the conservation of linear momentum.

Exercise 13.3 *Example: Rotating Superfluids*
Certain fluids at low temperature undergo a phase transition to a superfluid state. A good example is ^{4}He for which the transition temperature is 2.2K. As a superfluid has no viscosity, it cannot develop vorticity. How then can it rotate? The answer (e.g. Feynman 1972) is that not all the fluid is in a superfluid state; some of it is normal and can have vorticity. When the fluid rotates, all the vorticity is concentrated within microscopic vortex cores of normal fluid that are parallel to the rotation axis and have quantized circulations $\Gamma = h/m$, where m is the mass of the atoms and h is Planck's constant. The fluid external to these vortex cores is irrotational. These normal fluid vortices may interact with the walls of the container.

(i) Explain, using a diagram, how the vorticity of the macroscopic velocity field, averaged over many vortex cores, is twice the mean angular velocity of the fluid.

(ii) Make an order of magnitude estimate of the spacing between these vortex cores in a beaker of superfluid helium on a turntable rotating at 10 rpm.

(iii) Repeat this estimate for a millisecond neutron star, which mostly comprises superfluid neutron pairs at the density of nuclear matter and spins with a period of order a millisecond. (The mass of the star is roughly 3×10^{30}kg.)

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13.3 Low Reynolds' Number Flow – Stokes' Flow, Sedimentation and Nuclear Winter

In the last chapter, we defined the Reynolds' number, R, to be the ratio of the product of the characteristic speed and lengthscale of the flow to its the kinematic viscosity. The significance of the Reynold's number follows from the fact that in the Navier-Stokes equation (13.2), the ratio of the magnitude of the inertial term $|(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}|$ to the viscous acceleration, $|\nu\nabla^2\mathbf{v}|$ is approximately equal to R. Therefore, when $R \ll 1$, the inertial acceleration can often be ignored and the velocity field is determined by balancing the pressure against the viscous stress. The velocity will then scale linearly with the magnitude of the pressure gradient and will vanish when the pressure gradient vanishes. This has the amusing consequence that a low Reynolds' number flow driven by a solid object moving through a fluid at rest is effectively reversible; if the motion of the object is reversed, then the fluid elements will return almost to their original positions.

From the magnitudes of viscosities of real fluids [Table 12.2 (on page 12.28)], it follows that the low Reynolds' number limit is appropriate for either very small scale flows (e.g. the motion of micro-organisms) or for very viscous fluids (e.g. the earth's mantle). One example of small-scale flow arises in discussions of *nuclear winter*. Several scientists (e.g. Turco *etal* 1983) have pointed out that in the event of a major nuclear war, there is a danger that major fires in forested and urban areas will discharge large quantities of soot into the upper atmosphere where they will absorb solar optical and ultraviolet radiation while remaining reasonably transparent to infra-red radiation escaping from the earth's surface. The sedimentation of the soot back to the ground is controlled by the low-Reynolds-number flow of air around each soot particle, and as we shall see it is very slow. As a result, the soot, floating for a long time in the atmosphere, may cause solar heating of the atmosphere and dramatic cooling of the entire earth's surface, with catastrophic consequences for most species. (Major volcanos, like Mt. Pinatubo, have also ejected significant quantities of volcanic ash into the stratosphere, noticeably affecting the weather for times as long as a year or more.)

Relatively little soot is required to block out the sun. Micron-sized particles will absorb light with roughly their geometrical cross section. As the area of the earth's surface is roughly $7\times10^{14}\text{m}^2$, and the density of the particles is roughly 2000kg m^{-3}, a mass M of about 10^{12}kg, equivalent to one small mountain, is sufficient to accomplish this. (An impressively large energy $Mg_eH \sim 3\times10^{17}$J, equivalent to ~ 40 megatons of TNT, is however needed to elevate this amount of mass to an altitude $H \sim 30$km in the stratosphere.)

A key calculation in these predictions is to understand the rate at which the soot particles will settle out of the atmosphere. We model this process by computing the speed at which a spherical soot particle falls through quiescent air when the Reynolds' number is small. The speed is governed by a balance between the downward force of gravity and the speed-dependent upward drag force of the air. We shall compute this speed by first evaluating the force of the air on the moving particle ignoring gravity, and then, at the end of the calculation, inserting the influence of gravity.

13.3.1 Stokes Flow

We model the soot particle as a sphere of radius a. The flow of a viscous fluid past such a sphere is known as *Stokes' flow*. We will calculate this flow's velocity field, and then from it, the force of the fluid on the sphere. (This calculation also finds application in the famous Millikan oil drop experiment.)

It obviously is easiest to tackle this problem in the frame of the sphere and to seek a solution in which the flow speed, $\mathbf{v}(\mathbf{x})$, tends to a constant value $\mathbf{V}$ (the velocity of the sphere through the fluid), at large distances from the sphere's center. The asymptotic velocity $\mathbf{V}$ is presumed to be highly subsonic and so the flow is effectively incompressible, $\nabla \cdot \mathbf{v} = 0$. It is also stationary, $\partial \mathbf{v}/\partial t = 0$ by virtue of the stationarity of the sphere and the distant flow.

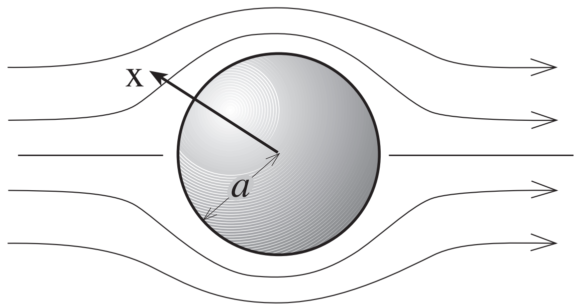

Fig. 13.7: Stokes flow around a sphere.

We define the Reynold's number for this flow by $R = \rho V a/\eta = Va/\nu$. As this is, by assumption, small, we can ignore the inertial term, which is $O(V\Delta v/a)$ in the Navier-Stokes Eq. (13.2) in comparison with the viscous term which is $O(\eta \Delta v/a^2\rho)$; here $\Delta v \sim V$ is the total velocity variation. The Navier-Stokes equation (13.2) can thus be well-approximated by

$$\nabla P = \eta \nabla^2 \mathbf{v} \ . \tag{13.18}$$

The full details of the flow are governed by this force-balance equation, the flow's incompressibility

$$\nabla \cdot \mathbf{v} = 0 \ , \tag{13.19}$$

and the boundary conditions $\mathbf{v} = 0$ at $r = a$ and $\mathbf{v} \to \mathbf{V}$ at $r \to \infty$.

From force balance (13.18) we infer that in order of magnitude the difference between the fluid's pressure on the front of the sphere and that on the back is $\Delta P \sim \eta V/a$. We also expect a viscous drag stress along the sphere's sides of magnitude $T_{r\theta} \sim \eta V/a$, where V/a is the magnitude of the shear. These two stresses, acting on the sphere's surface area $\sim a^2$ will produce a net drag force $F \sim \eta V a$. Our goal is to verify this order of magnitude estimate, compute the force more accurately, then balance this force against gravity and thereby infer the speed of fall V of a soot particle.

For a highly accurate analysis of the flow, we could write the full solution as a perturbation expansion in powers of the Reynolds' number R. We shall compute only the leading term in this expansion; the next term, which corrects for inertial effects, will be smaller than our solution by a factor $O(R)$.

