5

Buoyancy

'Buoy' mostly pronounced 'booe', probably of Germanic origin. A tethered floating object used to mark a location in the sea.

Fishes, whales, submarines, balloons and airships all owe their ability to float to *buoyancy*, the lifting power of water and air. The understanding of the physics of buoyancy goes back as far as antiquity and has probably sprung from the interest in ships and shipbuilding in classic Greece. The basic principle is due to Archimedes. His famous Law states that the buoyancy force on a body is equal and oppositely directed to the weight of the fluid that the body replaces. Actually the Law was not just one law, but a set of four propositions dealing with different configurations of body and liquid [7]. Before his time one had thought that the shape of a body determined whether it would sink or float.

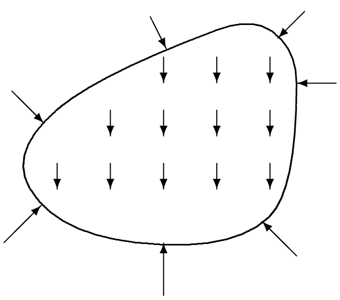
of Syracuse Archimedes (287–212 BC). *Greek mathematician. Discovered the formulas for area and volume of cylinders and spheres. Considered the father of fluid mechanics.*

The shape of a floating body and its mass distribution does determine whether it will float stably or capsize. Stability of floating bodies is of importance to shipbuilding, and to anyone who has ever tried to stand up in a small rowboat. Newtonian mechanics not only allows us to derive Archimedes' Principle for equilibrium of floating bodies, but also to characterize the deviations from equilibrium and calculate the restoring forces. Even if a body floating in or on water is in hydrostatic equilibrium, it will not be in complete mechanical balance in every orientation, because the center of mass of the body and the center of mass of the displaced water, also called the center of buoyancy, do not in general coincide.

The mismatch between the centers of mass and buoyancy for a floating body creates a moment of force, which tends to rotate the body towards a stable equilibrium. For submerged bodies, submarines, fishes and balloons, the stable equilibrium will always be with the center of gravity situated directly below the center of buoyancy. For bodies floating stably on the surface, ducks, ships, and dumplings, the center of gravity is mostly found directly above the center of buoyancy.

5.1 Archimedes' principle

Mechanical equilibrium takes a slightly different form than global hydrostatic equilibrium (4-15) when a body of another material is immersed in a fluid. If its material is incompressible, the body retains its shape and displaces an amount of fluid with exactly the same volume. If the body is compressible, as a rubber ball, the volume of displaced fluid will be smaller. The body may even take in fluid, like the piece of bread you dunk into your coffee, but then the physics becomes more complicated, and we shall disregard this possibility in the following. A body which is partially immersed may formally be viewed as a body that is fully immersed in a fluid for which the mass density and the equation of state vary from place to place. This also covers the case where part of the body is in vacuum which may be thought of as a fluid with the extreme properties, $\rho = p = 0$.



Gravity pulls at a body all over its volume, while pressure only acts the surface.

Weight and buoyancy

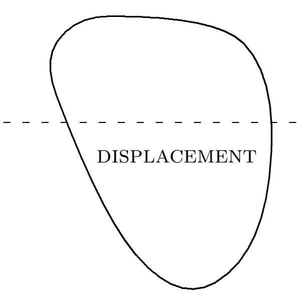
Let the actual, perhaps compressed, volume of the immersed body be V with surface S. In the field of gravity an unrestrained body is subject to two forces: its weight

$$\boldsymbol{\mathcal{F}}_G = \int_V \rho_{\text{body}}\, \boldsymbol{g}\, dV \ , \tag{5-1}$$

and the buoyancy due to pressure acting at its surface,

$$\boldsymbol{\mathcal{F}}_B = -\oint_S p\, d\boldsymbol{S} \ . \tag{5-2}$$

In general these two forces do not have to be in balance. The resultant $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B$ determines the direction that the unrestrained body will begin to move. In mechanical equilibrium the two forces must exactly cancel each other so that the body can remain in place.



For a body partially submerged in water the displacement is the amount of water that has been displaced by the volume of the body below the waterline.

Assuming that the body does not itself significantly contribute to the field of gravity, the local balance of forces in the fluid (4-19) will be the same as before the body was placed in the fluid. In particular the pressure in the fluid cannot depend on whether the volume V contains material that is different from the fluid itself. The pressure on the surface of the immersed body must for this reason be identical to the pressure on a body of fluid of the same shape. But then the global equilibrium condition (4-15) tells us that the buoyancy force will exactly balance the weight of the displaced fluid, so that

$$\boldsymbol{\mathcal{F}}_B = -\oint_S p\, d\boldsymbol{S} = -\int_V \rho_{\text{fluid}}\, \boldsymbol{g}\, dV \ . \tag{5-3}$$

This theorem is indeed Archimedes' principle: *the force of buoyancy equals (minus) the weight of the displaced fluid.*

The total force on the body may then be written

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = \int_V (\rho_{\rm body} - \rho_{\rm fluid}) \boldsymbol{g} \, dV \ , \tag{5-4}$$

explicitly confirming that when the body is made from the same fluid as its surroundings, so that $\rho_{\rm body} = \rho_{\rm fluid}$, the resultant force vanishes automatically. In general, however, the distributions of mass in the body and in the displaced fluid will be different.

Notice that Archimedes' principle is valid even if the gravitational field varies appreciably across the body. Archimedes principle fails, if the body is so large that its own gravitational field cannot be neglected, such as would be the case if an Earth-sized body fell into Jupiter's atmosphere. The extra compression of the fluid near the surface of the body generally increases the buoyancy force in the direction opposite to the ambient field of gravity. In semblance with Baron von Münchausen's adventure, the body in effect lifts itself by its bootstraps (see problems 5.6 and 5.7).

