Ph.136 Homework 11.

Alexander Putilin

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1 Problem A.(BT-11.3) Dispersion of Flexural Waves

We have the wave equation:

$$-D\frac{\partial^4\eta}{\partial z^4} + T\frac{\partial^2\eta}{\partial z^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} \tag{1}$$

Substitute $\eta = e^{ikz-i\omega t}$

$$-Dk^4\eta - Tk^2\eta = -\Lambda\omega^2\eta \tag{2}$$

This gives:

$$\omega(k) = c_s k\sqrt{1+\frac{k^2}{k_c^2}}, \quad \text{where} \tag{3}$$

$c_s = \sqrt{\frac{T}{\Lambda}}$, $k_c = \sqrt{\frac{T}{D}}$.

Phase velocity $V_{ph}(k) = \frac{\omega}{k} = c_s\sqrt{1+\frac{k^2}{k_c^2}}$

Group velocity $V_g(k) = \frac{d\omega}{dk} = \frac{c_s(1+\frac{2k^2}{k_c^2})}{\sqrt{1+\frac{k^2}{k_c^2}}}$

(The rest of this solution is by Kip:) A gaussian wave packet is a superposition of monochromatic waves with a Gaussian spread of their wave number. Let k_o be the wave number at the peak of the Gaussian and $1/L_o \ll k_o$ be the spread; then

$$\eta(z,t) \propto \int e^{-(k-k_o)^2L_o^2/2}e^{i[kz-\omega(k)t]}dk\ . \tag{4}$$

The packet's initial half width is L_o as one can easily see by computing the integral at $t=0$:

$$\eta(z,0) \propto e^{ik_oz}e^{-z^2/2L_o^2}\ . \tag{5}$$

This wave packet should propagate with the group velocity $V_{go} \equiv V_g(k_o)$. However, because there is a spread in the values of k, different Fourier components of the wave will propagate with slightly different group velocities. The spread of group velocities is $\Delta V_g \simeq V'_{go}/L_o$, where $V'_{go} = (dV_g/dk)_o$. Therefore, we expect the packet to widen as time passes, so its width is $L \simeq L_o + (V'_{go}/L_o)t$.

We can verify this analytically by expanding $\omega(k) = \omega_o + V_{go}(k - ko) + \frac{1}{2}V_{go}(k-ko)^2$ and inserting this into our equation above for $\eta(z,t)$. The integral is easy (just the Fourier transform of a Gaussian), and the result is

$$\eta(z,t) \propto e^{k_o z - \omega_o t} e^{-(z-V_{go}t)^2/L^2} \tag{6}$$

where

$$L = \sqrt{L_o^2 + V'_{go}t} \simeq L_o + \frac{V'_{go}t}{2L_o};. \tag{7}$$

thus our estimate of the spreading was off by a factor 2, and was correct only so long as the fraction increase in spread is small.

2 Problem B.(BT-11.4) Speed of Elastic Waves

We need to find sound velocities of 5 types of elastic waves. Consider them in turn.

2.1 Longitudinal waves along a rod.

The corresponding formula has been derived in BT. (BT-11.28)

$$c_1 = \sqrt{\frac{E}{\rho}} \tag{8}$$

2.2 Longitudinal waves along a sheet

Choose coordinate system so that z-axis is orthogonal to the sheet, x- and y-axes are parallel and wave is propagating in x-direction. Since wave is longitudinal it means

$$\xi_y = 0 \tag{9}$$

No force acts on the side of the sheet, i.e. $T_{zi} = 0$.

$$\Longrightarrow T_{zz} = -K\theta - 2\mu\Sigma_{zz} = 0, \qquad \Sigma_{zz} = -\frac{K}{2\mu}\theta \tag{10}$$

Tracelessness of Σ implies

$$\Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz} = 0 \tag{11}$$

But $\Sigma_{yy} = \xi_{y,y} - \frac{1}{3}\theta = -\frac{1}{3}\theta$. (Cf.(9)) Plugging into (11) we get

$$\Sigma_{xx} = \frac{1}{3}\theta + \frac{K}{2\mu}\theta = \frac{2-\nu}{3(1-2\nu)}\theta \tag{12}$$