Our solution to this classic problem is based on some general principles that ought to be familiar from other areas of physics. First we observe that the only way to form a scalar quantity such as the pressure variation which is linear in the velocity vector is by using the scalar product $\mathbf{V}\cdot\mathbf{x}$; there are no other vectors in the problem. Likewise, we anticipate that the velocity field will be a linear combination of terms involving $\mathbf{V}$ and $(\mathbf{V}\cdot\mathbf{x})\mathbf{x}$. Now take the divergence of Eq. (13.18), and conclude that the pressure must satisfy Laplace's equation, $\nabla^2 P = 0$. The solution should be axisymmetric about $\mathbf{V}$, and we know that axisymmetric solutions to Laplace's equation that decay as $r\to\infty$ can be expanded as a sum over Legendre polynomials, $\sum_{\ell=0}^{\infty} P_\ell(\mu)/r^{\ell+1}$, where μ is the cosine of the angle θ between $\mathbf{V}$ and $\mathbf{x}$, and r is $|\mathbf{x}|$. Clearly we only need the dipolar, $\ell=1$ term [for which $P_1(\mu)=\mu=\mathbf{V}\cdot\mathbf{x}/(Vr)$], so we can write

$$P = P_\infty + \frac{k\eta(\mathbf{V}\cdot\mathbf{x})a}{r^3} + \ldots \; . \tag{13.20}$$

Here k is a numerical constant which we must determine, we have introduced a factor η to make the k dimensionless, and P_∞ is the pressure far from the sphere.

We can use similar arguments to compute the vorticity by observing that it is an axial vector and the only option we have is

$$\boldsymbol{\omega} = \frac{\mathbf{V}\times\mathbf{x}}{a^2} f(r/a) \; . \tag{13.21}$$

The factor a appears in the denominator to make the unknown function $f(\xi)$ dimensionless. We determine this unkown function by using $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, rewriting Eq. (13.18) in the form

$$\boldsymbol{\nabla} P = -\eta\boldsymbol{\nabla}\times\boldsymbol{\omega} \; , \tag{13.22}$$

and substituting Eq. (13.21) to obtain $f(\xi)=k\xi^{-3}$, whence

$$\boldsymbol{\omega} = \frac{k(\mathbf{V}\times\mathbf{x})a}{r^3} \tag{13.23}$$

Now, Eq. (13.23) for the vorticity looks familiar. It has the form of the Biot-Savart law for the magnetic field from a current element. We can therefore write down immediately a formula for its associated "vector potential", which in this case is the velocity

$$\mathbf{v}(\mathbf{x}) = \frac{ka\mathbf{V}}{r} + \boldsymbol{\nabla}\psi \; . \tag{13.24}$$

The addition of the $\boldsymbol{\nabla}\psi$ term corresponds to the familiar gauge freedom in defining the vector potential. However in the case of fluid dynamics, where the velocity is a directly observable quantity, the choice of the scalar ψ is fixed by the boundary conditions instead of being free. By an argument analogous to those that led to Eqs. (13.20) and (13.21), the scalar ψ must be expressible in terms of a second dimensionless function $g(\xi)$ as

$$\psi = g(r/a)\mathbf{V}\cdot\mathbf{x} \; . \tag{13.25}$$

Next we recall that the flow is incompressible, i.e. $\boldsymbol{\nabla}\cdot\mathbf{v}=0$. Substituting Eq. (13.25) into Eq. (13.24) and setting the divergence expressed in spherical polar coordinates to zero, we obtain an ordinary differential equation for g

$$\frac{d^2g}{d\xi^2} + \frac{4}{\xi}\frac{dg}{d\xi} - \frac{k}{\xi^3} = 0. \tag{13.26}$$

This has the solution,

$$g(\xi) = A - \frac{k}{2\xi} + \frac{B}{\xi^3} , \tag{13.27}$$

where A and B are integration constants. As $\mathbf{v} \to \mathbf{V}$ far from the sphere, the constant $A = 1$. The constants B, k can be found by imposing the boundary condition $\mathbf{v} = 0$ for $r = a$. We thereby obtain $B = -1/4, k = -3/2$ and after substituting into Eq. (13.24) we obtain for the velocity field

$$\mathbf{v} = \left[1 - \frac{3}{4}\left(\frac{a}{r}\right) - \frac{1}{4}\left(\frac{a}{r}\right)^3\right]\mathbf{V} - \frac{3}{4}\left(\frac{a}{r}\right)^3\left[1 - \left(\frac{a}{r}\right)^2\right]\frac{(\mathbf{V}\cdot\mathbf{x})\mathbf{x}}{a^2} . \tag{13.28}$$

The associated pressure and vorticity are given by

$$\begin{aligned} P &= P_\infty - \frac{3\eta a(\mathbf{V}\cdot\mathbf{x})}{2r^3} , \\ \boldsymbol{\omega} &= \frac{3a(\mathbf{x}\times\mathbf{V})}{2r^3}. \end{aligned} \tag{13.29}$$

The pressure is seen to be largest on the upstream hemisphere as expected. However, the vorticity, which points in the direction of $\mathbf{e}_\phi$, is seen to be symmetric between the front and the back of the sphere. This is because, under our low Reynolds' number approximation, we are neglecting the convection of vorticity by the velocity field and only retaining the diffusive term. Vorticity is generated on the front surface of the sphere at a rate $\nu\omega_{\phi,r} = 3\nu V \sin\theta/a^2$ per unit length and diffuses into the surrounding flow; then, after the flow passes the sphere's equator, the vorticity diffuses back inward and is absorbed onto the sphere's back face, according to (13.28). An analysis that includes higher orders in the Reynold's number would show that not all of the vorticity is reabsorbed; some is left in the fluid, downstream from the sphere.

We have been able to obtain a simple solution for low Reynolds' number flow past a sphere. Although closed form solutions like this are not so common, the methods that were used to derive it are of widespread applicability. Let us recall them. First, we approximated the equation of motion by omitting the sub-dominant inertial term and invoked a symmetry argument. We used our knowledge of elementary electrostatics to write the pressure in the form (13.20). We then invoked a second symmetry argument to solve for the vorticity and drew upon another analogy with electromagnetic theory to derive a differential equation for the velocity field which was solved subject to the no-slip boundary condition on the surface of the sphere.

Having obtained a solution for the velocity field and pressure, it is instructive to re-examine our approximations. The first point to notice is that the velocity perturbation, given by Eq. (13.28) dies off slowly, inversely proportional to distance r from the sphere. This implies that the region through which the sphere is moving must be much larger than the sphere; otherwise the boundary conditions at $r \to \infty$ have to be modified. This is not a concern for a soot particle in the atmosphere. A second, related point is that, if we compare the sizes of the inertial term (which we neglected) and the pressure gradient (which we kept) in the full Navier-Stokes equation, we find

$$|(\mathbf{v}\cdot\nabla)\mathbf{v}| \sim \frac{V^2 a}{r^2} , \quad \left|\frac{\boldsymbol{\nabla} P}{\rho}\right| \sim \frac{\eta a V}{\rho r^3} . \tag{13.30}$$

Evidently, the inertial term becomes significant at a distance $r \gtrsim \eta/\rho V \sim a/R$ from the sphere. In fact, in order to improve upon our zero order solution we must perform a second expansion at large r including inertial effects and then match it asymptotically to a near zone expansion. This technique of *matched asymptotic expansions* is a very powerful and general way of finding approximate solutions valid over a wide range of length scales where the dominant physics changes from one scale to the next.

