Connection Coefficients for Spherical Polar Coordinates, Computed Via GRTensorM in Mathematica

Example by Kip, for Ph136c, 25 April 2000

■ Import GRTensor into Mathematica, and Set Directory

```
<< grii/grt.m
```

```
 GRTensorM Version 1.2 for Mathematica 3.x

 July 20, 1998

 Copyright 1996-98 by P. Musgrave, D. Pollney and K. Lake

 Update by Mustapha Ishak B.

 Use qload[] or grload[] to load a metric.

 ?grtensor for help.

 Metric directory = "/usr/local/mathematica/AddOns/ExtraPackages/grii/metrics/"
```

```
SetOptions[grii, MetricPath → "/home/kip/courses/ph136/1999.00/chapters/chap23/"]
```

```
{DisplayLimit → 5000, TimeStamp → True, DefaultSimplification → Together,
 Verbose → True, MetricPath → /home/kip/courses/ph136/1999.00/chapters/chap23/}
```

■ Define Coordinates, Basis Vectors, and Metric

I shall use as my example, the connection coefficients for spherical polar coordinates in 3−dimensionalEuclidean space. The coordinates are denoted r, th (for theta) and ph (for phi). The input of coordinates, basis vectors and metric is initiated via the command

```
makeg[sphere3d]
```

Here "sphere3d" is the name that I am giving to this metric and basis. This command brings up a series of prompt windows into which I type the appropriate information:

■ First Prompt Window: Choice of what to enter

>Do you wish to enter a
> 1) metric: g[dn,dn]
> 2) line element [ds],
> 3)non−holonomicbasis e1..e4, or
> 4) null tetrad [l,n,m,mbar]?

3

[Note: The phrase "non−holonomicbasis" means "non−coordinatebasis". I plan to use the orthonormal basis associated with spherical coordinates, so I choose this.]

■ Second Prompt Window: Names of coordinates

>Enter coordinates as a LIST (e.g. {r,thetha,phi,t})

{r,th,ph}

■ Third Prompt Window: Choice of components of basis vectors to enter

>Would you like to enter
> 1) covariant components,
> 2) contravariant components, or
> 3) both.

2

[Note: If we write our orthonormal basis vectors as $\boldsymbol{e}_{[j]} = \boldsymbol{e}^{k}_{[j]}\,\partial/\partial x^{k}$, then $\boldsymbol{e}^{k}_{[j]}$ are the "contravariant components" (of the non−coordinate basis vector $e_{(j)}$) to which the above query refers. I have chosen to enter these contravariant components.]

■ Fourth , Fifth and Sixth Prompt Windows: Values of components of basis vectors

>Enter the contravariant components of basis vector e[1] as a List (e.g. {1,0,0,0})

{1,0,0}

[Note: This says that $\boldsymbol{e}_{[r]}$ is equal to ∂/∂r]

>Enter the contravariant components of basis vector e[2] as a List (e.g.{1,0,0,0})

{0,1/r,0}

[Note: This says that $\boldsymbol{e}_{[\theta]}$ is (1/r)$\partial/\partial\theta$]

>Enter the contravariant components of basis vector e[2] as a List (e.g.{1,0,0,0}

{0,0,1/(r Sin[th])}

[Note: This says that $\boldsymbol{e}_{[\phi]}$ is 1/(r Sin[θ]) $\partial/\partial\phi$]

■ Remaining Prompt Windows: Metric components in the non−coordinatebasis, i.e. inner product of the basis vectors with each other

>Is the basis inner product
> 1) Diagonal, or
> 2) Symmetric?

1

>Enter eta[1,1]

1

[Note: eta[j,k] is GRTensor's notation for the components of the metric in the non−coordinatebasis. My response says that eta[1,1] = (the inner product of $\boldsymbol{e}_{[r]}$ with itself) is unity −−as is dictated by the fact that my chosen basis is orthonormal.

>Enter eta[2,2]

1

>Enter eta[3,3]

1

■ GRTensor's Response to this Input:

```
Makeg 2.0; GRTensor metric/basis entry utility


To quit makeg, type 'Exit' at any prompt.


The values you have entered are:

Coordinates ={r, th, ph}

Basis vectors:
```

$e_{(1)}{}^{r} = 1$

$e_{(1)}{}^{th} = 0$

$e_{(1)}{}^{ph} = 0$

$e_{(2)}{}^{r} = 0$

$e_{(2)}{}^{th} = \frac{1}{r}$

$e_{(2)}{}^{ph} = 0$

$e_{(3)}{}^{r} = 0$

$e_{(3)}{}^{th} = 0$

$e_{(3)}{}^{ph} = \frac{\text{Csc}[th]}{r}$

The inner product:

$eta_{(1)\,(1)} = 1$

$eta_{(1)\,(2)} = 0$

$eta_{(1)\,(3)} = 0$

$eta_{(2)\,(2)} = 1$

$eta_{(2)\,(3)} = 0$

$eta_{(3)\,(3)} = 1$

[Note: At this point, in response to a window, I tell GRTensor to save the metric; it saves it in my chosen directory, so I can use it in the future. In response to another window, I tell GRTensor that I want to use this metric in a calculation. It then responds with the following summary of the metric and basis:]

Information written to /home/kip/courses/ph136/1999.00/chapters/chap23/sphere3d.g

Default metric = sphere3d

For the sphere3d metric.