Recall that

$$K = \frac{E}{3(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$

Find the relation between θ and $\xi_{x,x}$

$$\Sigma_{xx} = \xi_{x,x} - \frac{1}{3}\theta \Rightarrow \theta = \frac{1-2\nu}{1-\nu}\xi_{x,x} \tag{13}$$

Then

$$T_{xx} = -K\theta - 2\mu\Sigma_{xx} = -\frac{E}{1-\nu^2}\xi_{x,x} \tag{14}$$

Equation of motion is

$$\rho\frac{\partial^2\xi_x}{\partial t^2} = T_{xx,x} + T_{xy,y} + T_{xz,z} = -\frac{E}{1-\nu^2}\frac{\partial^2\xi_x}{\partial x^2} \tag{15}$$

It gives the sound velocity

$$c_2 = \sqrt{\frac{E}{\rho(1-\nu^2)}}, \qquad \frac{c_1}{c_2} = \left(1-\nu^2\right)^{1/2} \tag{16}$$

2.3 Longitudinal waves along a rod embedded in incompressible medium.

Orient x-axis of Cartesian coordinate system along a rod. Incompressibility of medium means that there is no strain in transverse direction, i.e. $\xi_y = \xi_z = 0$.

$$\Rightarrow \theta = \xi_{x,x}, \qquad \Sigma_{xx} = \xi_{x,x} - \frac{1}{3}\theta = \frac{2}{3}\xi_{x,x} \tag{17}$$

Using (17) we get

$$T_{xx} = -K\theta - 2\mu\Sigma_{xx} = -E\frac{1-\nu}{(1+\nu)(1-2\nu)}\xi_{x,x} \tag{18}$$

$$c_3 = \sqrt{\frac{E}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)}}, \qquad \frac{c_1}{c_3} = \left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)^{-1/2} \tag{19}$$

2.4 Shear waves in an extended solid.

The sound velocity has been calculated in BT. (BT-11.20)

$$c_4 = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{\rho 2(1+\nu)}} \tag{20}$$

$$\frac{c_1}{c_4} = (2(1+\nu))^{1/2} \tag{21}$$

2.5 Torsional waves along a rod.

This has been also calculated in BT. (BT-11.34)

$$c_5 = c_4 = \sqrt{\frac{\mu}{\rho}}, \qquad \frac{c_1}{c_5} = (2(1+\nu))^{1/2} \tag{22}$$

3 Problem C.(BT-11.5) Xylophones

We start with the wave equation (BT-11.37)

$$\frac{\partial^4 \eta}{\partial z^4} + \frac{\Lambda}{D}\frac{\partial^2 \eta}{\partial t^2} = 0 \tag{23}$$

We are looking for the standing-wave solution of the form

$$\eta(t,z) = e^{-i\omega t} f(z) \tag{24}$$

Substituting into (23) we get the O.D.E. for $f(z)$

$$\frac{d^4 f}{dz^4} = \frac{\Lambda\omega^2}{D} f \tag{25}$$

It has the general solution

$$f(z) = A\sin kz + B\cos kz + C\sinh kz + D\cosh kz \tag{26}$$

where A, B, C, D are constants and

$$\omega^2 = \frac{D}{\Lambda} k^4 \tag{27}$$

To define the spectrum of k we should impose boundary conditions. We have two different cases.

3.1 Ends are free.

The appropriate boundary conditions are:

$$f(0) = f(l) = 0 \quad \text{- beam is supported at the ends} \tag{28}$$

$$f''(0) = f''(l) = 0 \quad \text{longitudinal stress vanishes} \tag{29}$$

$$\Longrightarrow B = C = D = 0, \qquad kl = \pi n, \qquad n = 1, 2, 3, \ldots \tag{30}$$

So the frequency spectrum is

$$\omega_n = \sqrt{\frac{D}{\Lambda}} k^2 = \sqrt{\frac{D}{\rho A}} \left(\frac{\pi n}{l}\right)^2 \tag{31}$$