Let us return to the problem which motivated this calculation, estimating the drag force on the sphere. This can be computed by integrating the stress tensor $\mathbf{T} = P\mathbf{g} - 2\eta\boldsymbol{\sigma}$ over the surface of the sphere. If we introduce a local orthonormal basis, $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$, we readily see that the only non-zero viscous contribution to the surface stress tensor is $T_{r\theta} = T_{\theta r} = \eta\partial v_\theta/\partial r$. The net resistive force along the direction of the velocity is then given by

$$\begin{aligned} \mathbf{F} &= \int_{r=a} \frac{d\boldsymbol{\Sigma}\cdot\mathbf{T}\cdot\mathbf{V}}{V} \\ &= -\int_0^{2\pi} 2\pi a^2 \sin\theta d\theta \left[-P_\infty \cos\theta + \frac{3\eta V\cos^2\theta}{2a} + \frac{3\eta V \sin^2\theta}{2a} \right] . \end{aligned} \tag{13.31}$$

The integrals are elementary; they give for the total resistive force

$$\mathbf{F} = -6\pi\eta a\mathbf{V} . \tag{13.32}$$

This is Stokes Law for the drag force in low Reynolds number flow. Two-thirds of the force comes from the viscous stress and one third from the pressure.

13.3.2 Sedimentation Rate

In order to compute the sedimentation rate of soot particles relative to the air we must restore gravity to our analysis. We can do so by regarding $\boldsymbol{\nabla} P$ throughout the above analysis as that portion of the gradient of P which is not balancing the gravitational force on the fluid. Having done so, we obtain precisely the same flow, in the (spherical) soot particle's reference frame as we obtained in the absence of gravity, and precisely the same viscous force on the particle's surface. This viscous force $6\pi\eta a V$ must balance the gravitational force on the particle, $4\pi\rho_s a^3 g_e/3$, where $\rho_s \sim 2000$kg m^{-3} is the density of soot. Hence,

$$V = \frac{2\rho_s a^2 g_e}{9\eta} \tag{13.33}$$

Now, the kinematic viscosity of air at sea level is, according to Table 12.2, $\nu \sim 10^{-5}$ m^{2}s^{-1} and the density is $\rho_a \sim 1$kg m^{-3}, so the coefficient of viscosity is $\eta = \rho_a\nu \sim 10^{-5}$kg m^{-1}s^{-1}. This viscosity is proportional to the square root of temperature and independent of the density [cf. Eq. (12.68)]; however, the temperature does not vary by more than about 25 per cent through the atmosphere, so for order of magnitude calculations we can use its value at sea level. Substituting the above values into (13.33), we obtain an equilibrium velocity of

$$V \sim 0.5(a/1\mu)^2 \text{mm s}^{-1} \tag{13.34}$$

We should also, for self-consistency, estimate the Reynolds' number; it is

$$R \sim \frac{2aV}{\eta} \sim 10^{-4}\left(\frac{a}{1\mu}\right)^3 \tag{13.35}$$

Our analysis is therefore only likely to be adequate for particles of radius $a \lesssim 10\mu$.

The drift velocity (13.34) is much smaller than wind speeds in the upper atmosphere $v_{\rm wind} \sim 30{\rm m\ s}^{-1}$. However, as the stratosphere is reasonably stratified, the net vertical motion due to the winds is quite small and so we can estimate the settling time by dividing the height $\sim$ 30km by the speed (13.34) to obtain

$$t_{\rm settle} \sim 6 \times 10^7 \left(\frac{a}{1\mu}\right)^{-2} {\rm s} \sim 2\left(\frac{a}{1\mu}\right)^{-2} \text{months}. \tag{13.36}$$

Of course rain and convection driven by the heating might cause a much more rapid precipitation. Nevertheless, this simple estimate is sufficient to conclude that if sub-micron particles were raised into the stratosphere there is considerable danger that they could remain for long enough to cause a severe, long-lasting change of weather. Scientific curiosity stops well short of wanting to perform the experiment.

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EXERCISES

Exercise 13.4 *Problem: Oseen's Paradox*
Consider low Reynolds' number flow past an infinite cylinder and try to repeat the analysis we used for a sphere to obtain an order of magnitude estimate for the drag force per unit length. Do you see any difficulty, especially concerning the analog of Eq. (13.30)?

Exercise 13.5 *Problem: Viscosity of the Earth's mantle.*
Episodic glaciation subjects the earth's crust to loading and unloading by ice. The last major ice age was 10,000 years ago and the subsequent unloading produces a non-tidal contribution to the acceleration of the earth's rotation rate of order

$$\frac{|\mathbf{\Omega}|}{|\dot{\mathbf{\Omega}}|} \simeq 6 \times 10^{11}{\rm yr}\ ,$$

detectable from observing the positions of distant stars. Corresponding changes in the earth's oblateness produce a decrease in the rate of nodal line regression of the geodetic satellite LAGEOS. Estimate the speed with which the polar regions (treated as spherical caps of radius $\sim$ 1000km) are rebounding now. Do you think the speed was much greater in the past?

Geological evidence suggests that a particular glaciated region of radius about 1000km sank in $\sim$ 3000yr during the last ice age. By treating this as a low Reynolds' number viscous flow, make an estimate of the coefficient of viscosity for the mantle.

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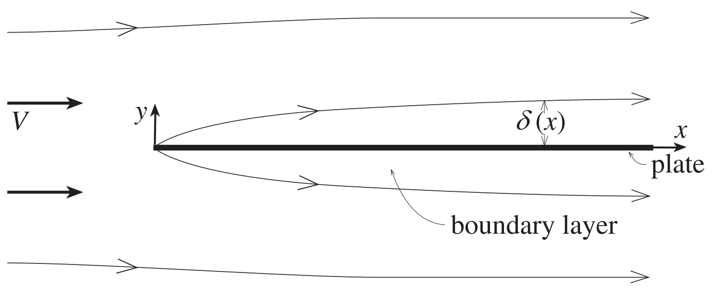

Fig. 13.8: Laminar boundary layer formed by a long, thin plate in a flow with asymptotic speed V. The length ℓ of the plate must give a Reynold's number $R_\ell \equiv V\ell/\nu$ in the range $10 \lesssim R_\ell \lesssim 10^6$; if R_ℓ is much less than 10, the plate will be in or near the regime of low Reynolds number flow (Sec. 13.3 above), and the boundary layer will be so thick everywhere that our analysis will fail. If R_ℓ is much larger than 10^6, then at sufficiently great distances x down the plate ($R_x = Vx/\nu \gtrsim 10^6$), the boundary layer will become unstable and its simple laminar structure will be destroyed; see Chap. 14.