Coordinates

$x^1 = r, \quad x^2 = th, \quad x^3 = ph$

For the sphere3d metric.

Basis inner product

$eta^{(r)\,(r)} = 1$

$eta^{(th)\,(th)} = 1$

$eta^{(ph)\,(ph)} = 1$

For the sphere3d metric.

Basis vectors

$e_{(r)}{}^{r} = 1$

$$e_{(th)}{}^{th} = \frac{1}{r}$$

$$e_{(ph)}{}^{ph} = \frac{\text{Csc[th]}}{r}$$

```
grmake completed.
```

■ Compute and Display Connection Coefficients in Non−CoordinateBasis

I now compute the connection coefficients in my orthonormal basis using the following command:

```
grcalc[Chr[bdn, bdn, bup]]

Created a definition for Chr[bdn, bdn, bup]

For the sphere3d metric.

Calculated e[bup, up]( 0.01 sec.)

Calculated g[up, up]( 0.02 sec.)

Calculated detg( 0. sec.)

Calculated g[dn, dn]( 0.02 sec.)

Calculated g[dn, dn, pdn]( 0.02 sec.)

Calculated Chr[dn, dn, dn]( 0.01 sec.)

Calculated Chr[bdn, bdn, bup]( 0.07 sec.)

CPU Time = 0.15 sec.
```

[Note: "Chr" is GRTensor's name for the connection coefficients (which it calls "Christoffel Symbols"); GRTensor uses a different convention from mine (and MTW's) for the ordering of indices on the connection coefficients: it puts the up index last instead of first, i.e. it permutes the indices leftward by one slot. The notation "bdn" means the down (covariant) index in the non−coordinatebasis ("b"); "bup" means the up (contravariant) index in the non−coordinatebasis. I print out the connection coefficients for the non−coordinatebasis with the command

```
grdisplay[Chr[bdn, bdn, bup]]
```

```
For the sphere3d metric.
```

```
Chr[bdn, bdn, bup]
```

$$\text{Chr}_{(r)\,(th)}^{\quad (th)} = \frac{1}{r}$$

$$\text{Chr}_{(r)\,(ph)}^{\quad (ph)} = \frac{1}{r}$$

$$\text{Chr}_{(th)\,(th)}^{\quad (r)} = -\frac{1}{r}$$

$$\text{Chr}_{(th)\,(ph)}^{\quad (ph)} = \frac{\text{Cot}[th]}{r}$$

$$\text{Chr}_{(ph)\,(ph)}^{\quad (r)} = -\frac{1}{r}$$

$$\text{Chr}_{(ph)\,(ph)}^{\quad (th)} = -\frac{\text{Cot}[th]}{r}$$

[Note: These agree with Eqs. (XXXXX) in Chapter 23.]

■ Compute and Display Connection Coefficients in Coordinate Basis

GRTensor denotes the connectin coefficients in the coordinate basis by Chr[dn,dn,up] (i.e. the same as in the non−coordinate basis, but with the "b" omitted from bdn,bdn,bup). I compute them with the following command:

```
grcalc[Chr[dn, dn, up]]
```

```
For the sphere3d metric.

Calculated Chr[dn, dn, up]( 0.02 sec.)

CPU Time = 0.03 sec.
```

I then print them out:

grdisplay[Chr[dn, dn, up]]

For the sphere3d metric.

Christoffel symbol of the second kind (symmetric in first two indices)

$$\text{Chr}_{\text{r th}}^{\text{th}} = \frac{1}{r}$$

$$\text{Chr}_{\text{r ph}}^{\text{ph}} = \frac{1}{r}$$

$$\text{Chr}_{\text{th th}}^{\text{r}} = -r$$

$$\text{Chr}_{\text{th ph}}^{\text{ph}} = \text{Cot}[\text{th}]$$

$$\text{Chr}_{\text{ph ph}}^{\text{r}} = -r\,\text{Sin}[\text{th}]^2$$

$$\text{Chr}_{\text{ph ph}}^{\text{th}} = -\text{Cos}[\text{th}]\,\text{Sin}[\text{th}]$$