Freiherr Karl Friedrich Hieronymus von Münchhausen (1720-1797). *German (Hanoveran) soldier, hunter, nobleman, and delightful story-teller. The stories of his travels to Russia were retold and further embroidered by others and published as "The Adventures of Baron Munchausen" in 1793. In one of these, he lifts himself (and his horse) out of deep snow by his bootstraps. Incidentally, this story is also the origin of the expression "bootstrapping", or more recently just "booting", a computer.*

Constant field of gravity

If the gravitational field is constant, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, the weight of the body is,

$$\boldsymbol{\mathcal{F}}_G = M_{\rm body} \, \boldsymbol{g}_0 \ , \tag{5-5}$$

and the buoyancy force becomes

$$\boldsymbol{\mathcal{F}}_B = -M_{\rm fluid} \, \boldsymbol{g}_0 \ . \tag{5-6}$$

Since the total force is the sum of these contributions, one might say that buoyancy acts as if the displacement were filled with fluid of negative mass $-M_{\rm fluid}$. In effect the buoyancy force acts as a kind of antigravity.

The total force on an unrestrained object is now,

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = (M_{\rm body} - M_{\rm fluid}) \boldsymbol{g}_0 \ . \tag{5-7}$$

If the body mass is smaller than the mass of the displaced fluid, the total force is directed upwards, and the unrestrained body will begin to move upwards. Alternatively, if the body is kept in place, the restraints must deliver a force $-\boldsymbol{\mathcal{F}}$ to prevent the object from moving.

A body can only hover motionlessly in a fluid if its mass equals the mass of the displaced fluid,

$$M_{\rm body} = M_{\rm fluid} \ . \tag{5-8}$$

Fish achieve this balance by adjusting the amount of water they displace through contraction and expansion of an internal air-filled bladder. Submarines on the contrary change their mass by pumping water in and out of ballast tanks. Curiously, no animals seem to have developed balloons for floating in the atmosphere, although both the physics and chemistry of ballooning appears to be within reach of biological evolution.

5.2 The gentle art of ballooning

Joseph Michel Montgolfier (1740-1810). *Experimented (together with his younger brother Jacques Étienne (1745-1799)) with hot-air balloons. On November 21, 1783, the first human flew in such a balloon for a distance of 9 kilometers at a height of 100 meter above Paris. Only one of the brothers ever flew, and then only once!*

Apart from large kites used in ancient China, balloons were the earliest flying machines. The first balloons made by the Montgolfier brothers in 1783 contained hot air which is lighter than cold. Hot-air balloons were a century later replaced by balloons containing light gases, hydrogen or helium, with greater lifting power. This also eliminated the need for a constant heat supply and made possible the huge (and dangerous) hydrogen airships of the 1930's. In the last half of the twentieth century hot-air balloons again came into vogue, especially for sports, because of the availability of modern strong lightweight materials (nylon) and fuel (propane).

Gas balloons

A large hydrogen or helium balloon typically begins its ascent being only partially filled, assuming an inverted tear-drop shape. During the ascent the gas expands because of the fall in ambient air pressure, and eventually the balloon becomes nearly spherical and stops expanding (or bursts) because the "skin" of the balloon cannot stretch further. Since the density of the displaced air falls with height, the balloon will reach a maximum height, a ceiling where it could hover permanently if it did not lose gas. In the end no balloon stays aloft forever.

Let the total mass of the balloon be M_0, including the mass of the gas, the balloon skin, the gondola, people, and what not. The condition for upwards flight is then that $M_0 \leq V\rho$ where V is the total volume of air that the balloon displaces and ρ the air density at its actual position. In the homentropic atmospheric model the air density is given by (4-44), and the condition for flight at height z becomes,

$$M_0 \leq \rho_0 V \left(1 - \frac{z}{h_2}\right)^{1/(\gamma-1)} \tag{5-9}$$

where $\gamma \approx 7/5$ is the adiabatic index of air, $\rho_0 \approx 1.2\ \mathsf{kg/m^3}$ its density at sea level, and $h_2 \approx 30$ km the isentropic scale height (4-43). If this inequality is fulfilled on the ground, the balloon will start to rise. During the rise the volume may expand towards a maximal value while the air density falls, and the balloon will keep rising until the inequality is no more be fulfilled, and the balloon has reached its ceiling.

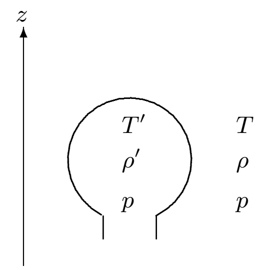
Example 5.2.1: A gas balloon has a maximal spherical diameter of 10 m yielding a volume $V \approx 524$ m^3. For the balloon to lift off at all, its mass must be smaller than $\rho_0 V = 628$ kg. Taking $M_0 = 400$ kg the ceiling becomes $z \approx 5$ km. At this height the air pressure and temperature are $p = 0.53$ atm and $T = 245$ K $= -29^\circ$ C. Assuming that the balloon contains hydrogen H_2 (with $M_{H_2} = 2$ g/mol and $\gamma = 7/5$) at this temperature and pressure, the total mass of the hydrogen is merely 28 kg. The surface area of the balloon is 314 m^2, so if the skin has thickness 2 mm and density 300 kg/m^3, its mass becomes 188 kg, which leaves about $400 - 28 - 188 = 184$ kg for the proper payload. Filled with helium He (with $M_{mol} = 4$ g/mol and $\gamma = 5/3$), the proper payload would be reduced to 156 kg.