13.4 High Reynolds Number Flow – Laminar Boundary Layers

As we have described, flow near a solid surface creates vorticity and, consequently, the velocity field near the surface cannot be derived from a scalar potential, $\mathbf{v} = \boldsymbol{\nabla}\psi$. However, if the Reynold's number is high, then the vorticity may be localized within a thin boundary layer adjacent to the surface, as in Fig. 13.4 above; and the flow may be very nearly of potential form $\mathbf{v} = \boldsymbol{\nabla}\psi$ outside that boundary layer. In this section we shall use the equations of hydrodynamics to model the flow in the simplest example of such a boundary layer: that formed when a long, thin plate is placed in a steady, uniform flow $V\mathbf{e}_x$ with its surface parallel to the flow (Fig. 13.8).

If the plate is not too long (see caption of Fig. 13.8), then the flow will be *laminar*, i.e. steady and two-dimensional—a function only of the distances x along the plate's length and y perpendicular to the plate (both being measured from an origin at the plate's front). We assume the flow to be very subsonic, so it can be regarded as incompressible. As the viscous stress decelerates the fluid close to the plate, it must therefore be deflected away from the plate to avoid accumulating, thereby producing a small y component of velocity along with the larger x component. As the velocity is uniform well away from the plate, the pressure is constant outside the boundary layer. We use this to motivate the approximation that P is also constant within the boundary layer. After solving for the flow, we will check the self-consistency of this *ansatz*. With P =constant and the flow stationary, only the inertial and viscous terms remain in the Navier-Stokes equation (13.2):

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \simeq \nu\nabla^2\mathbf{v} \ . \tag{13.37}$$

This equation must be solved in conjunction with $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and the boundary conditions $\mathbf{v} \to V\mathbf{e}_x$ as $y \to \infty$ and $\mathbf{v} \to 0$ as $y \to 0$.

The fluid first encounters the no-slip boundary condition at the front of the plate, $x = y = 0$. The flow there abruptly decelerates to vanishing velocity, creating a sharp velocity gradient that contains a sharp spike of vorticity. This is the birth of the boundary layer. As the fluid flows on down the plate, from $x = 0$ to larger x, the vorticity gradually diffuses outward from the wall into the flow, thickening the boundary layer.

Let us compute, in order of magnitude, the boundary layer's thickness $\delta(x)$ as a function of distance x down the plate. Incompressibility, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, implies that $v_y \sim v_x \delta/x$. Using this to estimate the relative magnitudes of the various terms in the x component of the force balance equation (13.37), we see that the dominant inertial term (left hand side) is $\sim V^2/x$ and the dominant viscous term (right hand side) is $\sim \nu V/\delta^2$. We therefore obtain the estimate $\delta \sim \sqrt{\nu x/V}$. This motivates us to define the function

$$\delta(x) \equiv \left(\frac{\nu x}{V}\right)^{1/2} . \tag{13.38}$$

for use in our quantitative analysis. Our analysis will reveal that the actual thickness of the boundary layer is several times larger than this $\delta(x)$.

Equation (13.38) shows that the boundary layer has a parabolic shape. To keep our analysis manageable, we shall confine ourselves to the region, not too close to the front of the plate, where the layer is thin, $\delta \ll x$, and the velocity is nearly parallel to the plate, $v_y \sim (\delta/x)v_x \ll v_x$.

To proceed further we use a technique of widespread applicability in fluid mechanics: we make a *similarity anszatz*. We suppose that, once the boundary layer has become thin ($\delta \ll x$), the cross sectional shape of the flow is independent of distance x down the plate (it is "similar" at all x). Stated more precisely, we assume that $v_x(x, y)$ (which has magnitude $\sim V$) and $(x/\delta)v_y$ (which also has magnitude $\sim V$) are functions only of the single dimensionless variable

$$\xi = \frac{y}{\delta(x)} = y\sqrt{\frac{V}{\nu x}} . \tag{13.39}$$

Our task, then, is to compute $\mathbf{v}(\xi)$ subject to the boundary conditions $\mathbf{v} = 0$ at $\xi = 0$, and $\mathbf{v} = V\mathbf{e}_x$ at $\xi \gg 1$. We do so with the aid of a second, very useful calculational device. Recall that any vector field [$\mathbf{v}(\mathbf{x})$ in our case] can be expressed as the sum of the gradient of a scalar potential and the curl of a vector potential, $\mathbf{v} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \mathbf{A}$. If our flow were irrotational $\boldsymbol{\omega} = 0$, we would need only $\boldsymbol{\nabla}\psi$, but it is not; the vorticity in the boundary layer is large. On the other hand, to high accuracy the flow is incompressible, $\theta = \boldsymbol{\nabla} \cdot \mathbf{v} = 0$, which means we need only the vector potential, $\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{A}$; and because the flow is two dimensional (depends only on x and y and has $\mathbf{v}$ pointing only in the x and y directions), the vector potential need only have a z component, $\mathbf{A} = A_z \mathbf{e}_z$. We denote its nonvanishing component by $A_z \equiv \eta(x, y)$ and give it the name *stream function*, since it governs how the laminar flow streams. In terms of the stream function, the relation $\mathbf{v} = \boldsymbol{\nabla} \times A$ takes the simple form

$$v_x = \frac{\partial \eta}{\partial y} , \quad v_y = -\frac{\partial \eta}{\partial x} . \tag{13.40}$$

This, automatically, satisfies $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$.

Since the stream function varies on the lengthscale δ, in order to produce a velocity field with magnitude $\sim V$, it must have magnitude $\sim V\delta$. This motivates us to *guess* that it has the functional form

$$\eta = V\delta(x)f(\xi) , \tag{13.41}$$

where $f(\xi)$ is some dimensionless function of order unity. This will be a good guess if, when inserted into Eq. (13.40), it produces a self-similar flow, i.e. one with v_x and $(x/\delta)v_y$ depending only on ξ. Indeed, inserting Eq. (13.41) into Eq. (13.40), we obtain

$$v_x = Vf' , \quad v_y = \frac{\delta(x)}{2x}V(\xi f' - f) , \tag{13.42}$$

where the prime means $d/d\xi$. This has the desired self-similar form.

By inserting these self-similar v_x and v_y into the x component of the force-balance equation (13.40), we obtain a non-linear third order differential equation for $f(\xi)$

$$\frac{d^3f}{d\xi^3} + \frac{f}{2}\frac{d^2f}{d\xi^2} = 0 \tag{13.43}$$

The fact that this equation involves x and y only in the combination $\xi = y\sqrt{V/\nu x}$ confirms that our self-similar anszatz was a good one. Equation (13.43) must be solved subject to the boundary condition that the velocity vanish at the surface and approach V as $y \to \infty$; i.e. [cf. Eqs. (13.42)] that

$$f(0) = f'(0) = 0 , \quad f'(\infty) = 1 . \tag{13.44}$$

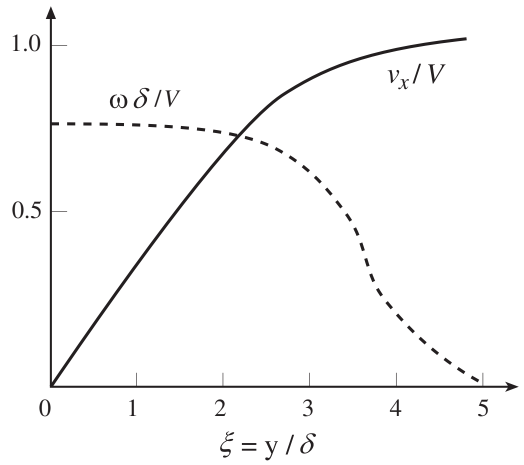

Fig. 13.9: Scaled velocity profile $v_x/V = f'(\xi)$ (solid line) for a laminar boundary layer as a function of scaled perpendicular distance $\xi = y/\delta$. Note that the flow speed is 90 per cent of V at a distance of 3δ from the surface and so δ is a good measure of the thickness of the boundary layer. Also shown is the scaled vorticity profile $\omega\delta/V$. [These figures are hand-sketched; a more accurate numerically generated solution will be provided in future version of this chapter.]