3.2 Clamped ends.

In this case the boundary conditions are:

$$f(0) = f(l) = 0 \tag{32}$$
$$f'(0) = f'(l) = 0 \tag{33}$$

They can be nontrivially satisfied only if

$$\cosh(kl)\cos(kl) = 1 \tag{34}$$

so the spectrum is

$$\omega_n = \sqrt{\frac{D}{\rho A}}\left(\frac{\alpha_n}{l}\right)^2, \tag{35}$$

where α_n are solution of the eqn.

$$\cosh(\alpha)\cos(\alpha) = 1 \tag{36}$$

If $\alpha \gg 1$, then $\cosh\alpha \gg 1$ and $\cos\alpha \ll 1$ i.e. $\alpha_n \approx \pi(n+\frac{1}{2})$. Actually this formula gives good approximation for all $n = 1, 2, 3, \ldots$.

$$\Longrightarrow \omega_n \approx \sqrt{\frac{D}{\rho A}}\left[\frac{\pi}{l}(n+\frac{1}{2})\right]^2, \tag{37}$$

The ratios of eigen-frequencies are

$$\omega_1 : \omega_2 : \omega_3 : \ldots \approx 9 : 25 : 49 : \ldots$$

The ratios are noninteger and the frequencies don't make up any harmonic chord. A xylophone with clamped ends would sound hollow.

4 Problem D.(BT-11.7) Reflection and Transmission of Normal, Longitudinal Waves at a Boundary

Assume that the wave is propagating along z-axis and $z = 0$ is a boundary between two media. ($z < 0$ corresponds to medium 1, and $z > 0$ - to medium 2) The longitudinal displacement is given by

$$\xi(z) = \begin{cases} e^{ikz} + re^{-ikz} & z < 0 \\ te^{ik'z} & z > 0 \end{cases} \tag{38}$$

where $k = \frac{\omega}{c_1}$, $k' = \frac{\omega}{c_2}$, ω is frequency, $c_{1,2}$ are sound velocities.

$$c_{1,2} = \sqrt{\frac{K_{1,2} + \frac{4}{3}\mu_{1,2}}{\rho_{1,2}}} \tag{39}$$

r is the ratio of the reflected wave amplitude to the incident amplitude, t is the corresponding ratio for the transmitted wave amplitude. (We‘ve chosen the incident wave to have a unit amplitude).

Time dependence is given by factor $e^{-i\omega t}$ in both cases.

$$\theta = \xi(z)_{,z} = \xi'(z), \qquad \Sigma_{zz} = \xi_{,z} - \frac{1}{3}\theta = \frac{2}{3}\xi' \tag{40}$$

$$T \equiv T_{zz} = -K\theta - 2\mu\Sigma_{zz} = -(K + \frac{4}{3}\mu)\xi'(z) \tag{41}$$

$$T = \begin{cases} -(K_1 + \frac{4}{3}\mu_1)(ike^{ikz} - ikre^{-ikz}) & z < 0 \\ -(K_2 + \frac{4}{3}\mu_2)ik'te^{ik'z} & z > 0 \end{cases} \tag{42}$$

Matching the displacement and the stress on the boundary ($z = 0$) we get the equations:

$$1 + r = t \tag{43}$$

$$(K_1 + \frac{4}{3}\mu_1)ik(1 - r) = (K_2 + \frac{4}{3}\mu_2)ik't \tag{44}$$

which give (after simple algebra):

$$t = \frac{2Z_1}{Z_1 + Z_2} \tag{45}$$

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{46}$$

$Z_{1,2} = \rho_{1,2}c_{1,2}$

Conservation of energy flux.

We want to show that (Incident Flux)=(Transmitted Flux)+ (Reflected Flux). We have the formula for the flux

$$F = \rho\omega^2 c\langle\xi^2\rangle \tag{47}$$

Thus conservation of energy flux reduces to

$$\rho_1 c_1(1 - |r|^2) = \rho_2 c_2 |t|^2, \qquad \text{or} \qquad Z_1(1 - |r|^2) = Z_2|t|^2 \tag{48}$$

Substituting t and r from (45),(46) we see that (48) is an identity.