Hot-air balloons

A hot-air balloon is open at the bottom so that the inside pressure is always the same as the atmospheric pressure outside. The air in the balloon is warmer ($T' > T$) than the outside temperature and the density is correspondingly lower ($\rho' < \rho$). If M_0 denotes the total mass of the balloon, the condition for flight is now that $M_0 < (\rho - \rho')V$. From the ideal gas law (4-27) and the equality of the inside and outside pressures it follows that $\rho' T' = \rho T$, so that the inside density is $\rho' = \rho T/T'$. The condition for flight at height z becomes,

$$M_0 \leq \left(1 - \frac{T}{T'}\right)\rho V = \left(1 - \frac{T_0}{T'}\left(1 - \frac{z}{h_2}\right)\right)\left(1 - \frac{z}{h_2}\right)^{\frac{1}{\gamma-1}} \rho_0 V \ . \quad (5\text{-}10)$$

On the right hand side we have inserted the expressions (4-42) and (4-44) for the homentropic atmospheric temperature and density.



A hot-air balloon has higher temperature $T' > T$ and lower density $\rho' < \rho$ but the same pressure as the surrounding atmosphere because it is open below.

Example 5.2.2: A spherical hot-air balloon with diameter $d = 10$ m is desired to reach a ceiling of $z = 1000$ m with air temperature $T' = 100°$ C = 373 K. When the ground temperature is $T_0 = 20$ °C = 293 K and the density $\rho_0 = 1.2$ kg/m^3, it follows that this balloon would be capable of lifting $M_0 \approx 140$ kg to the ceiling.

5.3 Stability of floating bodies

Although a body may be in buoyant equilibrium, so that the total force composed of gravity and buoyancy vanishes, $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = \mathbf{0}$, it may not be in complete mechanical equilibrium. The total moment of all the forces acting on the body must also vanish; for else an unrestrained body will start to rotate.

Moments of weight and buoyancy

The total moment is like the total force a sum of two contributions,

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_G + \boldsymbol{\mathcal{M}}_B \ , \quad (5\text{-}11)$$

with one contribution from gravity,

$$\boldsymbol{\mathcal{M}}_G = \int_V \boldsymbol{x} \times \rho_{\mathsf{body}} \boldsymbol{g} \, dV \ , \quad (5\text{-}12)$$

and the other from pressure, *i.e.* buoyancy,

$$\boldsymbol{\mathcal{M}}_B = \oint_S \boldsymbol{x} \times (-p \, d\boldsymbol{S}) \ . \quad (5\text{-}13)$$

If the total force vanishes, $\boldsymbol{\mathcal{F}} = \mathbf{0}$, the total moment will be independent of the origin of the coordinate system (page 41).

Assuming again that the presence of the body does not change the pressure distribution in the fluid, the moment of buoyancy is independent of the nature of the material inside V. In hydrostatic equilibrium the total moment on the same volume of fluid must vanish, $\boldsymbol{\mathcal{M}}_G^{\text{fluid}} + \boldsymbol{\mathcal{M}}_B = \mathbf{0}$, such that we get

$$\boldsymbol{\mathcal{M}}_B = -\int_V \boldsymbol{x} \times \rho_{\text{fluid}} \boldsymbol{g} \, dV \ . \tag{5-14}$$

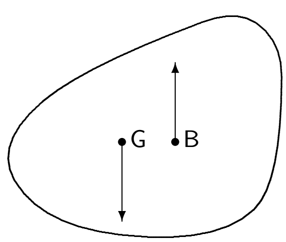
The moment of buoyancy equals the (minus) moment of gravity of the displaced fluid. This result is a natural corollary to Archimedes' principle, and of immense help in calculating the buoyancy moment. A formal proof of the theorem is found in problem 5.8.

Constant gravity and buoyant equilibrium

In the remainder of this chapter we assume that gravity is constant, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, and that the body is in buoyant equilibrium so that it displaces exactly its own mass of fluid, $M_{\text{fluid}} = M_{\text{body}} = M$. The densities of body and displaced fluid will, however, in general be different, $\rho_{\text{body}} \neq \rho_{\text{fluid}}$.

The moment of gravity (5-12) may as before (page 41) be expressed in terms of the center of the body mass distribution (here called the center of gravity),

$$\boldsymbol{\mathcal{M}}_G = \boldsymbol{x}_G \times M\boldsymbol{g}_0 \ , \qquad \boldsymbol{x}_G = \frac{1}{M}\int \boldsymbol{x}\rho_{\text{body}} \, dV \ . \tag{5-15}$$



Body in buoyant equilibrium but with non-vanishing total moment which here sticks out of the paper. The moment will for a submerged body tend to rotate it in the anticlockwise direction and thus bring the center of gravity below the center of buoyancy.

Similarly the moment of the mass distribution of the displaced fluid (5-14) is,

$$\boldsymbol{\mathcal{M}}_B = -\boldsymbol{x}_B \times M\boldsymbol{g}_0 \ , \qquad \boldsymbol{x}_B = \frac{1}{M}\int \boldsymbol{x}\rho_{\text{fluid}} \, dV \ . \tag{5-16}$$

Although each of these moments depends on the choice of origin of the coordinate system, the total moment,

$$\boldsymbol{\mathcal{M}} = (\boldsymbol{x}_G - \boldsymbol{x}_B) \times M\boldsymbol{g}_0 \ , \tag{5-17}$$

will be independent, as witnessed by the appearance of the difference of the two center positions.