Not surprisingly, Eq. (13.43) does not admit an analytic solution. However, it is simple to compute a numerical solution with the boundary conditions (13.44). The result for $v_x/V = f'(\xi)$ is shown in Fig. 13.9. This solution, the *Blasius profile*, has qualitatively the form we expected: the velocity v_x rises from 0 to V in a smooth manner as one moves outward from the plate, achieving a sizable fraction of V at a distance several times larger than $\delta(x)$.

This Blasius profile is our first example of a common procedure in fluid dynamics: taking account of a natural scaling in the problem to make a self-similar anzatz and thereby transform the partial differential fluid equations into ordinary differential equations. Solutions of this type are known as *similarity solutions*.

The motivation for using similarity solutions is obvious. The non-linear partial differential equations of fluid dynamics are much harder to solve, even numerically, than ordinary differential equations. Elementary similarity solutions are especially appropriate for problems where there is no characteristic length or timescale associated with the relevant physical quantities except those explicitly involving the spatial and temporal coordinates. Large Reynolds' number flow past a large plate has a useful similarity solution, whereas flow with $R_\ell \sim 1$, where the size of the plate is clearly a significant scale in the problem, does not. We shall encounter more examples of similarity solutions in the following chapters.

Now that we have a solution for the flow, we must examine a key approximation that underlies it: constancy of the pressure P. To do this, we begin with the y component of the force-balance equation (13.37) (a component that we never used explicitly in our analysis). The inertial and viscous terms are both $O(V^2\delta/x^2)$, so if we re-instate a term $-\boldsymbol{\nabla} P/\rho \sim -\Delta P/\rho\delta$, it can be no larger than the other two terms. From this we estimate that the pressure difference across the boundary layer is $\Delta P \lesssim \rho V^2\delta^2/x^2$. Using this estimate in the x component of force balance (13.37) (the component on which our analysis was based), we verify that the pressure gradient term is smaller than those we kept by a factor $\lesssim \delta^2/x^2) \ll 1$. For this reason, when the boundary layer is thin we can, indeed, neglect pressure gradients across it in computing its structure from longitudinal force balance.

It is of interest to compute the total drag force exerted on the plate. Letting ℓ be the plate's length and $w \gg \ell$ be its width, and noting that the plate has two sides, the drag force produced by the viscous stress acting on the plate's surface is

$$F = 2w \int_0^\ell \rho\nu \left(\frac{\partial v_x}{\partial y}\right)_{y=0} dx \ . \tag{13.45}$$

Inserting $\partial v_x/\partial y = (V/\delta)f''(0) = V\sqrt{\nu/Vx}f''(0)$ from Eq. (13.42), and performing the integral, we obtain

$$F = \frac{1}{2}\rho v_0^2 \times (2\ell w) \times C_D \ , \tag{13.46}$$

where

$$C_D = 4f''(0)R_\ell^{-1/2} \ . \tag{13.47}$$

Here we have introduced an often-used notation for expressing the drag force of a fluid on a solid body: we have written it as half the incoming fluid's kinetic stress ρV^2, times the surface area of the body $2\ell w$ on which the drag force acts, times a dimensionless drag coefficient C_D, and we have expressed the drag coefficient in terms of the Reynold's number

$$R_\ell = \frac{V\ell}{\nu} \tag{13.48}$$

formed from the body's relevant dimension, ℓ, and the speed and viscosity of the incoming fluid.

From Fig. 13.9, we estimate that $f''(0) \simeq 0.3$ (an accurate numerical value is 0.332), and so $C_D \simeq 1.328R^{-1/2}$. Note that the drag coefficient decreases as the viscosity decreases and the Reynolds' number increases. However, as we shall discuss in the next section, this model breaks down for very large Reynold's numbers $R_\ell \gtrsim 10^6$ because the boundary layer becomes turbulent.

13.4.1 Vorticity

It is illuminating to consider the structure of the boundary layer in terms of the vorticity. Since the flow is two-dimensional with velocity $\mathbf{v} = \boldsymbol{\nabla} \times (\eta \mathbf{e}_z)$, its vorticity is $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\eta \mathbf{e}_z) = -\nabla^2(\eta \mathbf{e}_z)$, which has as its only nonzero component

$$\omega \equiv \omega_z = -\nabla^2 \eta \simeq -\frac{V}{\delta} f''(\xi) \; . \tag{13.49}$$

This vorticity is exhibited in Fig. 13.9.

From Eq. (13.43), we observe that the gradient of vorticity vanishes at the plate. This means that the vorticity is not diffusing out of the plate's surface. Neither is it being convected away from the plate's surface, as the perpendicular velocity vanishes there. However, there is no vorticity upstream so it must have been created somewhere. Its source is the plate's leading edge, where, as we have already remarked, our approximations must break down. The vorticity created impulsively there diffuses away from the plate in the downstream flow. If we transform into a frame moving with an intermediate speed $\sim V/2$, and measure time t since passing the leading edge, the vorticity will diffuse a distance $\sim (\nu t)^{1/2}$ away from the surface after time t just like heat in a thermal conductor. This accounts for the parabolic shape of the boundary layer.

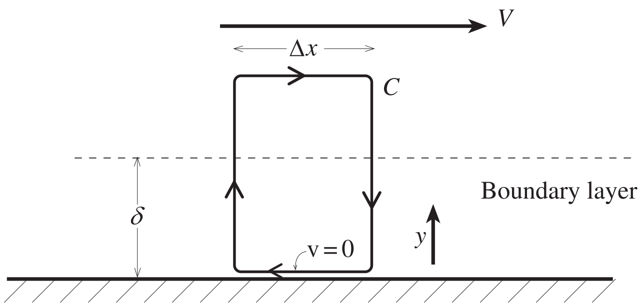

Fig. 13.10: The circulation $\Gamma = \int_C \mathbf{v} \cdot d\mathbf{x}$. The circuit consists of a rectangle with one side of length Δx parallel to the surface and the other extending from the surface perpendicularly through the boundary layer and into the surrounding homogeneous flow. The circulation is therefore given simply by $\Gamma = V\Delta x$.

Equivalently we can consider what happens to the circulation Γ. The circulation associated with the circuit shown in Figure 13.10 is just $V\Delta x$, or simply V per unit length of the plate. Since V is constant along the surface (independent of x), no vorticity is created along solid surface; it must all be be created at the leading edge.

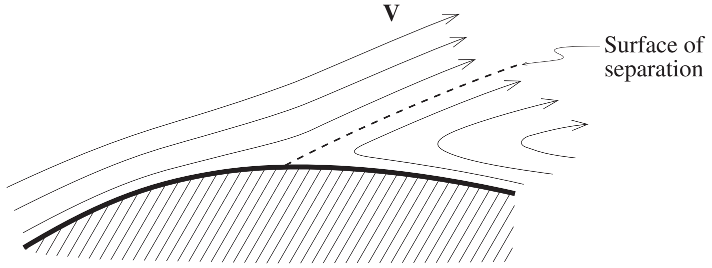

Fig. 13.11: Separation of a boundary layer in the presence of an adverse pressure gradient.