5 Problem E.(BT-11.8) Earthquakes

Energy release $E = 10^{5.2+1.44M}J$, for $M = 8.5$ it gives

$$E \approx 10^{17.5}J \approx 3 \cdot 10^{17}J \tag{49}$$

This should be equal to the elastic energy

$$E_{el} \approx \mu\Sigma^2(VT)^2\lambda \tag{50}$$

where Σ - typical strain, $\mu \sim 30GPa$ - shear coefficient, $V \sim 3\frac{km}{s}$, $T \sim 100s$, $\lambda \sim \frac{V}{\nu} \sim \frac{3\frac{km}{s}}{1s^{-1}} \sim 3km$ - wavelength.

$(VT)^2\lambda$ factor is the volume of the earthquake region. Equating (49) and (50) we get

$$\Sigma \sim \sqrt{\frac{E}{\mu(VT)^2\lambda}} \sim 10^{-4} \tag{51}$$

This is a reasonable estimate. Since the Earth crust is inhomogenous and has a lot of defects, relatively small strain is sufficient to produce a fault. (For a perfect crystal the corresponding strain would be $\sim 10^{-2}$).

The typical displacement in the vicinity of the fault

$$\xi \sim VT\Sigma \sim 30m \tag{52}$$

Now let's estimate amplitude (and acceleration) of elastic waves far away from the earthquake region. The energy density of the waves is (neglecting the density stratification) $\rho\omega^2\xi^2$, where $\rho \sim 5 \cdot 10^3 \frac{kg}{m^3}$, $\omega = 2\pi\nu \sim 6s^{-1}$, ξ is wave amplitude. The total energy is $\rho\omega^2\xi^2$(Volume). There are two types of waves: Rayleigh waves that propagate near surface and S,P-waves propagating in the bulk.

For surface waves: Volume$\sim 2\pi rVT\lambda$ while for bulk waves Volume$\sim r^2VT$. r is the distance between the earthquake and the observation point. Assume quite arbitrary that surface waves carry 1/3 of all the released energy and bulk waves the remaining 2/3. Then the amplitudes of surface and bulk waves and corresponding accelerations can be estimated as

$$\xi_{surf}^2 \sim \frac{E}{6\pi rVT\lambda\rho\omega^2} \quad a_{surf} \sim \xi_{surf}\omega^2 \tag{53}$$

$$\xi_{bulk}^2 \sim \frac{2E}{3r^2VT\rho\omega^2} \quad a_{bulk} \sim \xi_{bulk}\omega^2 \tag{54}$$

If observer is in the next state

$$r \sim 500km, \ \ a_{surf} \sim 40\frac{cm}{s^2}, \ \ a_{bulk} \sim 10\frac{cm}{s^2} \tag{55}$$

If observer is in the next continent

$$r \sim 10^4km, \ \ a_{surf} \sim 10\frac{cm}{s^2}, \ \ a_{bulk} \sim 0.5\frac{cm}{s^2} \tag{56}$$

We see that surface waves are more important in both cases.

Note added by Kip: These conclusions are modified somewhat by attenuation of the waves with distance. The wave amplitude attenuates $\propto e^{-\pi r/\lambda Q}$, where Q is the quality factor. It turns out (contrary to what I said in class on Wednesday) that the Q's vary strongly depending on the nature of the rock through which they propagate; they can range from as low as $Q \sim 20$ for alluvial materials near the earth's surface to as high as ~ 3000 in the mantle at depths of a few hundred km. Even for the largest $Q \sim 3000$ the intercontinental attenuation is significant, $\exp[-\pi 10,000\mathrm{km}/3\mathrm{km}\times 3000] \sim 0.03$. It is my impression that the surface waves are typically subject to somewhat greater attenuation than the body waves, so they may not dominate at great distances in the manner suggested by the above estimates; but I've not checked this. For some measurements of Q's see, e.g., http://rses.anu.edu.au/seismology/ar99/atten.html.