As long as the total moment is non-vanishing, the body is not in mechanical equilibrium, but will start to rotate towards an orientation with vanishing moment. Except for the trivial case where the centers of gravity and buoyancy coincide, the above equation tells us that the total moment can only vanish if the centers lie on the same vertical line,

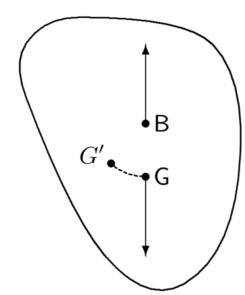
$$\boldsymbol{x}_G - \boldsymbol{x}_B \propto \boldsymbol{g}_0 \ . \tag{5-18}$$

For $\boldsymbol{x}_G \neq \boldsymbol{x}_B$, there are two possible orientations satisfying this condition: one where the center of gravity lies above the center of buoyancy, and another where the center of gravity is lowest. At least one of these will be stable.

Submerged body

For a fully submerged rigid body, for example a submarine, both centers are always in the same place relative to the body. If the center of gravity does not lie directly below the center of buoyancy, but displaced a bit horizontally, the direction of the moment will always tend to turn the body so that the center of gravity is lowered with respect to the center of buoyancy. The only stable orientation of the body is where the center of gravity lies vertically below the center of buoyancy. Any small perturbation away from this orientation will soon be corrected and the body brought back to the equilibrium orientation. A similar argument shows that the other equilibrium orientation with the center of gravity above the center of buoyancy is unstable and will flip the body over, if perturbed the tiniest amount.

> This is why the gondola hangs below an airship or balloon, and why a fish goes belly-up when it dies, because it loses control of the swim bladder which enlarges into the belly and reverses the positions of the centers of gravity and buoyancy. It mostly also loses buoyant equilibrium and floats to the surface.

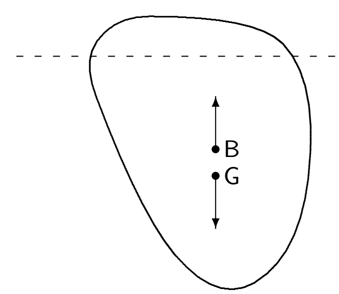


A fully submerged body in stable equilibrium must have the center of gravity directly below the center of buoyancy. If G is moved to G' a restoring moment is created which sticks out of the plane of the paper.

Body floating on the surface

At the surface of a liquid, a body such as a ship or an iceberg will according to Archimedes' principle always arrange itself so that the mass of displaced liquid exactly equals the mass of the body. Here we assume that there is vacuum or a very light fluid such as air above the liquid. The center of gravity is always in the same place relative to the body, but the center of buoyancy depends now on the orientation of the body, because the volume of displaced fluid changes place and shape (while keeping its mass constant) when the body orientation changes.

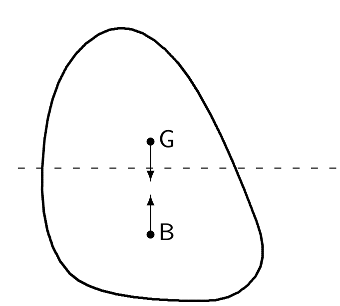
Stability can again only occur when the two centers lie on the same vertical line, but there may be more than one stable orientation. A sphere made of homogeneous wood floating on water, is stable in all orientations . None of them are in fact truly stable, because it takes no force to move from one to the other. This is however a marginal case.



A floating body may have a stable equilibrium with the center of gravity directly below the center of buoyancy.

A floating body may like a submerged body possess a stable orientation with the center of gravity directly *below* the center of buoyancy. A heavy keel is, for example, used to lower the center of gravity of a sailing ship so much that this orientation becomes the only stable equilibrium. In that case it becomes virtually impossible to capsize the ship, even in a very strong wind.

The stable orientation for most floating objects, such as ships, will in general have the center of gravity situated directly *above* the center of buoyancy. This happens always when an object of constant mass density floats on top of a liquid of constant mass density, for example an iceberg on water. The part of the iceberg that lies below the waterline must have its center of buoyancy in the same place as its center of gravity. The part of the iceberg lying above the water cannot influence the center of buoyancy whereas it always will shift the center of gravity upwards.



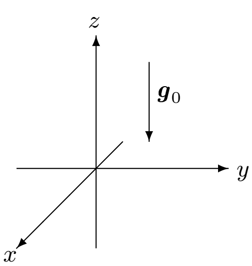
A floating body most often has a stable equilibrium with the center of gravity directly above *the center of buoyancy.*

How can that situation ever be stable? Will the restoring moment not be of the wrong sign? Why don't ducks and tall ships capsize spontaneously? The qualitative answer is that when the body is rotated away from such an equilibrium orientation, the volume of displaced water will change position and shift the center of buoyancy back to the other side of the center of gravity, reversing the direction of the restoring moment.

5.4 Ship stability

Sitting comfortably in a small rowboat, it is fairly obvious that the center of gravity lies above the center of buoyancy, and that the situation is stable with respect to small movements of the body. But many a fisherman has learnt that suddenly standing up may compromise the stability and send him out among the fishes. There is, as we shall see, a strict limit to how high the center of gravity may be above the center of buoyancy.

Most ships are mirror symmetric in a plane, but we shall be more general and consider a "ship" of an arbitrary shape. We shall assume that the ship initially is in full mechanical equilibrium and calculate the moment that arises when it is brought slightly out of equilibrium. If the moment tends to turn the ship back into equilibrium, the initial orientation is stable. To lowest order of approximation, the stability turns out to be an essentially two-dimensional problem, depending mainly on the shape of the outline of the ship's hull in the waterline.

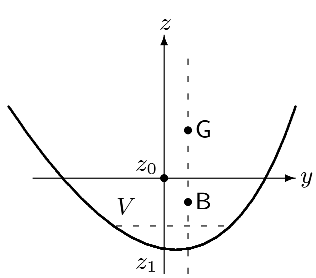


The flat-earth coordinate system.