13.4.2 Separation

Next consider flow past a non-planar surface, e.g. an aircraft wing. In this case, there in general will be a pressure gradient along the boundary layer, which cannot be ignored in contrast to the pressure gradient across the boundary layer. If the pressure decreases along the flow, the flow will accelerate and more vorticity will be created at the surface and will diffuse away from the surface. However, if there is an "adverse" pressure gradient and the flow decelerates, then negative vorticity must be created at the wall. For a sufficiently adverse gradient, the negative vorticity gets so strong that it cannot diffuse fast enough into and through the boundary layer to maintain a simple boundary-layer-type flow. Instead, the boundary layer *separates* from the surface, as shown in Fig. 13.11, and a backward-flow is generated beyond the separation point by the negative vorticity. This phenomenon can occur on an aircraft when the wings' angle of attack (i.e. the inclination of the wings to the horizontal) is too great. An unfavorable pressure gradient develops on the upper wing surfaces, the flow separates and the plane stalls. The designers of wings make great effort to prevent this, as we shall discuss briefly in the next chapter.

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EXERCISES

Exercise 13.6 *Example: Potential Flow around a Cylinder*
Consider stationary incompressible flow around a cylinder of radius a with sufficiently large Reynolds' number that viscosity may be ignored except in a thin boundary layer which is assumed to extend all the way around the cylinder. The velocity is assumed to have the uniform value $\mathbf{V}$ at large distances from the cylinder. Show that the velocity field outside the boundary layer may be derived from a scalar potential ψ which satisfies Laplace's equation, $\nabla^2\psi = 0$. Write down suitable boundary conditions for ψ. Now write the velocity potential in the form

$$\psi = \mathbf{V}\cdot\mathbf{x} + f(\mathbf{x})$$

and solve for f. Sketch the streamlines and equipotentials.

Next use Bernoulli's equation to compute the pressure distribution over the surface and the net drag force given by this solution. Does this seem reasonable? Finally, consider the effect of the pressure distribution on the boundary layer. How do you think the real flow will be different from the potential solution? How will the drag change?

Exercise 13.7 *Problem: Reynolds' Numbers*
Estimate the Reynolds' numbers for the following flows. Make sketches of the flow fields pointing out any salient features.

(i) A hang glider in flight.

(ii) Planckton in the ocean.

(iii) A physicist waving his hands.

Exercise 13.8 *Problem: Fluid Dynamical Scaling*
An auto manufacturer wishes to reduce the drag force on a new model by changing its design. She does this by building a one eighth scale model and putting it into a wind tunnel. How fast must the air travel in the wind tunnel to simulate the flow at 60 mph on the road?

Exercise 13.9 *Example: Stationary Laminar Flow down a Long Pipe*
Fluid flows down a long cylindrical pipe with length b much larger than radius a, from a reservoir maintained at pressure P_0, to a free end where the pressure is negligible. In this problem, we try to understand the velocity profile for a given "discharge" (i.e. mass flow per unit time) $\dot{M}$ of fluid along the pipe. We assume that the Reynolds' number is small enough for the flow to be treated as laminar.

(i) Close to the entrance of the pipe, the boundary layer will be very thin and the velocity will be nearly independent of radius. What will be the fluid velocity there in terms of its density and $\dot{M}$? How far must the fluid travel along the pipe before the vorticity diffuses into the center of the flow and the boundary layer becomes as thick as the radius? An order of magnitude calculation is adequate and you may assume that the pipe is much longer than your estimate.

(ii) Use the Poiseuille formula derived in Sec. 12.4 to derive $\dot{M}$ and the velocity profile. Hence sketch how the velocity profile changes along the pipe.

(iii) Outline a procedure for computing the discharge in a long pipe of arbitrary cross section.

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13.5 Kelvin-Helmholtz Instability — Excitation of Ocean Waves by Wind

A particularly simple type of vorticity distribution is one where the vorticity is confined to a thin, plane interface between two immiscible fluids. In other words, we suppose that one fluid is in uniform motion relative to the other. This type of flow arises quite frequently; for

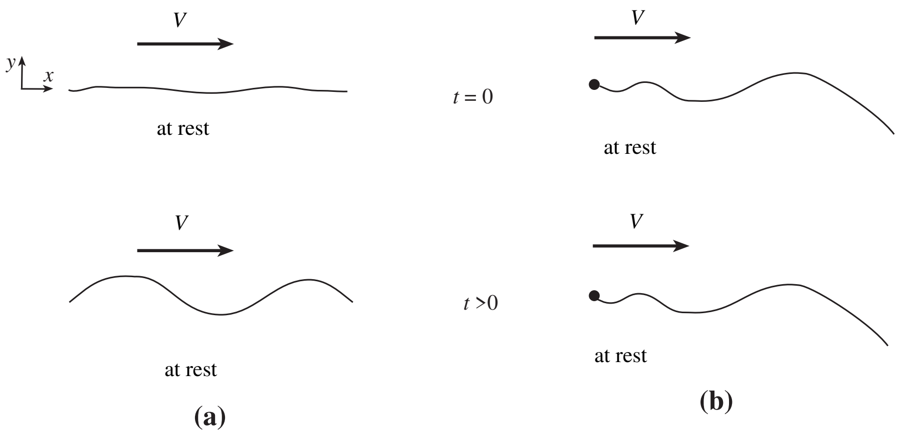

Fig. 13.12: Kelvin-Helmholtz Instability. a) Temporarily growing mode. b) Spatially growing mode.

example, when the wind blows over the ocean or when smoke from a smokestack discharges into the atmosphere. Following the discussion of the previous section, we expect that a boundary layer will form at the interface and the vorticity will slowly diffuse away from it. However, quite often the boundary layer is much thinner than the length scale associated with the flow and we can therefore treat it as a plane discontinuity in the velocity field. It turns out that this type of flow is unstable, as we shall now demonstrate. This instability is known as the *Kelvin-Helmholtz instability*. It provides another good illustration of the behavior of vorticity as well as an introduction to the techniques that are commonly used to analyze fluid instabilities.

We restrict attention to the simplest version of the instability. Consider an equilibrium with a fluid of density ρ_+ moving horizontally with speed V above a second fluid, which is at rest, with density ρ_-. Let x be a coordinate measured along the interface and let y be measured perpendicular to the surface.

The equilibrium contains a sheet of vorticity lying in the plane $y = 0$, across which the velocity changes discontinuously. Now this discontinuity ought to be treated as a boundary layer, with a thickness determined by the viscosity. However, in this problem, we shall analyze disturbances with length scales much greater than the thickness of the boundary layer. A corollary of this assumption is that we can ignore viscous stresses in the body of the flow. We also specialise to very subsonic speeds, for which the flow can be treated as incompressible, and we ignore the effects of surface tension and gravity.

