CRITICAL PHENOMENA, REN. THEORY

D16-2 Physics, Northwestern Univ.

take home exam, due 12:30 PM tuesday, June 1, 1998

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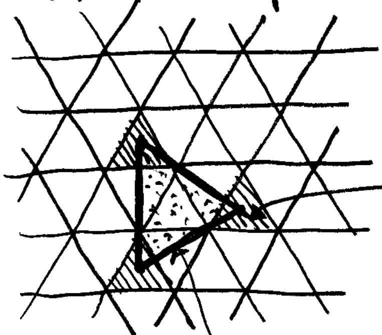
2-d ISING MODEL ON TRIANGULAR LATTICE

Short version: Implement the real-space renormalization on the Ising model on a 2-dimensional triangular lattice

Purpose: learn a systematic real-space renormalization technique

Guided tour version:

Triangular lattice:



$$\mathcal{H} = K \sum_{\langle ij \rangle} S_i S_j + h \sum_i S_i$$

$S_i = \{-1, +1\}$, a block $\sigma_I = \{S_1^I, S_2^I, S_3^I\}$

Block spin by majority rule

$$S_I = \text{sign}\{S_1^I + S_2^I + S_3^I\} \qquad \sigma_I =$$

Lattice constant $a \to a\sqrt{3}$.

(*) Write down the table of 8 distinct $\{\sigma_1 \sigma_2 \sigma_3\}$ configurations: for later use

$S_I = +1 \leftarrow$	↑↑↑,	↑↑↓,	,		(9.148)
$S_I = -1 \leftarrow$	,	,	,	↓↓↓	(9.149)

Coarse grained Hamiltonian:

$$e^{\mathcal{H}'[S_I]} = \sum_{\{\sigma_I\}} e^{\mathcal{H}[S_I,\sigma_I]} \qquad (9.150)$$

We concentrate on $h=0$, no external field case.

$$\mathcal{H} = \underbrace{\mathcal{H}_0}_{\substack{\text{within the} \\ \text{block}}} + \underbrace{V}_{\substack{\text{across the} \\ \text{blocks}}}$$

$$\mathcal{H}_0 = K \sum_I \sum_{ij \in I} S_i S_j \qquad V = K \sum_{I \neq J} \sum_{i \in I, j \in J} S_i S_J$$

We will develop perturbation theory using $\mathcal{H}_0$ as the "nonperturbed" Hamiltonian, with averages defined as

$$\langle A \rangle_0 = \frac{\sum_{\{\sigma_I\}} e^{\mathcal{H}_0[S_I,\sigma_I]} A[S_I,\sigma_I]}{\sum_{\{\sigma_I\}} e^{\mathcal{H}_0[S_I,\sigma_I]}}$$

⊗ Show that

$$\sum_{\{\sigma_I\}} e^{\mathcal{H}_0[S_I,\sigma_I]} = Z_0(K)^M$$

Total # blocks

$$Z_0(K) = e^{3K} + 3e^{-K}$$

⊗ Does this result depend on the block-spin S_I ?

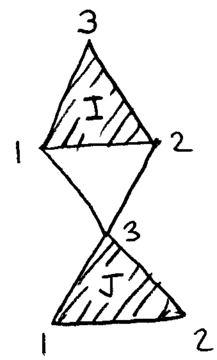
Perturbation theory

⊛ show that

$$\langle e^{V}\rangle_0 = e^{\langle V\rangle_0 + \frac{1}{2}\left(\langle V^2\rangle_0 - \langle V\rangle_0^2\right) + \cdots}$$

$$\mathcal{H}'[S_I] = M \ln Z_0(K) + \langle V\rangle_0 + \frac{1}{2}\left(\langle V_0^2\rangle - \langle V\rangle_0^2\right) + \cdots$$

First order in V



$$V = \sum_{I \neq J} V_{IJ}$$

$$V_{IJ} = K\, S_3^J \left(S_1^I + S_2^I\right)$$

$$\langle V_{IJ}\rangle_0 = 2K \langle S_3^J S_1^I\rangle_0$$

⊛ show that

$$\langle V_{IJ}\rangle_0 = 2K \langle S_3^J\rangle_0 \langle S_1^I\rangle_0$$

$$\langle S_3^J\rangle = S_J\, \Phi(K), \quad \Phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$

$$\mathcal{H}'[S_I] = M \ln Z_0(K) + K' \sum_{\langle IJ\rangle} S_I S_J + O(V^2)$$

$$K' = R(K) = 2K\,\Phi(K)^2$$

Fixed points, critical exponents

⊛ show that the fixed points

$$K^* = 2K^* \Phi(K^*)^2$$

are

$$= 0, \infty, K_c ; \quad \Phi(K_c) = \frac{1}{\sqrt{2}}$$

$$K_c = \frac{1}{4} \ln(1 + 2\sqrt{2})$$

how does that compare to Onsager's exact result $K_c = 0.27\ldots$?

⊛ show that the eigenvalue

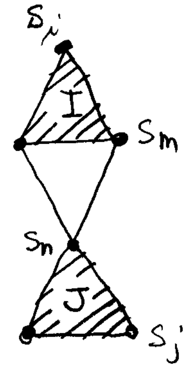
$$\Lambda_1 = \left.\frac{\partial R(K)}{\partial K}\right|_{K_c} \simeq \begin{matrix} 1.81\ldots \ (?) \\ 1.62\ldots \ (?) \\ -0.27\ldots \ (?) \end{matrix}$$

(Onsager says $\Lambda_1 = \sqrt{3}$)

⊛ discuss the behavior of correlation length at K_c

2nd order in V

$$\langle V^2\rangle_0 - \langle V\rangle_0^2 = K \sum_{i,j}\sum_{m,n}\Big(\langle S_i S_j S_m S_n\rangle_0 - \underbrace{\langle S_i S_j\rangle_0 \langle S_m S_n\rangle_0}_{\text{this you already know}}\Big)$$



(*) show that $\langle V^2\rangle_0 - \langle V\rangle_0^2 = 0$ unless

S_i, S_m in the same block

S_j, S_n in the same block

(or other way around)

Forces 2nd and 3rd nearest neighbor interactions!

(If you get next part done, I'll be impressed):

(*) show that in terms of 1st, 2nd and 3rd neighbor interactions there are 3 eigenvalues

$\Lambda_1 = 1{,}77\ldots$

$\Lambda_2 = 0{,}23\ldots$

$\Lambda_3 = -0{,}12\ldots$

(*) What is relevant, what is irrelevant in this renormalized theory as $K \to K_c$? Are you closer to the exact Onsager result?

end of exam