Ship's shape

In a flat Earth coordinate system with vertical z-axis the ship's shape may be described by its horizontal area $A(z)$ in some interval $z_1 < z < z_2$. The waterline is at $z = z_0$ and the deepest point of the ship lies $d = z_0 - z_1$ below the surface, also called the ship's *draught* (or *draft*). Usually one chooses $z_0 = 0$, but it is not required. The volume of water that the ship displaces becomes,

$$V_0 = \int_{z_1}^{z_0} A(z)\, dz \ . \tag{5-19}$$



The ship in an equilibrium orientation, stable or unstable, shown in a vertical plane containing the aligned centers of gravity and buoyancy. The horizontal dashed line indicates the area $A(z)$.

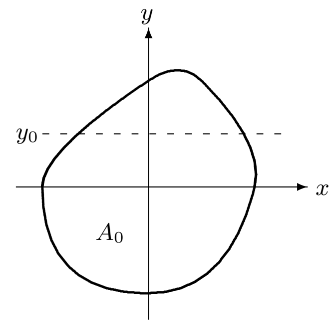
In equilibrium the total mass of the ship of course equals the mass of displaced water, $M = \rho_0 V_0$. Conversely, given the mass this equation determines the draught d.

Similarly the vertical position of the center of buoyancy is,

$$z_B = \frac{1}{V_0} \int_{z_1}^{z_0} z\, A(z) dz \ . \tag{5-20}$$

In equilibrium the horizontal positions of the center of buoyancy and gravity must be equal $x_B = x_G$ and $y_B = y_G$, whereas the vertical position z_G of its center of gravity depends on the actual mass distribution of the ship.

Center of roll



The area A_0 of the ship in the waterline may be of quite arbitrary shape. The z-axis is vertical, sticking out of the paper. The ship is tilted around the line $y = y_0$.

The ship is now tilted slightly through a small angle α around a line $y = y_0$, parallel to x-axis, so that the previous waterline area $A_0 = A(z_0)$ comes to lie in the plane $z = z_0 + \alpha(y - y_0)$. The net change in the displacement due to the tilt is to lowest order in α given by the difference in volumes of the two wedge-shaped regions between new and the old waterlines,

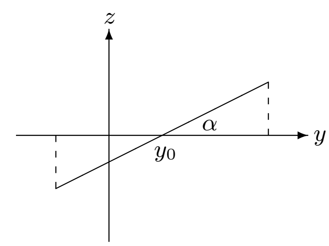
$$\delta V = -\int_{A_0} (z - z_0)\, dxdy = -\alpha \int_{A_0} (y - y_0)\, dxdy \ . \tag{5-21}$$

Here we have disregarded the small corrections of order α^2 due to the actual shape of the hull just above and below the waterline.

For the ship to remain in buoyant equilibrium after the tilt, the change in displacement must vanish, $\delta V = 0$, which is only possible for $y_0 = \frac{1}{A_0}\int_{A_0} y\, dA$. Including also tilts around a line parallel with the y-axis, this defines a unique point,

$$(x_0, y_0) = \frac{1}{A_0}\int_{A_0} (x, y)\, dxdy \ , \tag{5-22}$$

which we shall call the *center of roll*. It is not hard to show that a roll of the ship around any axis through this point will generate no change in displacement.



Tilt around the axis $y = y_0$. The change in displacement is negative in the wedge to the right and positive in the wedge to the left.

The restoring moment

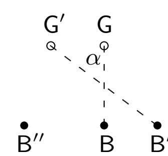
Without loss of generality we consider from now on only a roll around a line parallel with the x-axis. Such a roll generates a restoring moment, which may be calculated from (5-17),

$$\mathcal{M}_x = -(y_G - y_B) M g_0 \ . \tag{5-23}$$

Since we have $y_G = y_B$ in the original mechanical equilibrium, the difference in coordinates after the tilt may be written,

$$y_G - y_B = \delta y_G - \delta y_B \ , \tag{5-24}$$

where δy_G and δy_B are the small horizontal shifts of order α in the centers of gravity and buoyancy.

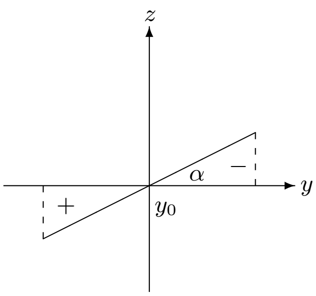


The tilt rotates the center of gravity from G to G′, and the center of buoyancy from B to B′. In addition, the change in displaced water shifts the center of buoyancy back to B″. In stable equilibrium this point must for $\alpha > 0$ lie to the left of the new center of gravity.

The center of gravity is (hopefully!) fixed with respect to the ship and is to first order in α shifted horizontally by a simple rotation,

$$\delta y_G = -\alpha(z_G - z_0) \ . \tag{5-25}$$

There will also be a vertical shift, $\delta z_G = \alpha(y_G - y_0)$, but that is of no importance to the stability in the lowest order of approximation.



The change in displacement consists in moving the water in the wedge to the right into the wedge to the left.

The center of buoyancy is at first shifted in the same way as the center of gravity by the tilt, but because the displacement also changes there will be another contribution Δy_B,

$$\delta y_B = -\alpha(z_B - z_0) + \Delta y_B \ . \tag{5-26}$$

The change in displacement consists in moving the water in wedge-shaped region from $y > y_0$ into the region $y < y_0$. Due to the choice of origin of the coordinates the two regions have equal volumes. Averaging $y - y_0$ over the volume of these regions, the horizontal change in the center of buoyancy becomes,

$$\Delta y_B = \langle y - y_0 \rangle = -\frac{1}{V_0} \int_{A_0} (y - y_0)(z - z_0) dx dy = -\frac{\alpha}{V_0} \int_{A_0} (y - y_0)^2 \, dx dy \ .$$

Finally, putting it all together we find the restoring moment

$$\boxed{\mathcal{M}_x = -\alpha \left(z_B + \frac{I_0}{V_0} - z_G \right) M g_0 \ ,} \tag{5-27}$$

where

$$\boxed{I_0 = \int_{A_0} (y - y_0)^2 \, dx dy} \ , \tag{5-28}$$

is the second "moment of inertia" around the x-axis of the hull area A_0 in the waterline. It is a purely geometric quantity which may be calculated from the outline of the ship in the waterline.