A full description of this flow requires solving the full equations of fluid dynamics, which are quite non-linear and must be attacked numerically. However, we can make progress analytically on an important sub-problem. This involves answering the question of whether or not the simple vortex sheet with uniform flow above and below it is stable to small perturbations. This allows us to linearize in the amplitude of these perturbations. If we discover that these perturbations grow with time then we will eventually be forced to consider the full non-linear problem. However, uncovering the conditions for instability is a very important first step in our analysis of the flow.

Let us imagine a small perturbation to the location of the interface by an amount $\xi(x)$

[Fig. 13.12(a)]. We denote the associated perturbations to the pressure and velocity by $\delta P, \delta\mathbf{v}$. That is to say we write

$$P(\mathbf{x}, t) = P_0 + \delta P(\mathbf{x}, t) \ , \quad \mathbf{v} = VH(y)\mathbf{e}_x + \delta v(\mathbf{x}, t) \ , \tag{13.50}$$

where P_0 is the constant pressure in the equilibrium flow about which we are perturbing, V is the constant speed of the flow above the interface, and $H(y)$ is the Heaviside step function (1 for $y > 0$ and 0 for $y < 0$). We substitute these $P(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ into the governing equations: the incompressibility relation

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \ , \tag{13.51}$$

and the viscosity-free Navier-Stokes equation, i.e., the Euler equation,

$$\frac{d\mathbf{v}}{dt} = \frac{-\boldsymbol{\nabla} P}{\rho} \ . \tag{13.52}$$

We then subtract off the equations satisfied by the equilibrium quantities to obtain, for the perturbed variables,

$$\boldsymbol{\nabla} \cdot \delta\mathbf{v} = 0, \tag{13.53}$$

$$\frac{d\delta\mathbf{v}}{dt} = -\frac{\boldsymbol{\nabla}\delta P}{\rho} \ . \tag{13.54}$$

Combining these two equations we find, once again, that the pressure satifies Laplace's equation

$$\nabla^2 \delta P = 0 \tag{13.55}$$

(cf. Sec. 13.3).

We now follow the procedure that we used in Sec. 11.5 when treating Rayleigh waves on the surface of an elastic medium: we seek a wave mode in which the perturbed quantities vary $\propto \exp[i(kx - \omega t)]f(y)$ with $f(y)$ dying out away from the interface. From Laplace's equation (13.55), we infer an exponential falloff with $|y|$:

$$\delta P = \delta P_0 e^{-k|y| + i(kx - \omega t)} \ , \tag{13.56}$$

where δP_0 is a constant.

Our next step is to substitute this δP into the perturbed Euler equation (13.54) to obtain

$$\begin{aligned} \delta v_y &= \frac{ik\delta P}{(\omega - kV)\rho_+} \quad \text{at } y > 0 \\ &= \frac{-ik\delta P}{\omega\rho_-} \quad \text{at } y < 0 \ . \end{aligned} \tag{13.57}$$

We must impose two boundary conditions at the interface between the fluids: continuity of the vertical displacement ξ of the interface (the tangential displacement need not be continuous since we are examining scales large compared to the boundary layer), and continuity of the pressure P across the interface. [See Eq. (11.44) and associated discussion for the analogous boundary conditions at a discontinuity in an elastic medium.] Now, the vertical interface displacement ξ is related to the velocity perturbation by $d\xi/dt = \delta v_y(y=0)$, which implies by Eq. (13.57) that

$$\begin{aligned}\xi &= \frac{i\delta v_y}{(\omega - kV)} \quad \text{at } y = 0_+ \quad \text{(immediately above the interface)} \\ &= \frac{i\delta v_y}{\omega} \quad \text{at } y = 0_- \quad \text{(immediately below the interface).}\end{aligned} \tag{13.58}$$

Then, by virtue of eqs. (13.56), (13.58), and (13.57), the continuity of pressure and vertical displacement at $y = 0$ imply that

$$\rho_+(\omega - kV)^2 + \rho_-\omega^2 = 0 \ , \tag{13.59}$$

where ρ_+ and ρ_- are the densities of the fluid above and below the interface. Solving for frequency ω as a function of horizontal wave number k, we obtain the following *dispersion relation for linear Kelvin-Helmholtz modes:*

$$\omega = kV\left(\frac{\rho_+ \pm i(\rho_+\rho_-)^{1/2}}{\rho_+ + \rho_-}\right) \ . \tag{13.60}$$

13.5.1 Temporal growth

Suppose that we create some small, localised disturbance at time $t = 0$. We can Fourier analyse the disturbance in space and, as we have linearized the problem, can consider the temporal evolution of each Fourier component. Now, what we ought to do is to solve the initial value problem carefully taking account of the initial conditions. However, when there are growing modes, we can usually infer the long-term behavior by ignoring the transients and just consider the growing solutions. In our case, we infer from Eqs. (13.56), (13.57), (13.58) and the dispersion relation (13.60) that a mode with spatial frequency k must grow as

$$\delta P,\ \xi_y \propto \exp\left[\left(\frac{kV(\rho_+\rho_-)^{1/2}}{(\rho_+ + \rho_-)}\right)t + ik\left(x - V\frac{\rho_+}{\rho_+ + \rho_-}t\right)\right] \tag{13.61}$$

Thus, this mode grows exponentially with time [Fig. 13.12(a)]. Note that the mode is non-dispersive, and if the two densities are equal, it e-folds in a few periods. This means that the fastest modes to grow are those which have the shortest periods and hence the shortest wavelengths. (However, the wavelength must not approach the thickness of the boundary layer and thereby compromise our assumption that the effects of viscosity are negligible.)

We can understand this growing mode somewhat better by transforming into the center of momentum frame, which moves with speed $\rho_+V/(\rho_+ + \rho_-)$ relative to the frame in which the lower fluid is at rest. In this (primed) frame, the velocity of the upper fluid is $V' = \rho_-V/(\rho_+ + \rho_-)$ and the perturbations evolve as

$$\delta P,\ \xi \propto \exp[kV'(\rho_+/\rho_-)^{1/2}t]\cos(ikx') \tag{13.62}$$

In this frame the wave is purely growing, whereas in our original frame it oscillated with time as it grew.

13.5.2 Spatial Growth

An alternative type of mode is one in which a small perturbation is excited temporally at some point where the shear layer begins [Fig. 13.12(b)]. In this case we regard the frequency ω as real and look for the mode with positive imaginary k corresponding to spatial growth. Using Eq. (13.60), we obtain

$$k = \frac{\omega}{V}\left[1 \pm i\left(\frac{\rho_+}{\rho_-}\right)^{1/2}\right] \tag{13.63}$$

The mode therefore grows exponentially with distance from the point of excitation.

13.5.3 Relationship between temporal and spatial growth; Excitation of ocean waves by wind

An illustrative application is to a wind blowing steadily over the ocean. In this case, $\rho_+/\rho_- \sim 10^{-3}$ and for temporal growth of the waves, the imaginary part of the frequency satisfies

$$\omega_i = \sqrt{\rho_+/\rho_-} \sim 0.03\omega \tag{13.64}$$

for the growing mode [cf. Eq. (13.61)]. For spatial growth, we have [Eq. (13.63)]

$$k_i = \frac{\sqrt{\rho_+/\rho_-}}{V} \sim 0.03\frac{\omega}{V}\ . \tag{13.65}$$

In other words it takes about 30 periods either spatially or temporally for the amplitude of the ocean waves to increase by a factor $\sim e$.