Rectangular waterline area: For a ship with rectangular waterline area with sides $2a$ and $2b$, the roll center coincides with the center of the rectangle, and the second moment around the x-axis becomes,

$$I_0 = \int_{-a}^{a} dx \int_{-b}^{b} dy \, y^2 = \frac{4}{3} a b^3 \ . \tag{5-29}$$

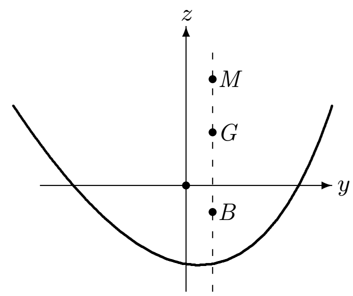
If $a > b$ this is the largest moment around any tilt axis.

Elliptic waterline area: If the ship has an elliptical waterline area with axes $2a$ and $2b$, the second moment around the x-axis becomes,

$$I_0 = \int_{-a}^{a} dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} y^2 \, dy = \frac{4}{3} a b^3 \int_0^1 (1 - t^2)^{3/2} \, dt = \frac{\pi}{4} a b^3 \ . \tag{5-30}$$

Notice that this is only about half of the rectangular result.

The metacenter



Stable ship. The metacenter lies above the center of gravity.

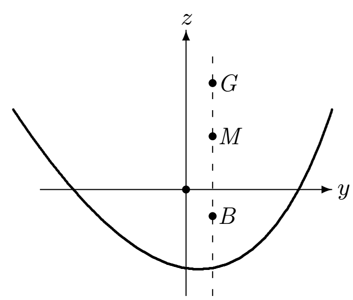
For the ship to be stable, the restoring moment must counteract the tilt and thus have opposite sign of the tilt angle α, which implies that the expression in parenthesis in (5-27) must be positive. The stability condition may thus be written,

$$z_M > z_G \ , \tag{5-31}$$

where

$$\boxed{z_M = z_B + \frac{I_0}{V_0}} \ , \tag{5-32}$$

is the z-coordinate of a fictitious point situated vertically above the original center of buoyancy. This point is called the *metacenter*, and *the ship is stable when the metacenter lies above the center of gravity.* A good captain should always know the positions of the center of gravity and the metacenter of his ship before he sails, or else he may capsize when casting off.



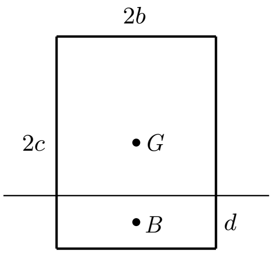
Unstable ship. The metacenter lies below the center of gravity.

The restoring moment (5-27) is proportional to the vertical distance, $z_M - z_G$, between the metacenter and the center of gravity. The closer the center of gravity comes to the metacenter, the smaller will the restoring moment be, and the longer will the period of rolling oscillations be. The actual roll period depends also on the true moment of inertia of the ship around the roll axis (see problem 5.11).

The orientation of the coordinate system with respect to the ship's hull was not specified in the analysis and is therefore valid for a tilt around any direction. For a ship to be fully stable, the stability condition must be fulfilled for all possible tilt axes. Since the displacement V is the same for all choices of tilt axis, the second moment of the area on the right hand side of (5-31) should be chosen to be the smallest one. Often it is quite obvious which moment is the smallest. Many modern ships are extremely long with the same cross section for most of their length and a mirror symmetry through a vertical plane. For such ships the smallest moment is clearly obtained with the roll axis parallel to the longitudinal axis of the ship.

Example 5.4.1: An elliptical rowboat with vertical sides has major axis $2a = 2$ m and minor axis $2b = 1$ m. The smallest moment of the rectangular area is $I = \frac{\pi}{4}ab^3 \approx 0.1$ m^4. If your mass is 75 kg and the boat's is 50 kg, the displacement will be $V_0 = 0.125$ m^3, and the draught $d \approx V_0/4ab = 6.25$ cm, ignoring the usually curved shape of the boat's hull. The coordinate of the center of buoyancy becomes $z_B = -3.1$ cm and the metacenter $z_M = 75$ cm. Getting up from your seat may indeed raise the center of gravity so much that it gets close to the metacenter and the boat begins to roll violently. Depending on your weight and mass distribution the boat may even become unstable and turn over.

Floating block



Floating block with height h, draught d, width 2b, and length 2a into the paper.

The simplest non-trivial case in which we may apply the stability criterion is that of a rectangular block of dimensions $2a$, $2b$ and $2c$ in the three coordinate directions. Without loss of generality we may assume that $a > b$. The center of the waterline area coincides with the roll center and the origin of the coordinate system with the waterline at $z_0 = 0$. The block is assumed to be made from a uniform material with constant density ρ_1 and floats in a liquid of constant density ρ_0.