This result is a simple application of a somewhat more general result. Consider any type of wave for which the dispersion relation $\omega(k)$ is an analytic function. Suppose that, among the solutions to this dispersion relation, there is a wave with real k that grows slowly with time; its growth rate can then be expressed as $\omega_i(k_r)$, where the subscripts denote "imaginary" and "real". There will be another solution, describing a wave with real frequency ω and a slow spatial growth of amplitude, with e-folding rate given by $k_i(\omega_r)$. The growth rates for these two solutions are related by the following equation, which follows from the Cauchy-Riemann equations for any complex analytic function:

$$\frac{\omega_i(k_r)}{k_i(\omega_r)} \simeq \frac{\partial\omega_i}{\partial k_i} = \frac{\partial\omega_r}{\partial k_r} = v_g\ . \tag{13.66}$$

This ratio is recognised as the group velocity. In the present application this is the wind velocity, $V \sim 10$ m s^{-1}.

The computational procedure that we have in studying the Kelvin-Helmholtz instability —seeking eigenfunction solutions to the equations of small perturbations—, although in widespread use, must be applied with care. For example, if the full temporal evolution of some initial configuration is needed, then there is no escape from solving the initial value problem, and the resulting transients may exhibit important behaviors that are absent in the individual growing modes.

13.5.4 Physical Interpretation

We have performed a normal mode analysis of a particular flow and discovered that there are unstable modes. However much we calculate the form of the growing modes, though, we cannot be said to understand the instability until we can explain it physically. In the case of the Kelvin-Helmholtz instability, this is a simple task.

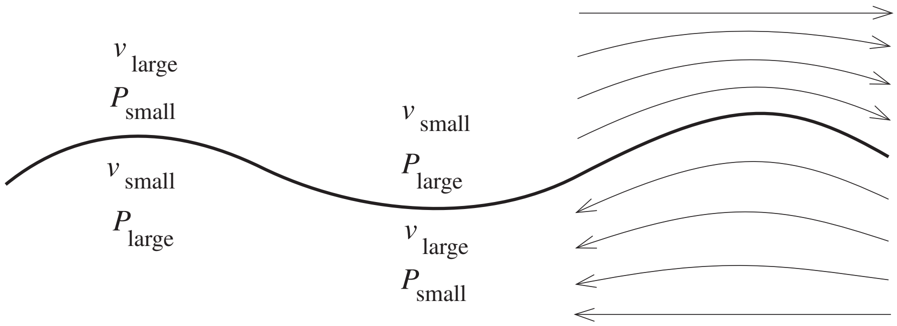

Fig. 13.13: Physical explanation for the Kelvin-Helmholtz instability.

The flow pattern is shown schematically in Fig. 13.13). The air will have to move faster when passing over a crest in the water wave as the cross sectional area of a flow tube diminishes and the flux of air must be conserved. By Bernoulli's theorem, the pressure will be lower than ambient at this point and so the water in the crest will rise even higher. Conversely in the trough of the wave, the air will travel slower and the pressure will increase. The pressure differential will push the trough downward, making it grow.

Equivalently, we can regard the boundary layer as a plane containing parallel vortex lines which interact one with another, much like magnetic field lines exert pressure on one on another. When the vortex lines all lie strictly in a plane, they are in equilibrium as the repulsive force exerted by one on its neighbor is balanced by an opposite force exerted by the opposite neighbor. However, when this equilibrium is disturbed, the forces become unbalanced and the vortex sheet effectively buckles.

More generally, whenever there is a large amount of relative kinetic energy available in a large Reynolds' number flow, there exists the possibility of unstable modes that can tap this energy and convert it into a spectrum of growing modes, which can then interact non-linearly to create fluid turbulence, which ultimately is dissipated as heat. However, the fact that free kinetic energy is available does not necessarily imply that the flow is unstable; sometimes it is stable. Instability must be demonstrated, often a very difficult task.

13.5.5 Rayleigh and Richardson Stability Criteria.

Let us conclude this chapter by examining the stability of two interesting types of shear flow. The analyses we shall give are more directly physical than our routine solution for small amplitude normal modes. Firstly, we consider *Couette* flow, i.e. the azimuthal flow of an incompressible, effectively inviscid fluid between two coaxial cylinders.

Let us consider the stability of the flow to purely axisymmetric perturbations by interchanging two rings. As there are no azimuthal forces, the interchange will occur at constant angular momentum. Now suppose that the ring that moves outward in radius r has lower specific angular momentum j than its surroundings; then it will have less centrifugal force per unit mass j^2/r^3 than its surroundings and will thus experience a restoring force that drives it back to its original position. Conversely, if the angular momentum decreases outward, the ring will continue to expand. We conclude on this basis that *Couette and similar flows are unstable when the angular momentum decreases outward).* This is known as the *Rayleigh criterion.* We shall return to Couette flow in the following chapter.

Compact stellar objects (black holes, neutron stars and white dwarfs) are sometimes surrounding by orbiting *accretion disks* of gas. The gas in these disks orbits roughly with the angular velocity dictated by Kepler's laws. Therefore the specific angular momentum of the gas increases radially outward, proportional to the square root of the orbital radius. Consequently accretion disks are stable to this type of instability.

We ignored the influence of gravity on the Kelvin-Helmholtz instability. This is not always valid; gravity can sometimes be quite important. Consider, as an example, the earth's stratosphere. Its density decreases upward much faster than if the stratosphere were isentropic. This means that when a fluid element moves upward (adiabatically), its pressure-induced density changes are small compared to those of its ambient surroundings, which in turn means that the air's motions can be regarded as incompressible. This incompressiblity, together with the upward decrease of density, guarantees that the stratosphere is stabily stratified. It may be possible, however, for the stratosphere to tap the relative kinetic energy in its horizontal winds so as to mix the air vertically. Consider two thin streams separated vertically by a distance δh and moving horizontally with relative speed δV. In the center of velocity frame, the streams each have speed $\delta V/2$ and they differ in density by $\delta\rho = |\rho'\delta h|$. To interchange these streams requires doing a work per unit mass $\delta W = g\delta\rho/2\rho$ (where the factor 2 comes from the fact that there are two unit masses involved in the interchange, one going up and the other coming down). This work can be supplied by the streams' kinetic energy, if the available kinetic energy per unit mass $\delta E_k = (\delta V/2)^2/2$ exceeds the required work. *A necessary condition for instability* is then that

$$\delta E_k = (\delta V)^2/8 > \delta W = g|\rho'/2\rho|\delta h^2 \tag{13.67}$$

or

$$Ri = \frac{|\rho'|g}{\rho V'^2} < \frac{1}{4} \tag{13.68}$$

where $V' = dV/dz$ is the velocity gradient. This is known as the Richardson criterion and Ri is the Richardson number. Under a wide variety of circumstances this criterion turns out to be not only necessary but also sufficient for instability.

The density scale height in the atmosphere is $\rho/|\rho'| \sim 10$ km. Therefore the maximum velocity gradient allowed by the Richardson criterion is

$$V' \lesssim 60 \frac{\mathrm{m\ s^{-1}}}{\mathrm{km}}\ . \tag{13.69}$$

Larger velocity gradients are rapidly disrupted by instabilities. This instability is responsible for much of the clear air turbulence encountered by airplanes and it is to a discussion of turbulence that we shall turn in the next chapter.

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