In hydrostatic equilibrium we must have $M = 4abd\rho_0 = 8abc\rho_1$, or

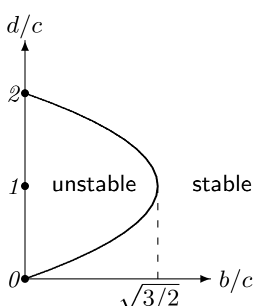
$$\frac{\rho_1}{\rho_0} = \frac{d}{2c} \ . \tag{5-33}$$

The position of the center of gravity is $z_G = c - d$ and the center of buoyancy $z_B = -d/2$. Using (5-29) and $V_0 = 4abd$, the position of the metacenter is

$$z_M = -\frac{d}{2} + \frac{b^2}{3d} \ . \tag{5-34}$$

Rearranging the stability condition, $z_M > z_G$, it may be written as

$$\left(\frac{d}{c} - 1\right)^2 > 1 - \frac{2b^2}{3c^2} \ . \tag{5-35}$$



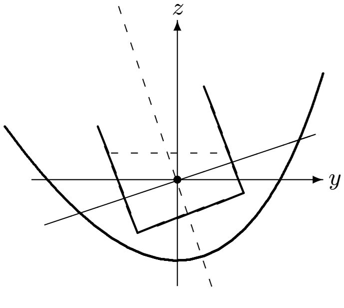
Stability diagram for the floating block.

When the block dimensions obey $a > b$ and $b/c > \sqrt{3/2} = 1.2247\ldots$, the right hand side becomes negative and the inequality is always fulfilled. On the other hand, if $b/c < \sqrt{3/2}$ there is a range of draught values around $d = c$ (*i.e.* $\rho_1/\rho_0 = \frac{1}{2}$),

$$1 - \sqrt{1 - \frac{2}{3}\left(\frac{b}{c}\right)^2} < \frac{d}{c} < 1 + \sqrt{1 - \frac{2}{3}\left(\frac{b}{c}\right)^2} \ , \tag{5-36}$$

for which the block is unstable. If the draught lies in this interval the block will keel over and come to rest in another orientation (see problem 5.13).

Ship with liquid cargo



Tilted ship with an open container filled with liquid.

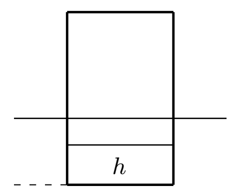
Many ships carry liquid cargos, oil, water, etc. When the tanks are not completely filled this kind of cargo may strongly influence the stability of the ship. In heavy weather or due to accidents, car ferries may inadvertently also get a layer of water on the car deck. The main effect of an open liquid surface inside the ship is that the center of mass is shifted in the same direction by the redistribution of real liquid as the shift in the center of buoyancy due to the change in displaced water, *i.e.* towards negative y-values. This disturbs the stability and creates a competition between the liquid carried by the ship and the water displaced by the ship.

For the case of a single open tank the calculation of the restoring moment must now include the liquid cargo. A similar analysis as before shows that there will be a change in center of gravity from the movement of a wedge of real liquid of density ρ_1,

$$\Delta y_G = -\alpha \frac{\rho_1 I_1}{M} = -\alpha \frac{\rho_1}{\rho_0} \frac{I_1}{V_0} \tag{5-37}$$

where I_1 is the second moment of the open liquid surface. The metacentric height now becomes

$$\boxed{z_M = z_B + \frac{I_0}{V_0} - \frac{\rho_1}{\rho_0} \frac{I_1}{V_0}} \tag{5-38}$$

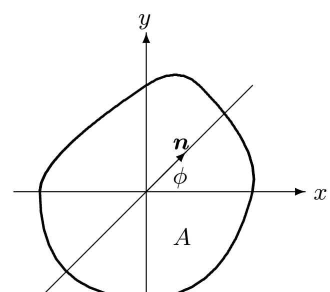


A "car ferry" with water on the deck is inherently unstable because the movement of the real water on the deck nearly cancels the stabilizing movement of the displaced water.

The effect of the moving liquid is to lower the metacentric height with possible destabilization as result. The unavoidable sloshing of the liquid may further compromise the stability. The destabilizing effect of a liquid cargo is often counteracted by dividing the hold into a number of smaller compartments by means of bulkheads along the ship's principal roll axis.

In car ferries almost any level h of water on the car deck may cause the ferry to capsize because $\rho_1 = \rho_0$ and $I_1 \approx I_0$, making $z_M \approx z_B$ independently of h. As several accidents have shown, car ferries are in fact highly susceptible to the destabilizing effects of water on the car deck. Water-proof longitudinal bulkheads on the car deck of a car ferry are usually avoided because it would hamper efficient loading of the cars.

∗ Principal roll axis



*Tilt axis **n** forming an angle ϕ with the x-axis.*

It has already been remarked that the metacenter for absolute stability is determined by the smallest second moment of the waterline area. Instead of tilting the ship around the x-axis, it is tilted around an axis $\boldsymbol{n} = (\cos\phi, \sin\phi, 0)$ forming an angle ϕ with the x-axis. Since this configuration is obtained by a simple rotation through ϕ around the z-axis, the transverse coordinate to be used in calculating the second moment becomes $y' = y\cos\phi - x\sin\phi$ (see eq. (2-35b)), and we find

$$I'_0 = \int_A (y')^2 \, dA = I_{xx}\cos^2\phi + I_{yy}\sin^2\phi + 2I_{xy}\sin\phi\cos\phi = \boldsymbol{n}\cdot\mathbf{I}\cdot\boldsymbol{n} \tag{5-39}$$

where I_{xx}, I_{yy} and I_{xy} are the elements of the matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{pmatrix} = \int_A \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} dA \tag{5-40}$$

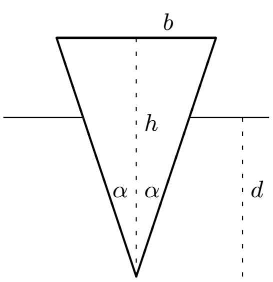
The extrema of the positive definite quadratic form $\boldsymbol{n}\cdot\mathbf{I}\cdot\boldsymbol{n}$ are found from the eigenvalue equation $\mathbf{I}\cdot\boldsymbol{n} = \lambda\boldsymbol{n}$ (see problem 5.10). The eigenvector corresponding to the smallest eigenvalue is called the *principal roll axis* of the ship and its eigenvalue determines the metacenter for absolute stability.

Problems

5.1 A stone weighs 1000 N in air and 600 N when submerged in water. Calculate the volume and average density of the stone.

5.2 A hydrometer with mass $M = 4$ g consists of a roughly spherical glass container and a long thin cylindrical stem of radius $a = 2$ mm. The sphere is weighed down so that the apparatus will float stably with the stem pointing vertically upwards and crossing the fluid surface at at some point. How much deeper will it float in alcohol with mass density $\rho_1 = 0.78$ g/cm^3 than in oil with mass density $\rho_2 = 0.82$ g/cm^3? You may disregard the tiny density of air.

5.3 A cylindrical wooden stick (density $\rho_1 = 0.65$ g/cm^3) floats in water (density $\rho_0 = 1$ g/cm^3). The stick is loaded down with a lead weight (density $\rho_2 = 11$ g/cm^3) at one end such that it floats in vertical position with a fraction $f = 1/10$ of its length out of the water. **(a)** What is the ratio (M_1/M_2) between the masses of the wooden stick and the lead weight? **(b)** How large a fraction can stick out of the water (disregarding questions of stability)?



Triangular ship of length L (into the paper) floating with its peak vertically downwards.

5.4 A ship of length L has a longitudinally invariant cross section in the shape of an isosceles triangle with half opening angle α and height h. It is made from homogeneous material of density ρ_1 an floats in a liquid of density $\rho_0 > \rho_1$. **(a)** Determine the stability condition on the mass ratio ρ_1/ρ_0 when the ship floats vertically with the peak downwards. **(b)** Determine the stability condition on the mass ratio when the ship floats vertically with the peak upwards. **(c)** What is the smallest opening angle that permits simultaneous stability in both directions?

5.5 A right rotation cone has half opening angle α and height h. It is made from homogeneous material of density ρ_1 an floats in a liquid of density $\rho_0 > \rho_1$. **(a)** Determine the stability condition on the mass ratio ρ_1/ρ_0 when the cone floats vertically with the peak downwards. **(b)** Determine the stability condition on the mass ratio when the cone floats vertically with the peak upwards. **(c)** What is the smallest opening angle that permits simultaneous stability in both directions?

5.6 A barotropic compressible fluid is in hydrostatic equilibrium with pressure $p(z)$ and density $\rho(z)$ in a constant external gravitational field with potential $\Phi = g_0 z$. A finite body having a "small" gravitational field $\Delta\Phi(\boldsymbol{x})$ is submerged into the fluid. **(a)** Show that the change in hydrostatic pressure to lowest order of approximation is

$$\Delta p(\boldsymbol{x}) = -\rho(z)\Delta\Phi(\boldsymbol{x}) \ . \tag{5-41}$$

(b) Show that for a spherically symmetric body of radius a and mass M, the extra surface pressure is $\Delta p = g_1 a \rho(z)$ where $g_1 = GM/a^2$ is the magnitude of surface gravity, and that the buoyancy force is increased.

5.7 Two identical homogenous spheres of mass M and radius a are situated a distance $D \gg a$ apart in a barotropic fluid. Due to their field of gravity, the fluid will be denser near the spheres. There is no other gravitational field present, the fluid density is ρ_0 and the pressure is p_0 in the absence of the spheres. One may assume that the pressure corrections due to the spheres are everywhere small in comparison with p_0. **(a)** Show that the spheres will repel each other and calculate its magnitude to leading order in a/D. **(b)** Compare with the gravitational attraction between the spheres. **(c)** Under which conditions will the total force between the spheres vanish.

∗ **5.8** Prove without assuming constant gravity that the hydrostatic moment of buoyancy equals (minus) the moment of gravity of the displaced fluid (corollary to Archimedes' law).

∗ **5.9** Assuming constant gravity, show that for a body not in buoyant equilibrium (*i.e.* for which the total force $\boldsymbol{\mathcal{F}}$ does not vanish), there is always a well-defined point $\boldsymbol{x}_0$ such that the total moment of gravitational and buoyant forces is given by $\boldsymbol{\mathcal{M}} = \boldsymbol{x}_0 \times \boldsymbol{\mathcal{F}}$.

∗ **5.10** Let $\mathbf{I}$ be a symmetric 2×2 matrix. Show that the extrema of the corresponding quadratic form $\boldsymbol{n} \cdot \mathbf{I} \cdot \boldsymbol{n} = I_{xx} n_x^2 + 2 I_{xy} n_x n_y + I_{yy} n_y^2$ where $n_x^2 + n_y^2 = 1$ are determined by the eigenvectors of $\mathbf{I}$ satisfying $\mathbf{I} \cdot \boldsymbol{n} = \lambda \boldsymbol{n}$.

∗ **5.11** Show that in a stable orientation the angular frequency of small oscillations around around a principal tilt axis of a ship is

$$\omega = \sqrt{\frac{M g_0}{J}(z_M - z_G)}$$

where J is the moment of inertia of the ship around this axis.

∗ **5.12** A ship has a waterline area which is a regular polygon with $n \geq 3$ edges. Show that the area moment tensor (5-40) has $I_{xx} = I_{yy}$ and $I_{xy} = 0$.

∗ **5.13** A homogeneous cubic block has density equal to half that of the liquid it floats on. Determine the stability properties of the cube when it floats **(a)** with a horizontal face below the center, **(b)** with a horizontal edge below the center, and **(c)** with a corner vertically below the center. Hint: problem 5.12 is handy for **(d)** c, which you should be warned is quite difficult.

