Exercises

Exercise 11.1 **Binary symbolic dynamics.** *Verify that the shortest prime binary cycles of the unimodal repeller of figure 11.8 are* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *Compare with table 11.1. Try to sketch them in the graph of the unimodal function* $f(x)$*; compare ordering of the periodic points with figure 11.9. The point is that while overlayed on each other the longer cycles look like a hopeless jumble, the cycle points are clearly and logically ordered by the alternating binary tree.*

Exercise 11.2 **3-disk fundamental domain symbolic dynamics.** *Try to sketch* $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. *in the fundamental domain, figure 11.6, and interpret the symbols* $\{0,1\}$ *by relating them to topologically distinct types of collisions. Compare with table 11.2. Then try to sketch the location of periodic points in the Poincaré section of the billiard flow. The point of this exercise is that while in the configuration space longer cycles look like a hopeless jumble, in the Poincaré section they are clearly and logically ordered. The Poincaré section is always to be preferred to projections of a flow onto the configuration space coordinates, or any other subset of phase space coordinates which does not respect the topological organization of the flow.*

Exercise 11.3 **Generating prime cycles.** *Write a program that generates all binary prime cycles up to given finite length.*

Exercise 11.4 **A contracting baker's map.** *Consider a contracting (or "dissipative") baker's defined in exercise 4.4.*

The symbolic dynamics encoding of trajectories is realized via symbols 0 $(y \le 1/2)$ *and* 1 $(y > 1/2)$. *Consider the observable* $a(x,y) = x$. *Verify that for any periodic orbit* p $(\epsilon_1 \ldots \epsilon_{n_p})$, $\epsilon_i \in \{0,1\}$

$$A_p = \frac{3}{4}\sum_{j=1}^{n_p} \delta_{j,1} \,.$$

Exercise 11.5 **Reduction of 3-disk symbolic dynamics to binary.**

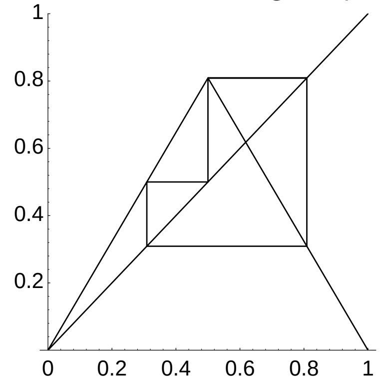
(a) *Verify that the 3-disk cycles*
$\{\overline{1\,2}, \overline{1\,3}, \overline{2\,3}\}$, $\{\overline{1\,2\,3}, \overline{1\,3\,2}\}$, $\{\overline{12\,13}$ + *2 perms.*$\}$,
$\{\overline{121\,232\,313}$ + *5 perms.*$\}$, $\{\overline{121\,323}$+ *2 perms.*$\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ *respectively.*

(b) *Check the reduction for short cycles in table 11.2 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 11.6.*

(c) *Optional: Can you see how the group elements listed in table 11.2 relate irreducible segments to the fundamental domain periodic orbits?*

Exercise 11.6 Unimodal map symbolic dynamics. *Show that the tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the sequence of s_n's into a binary number by the algorithm (11.11). This follows by inspection from the binary tree of figure 11.9.*

Exercise 11.7 "Golden mean" pruned map. *Consider a symmetrical tent map on the unit interval such that its highest point belongs to a 3-cycle:*



(a) *Find the absolute value Λ for the slope (the two different slopes $\pm\Lambda$ just differ by a sign) where the maximum at $1/2$ is part of a period three orbit, as in the figure.*

(b) *Show that no orbit of this map can visit the region $x > (1+\sqrt{5})/4$ more than once. Verify that once an orbit exceeds $x > (\sqrt{5}-1)/4$, it does not reenter the region $x < (\sqrt{5}-1)/4$.*

(c) *If an orbit is in the interval $(\sqrt{5}-1)/4 < x < 1/2$, where will it be on the next iteration?*

(d) *If the symbolic dynamics is such that for $x < 1/2$ we use the symbol 0 and for $x > 1/2$ we use the symbol 1, show that no periodic orbit will have the substring $\_00\_$ in it.*

(e) *On the second thought, is there a periodic orbit that violates the above $\_00\_$ pruning rule?*

For continuation, see exercise 13.6 and exercise 13.8. See also exercise 13.7 and exercise 13.9.

Exercise 11.8 Binary 3-step transition matrix. *Construct [8×8] binary 3-step transition matrix analogous to the 2-step transition matrix (11.15). Convince yourself that the number of terms of contributing to $\mathrm{tr}\,T^n$ is independent of the memory length, and that this $[2^m \times 2^m]$ trace is well defined in the infinite memory limit $m \to \infty$.*

Exercises

Exercise 12.1 **A Smale horseshoe.** *The Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix} \tag{12.10}$$

maps the (x, y) plane into itself - it was constructed by Hénon [3.1] in order to mimic the Poincaré section of once-folding map induced by a flow like the one sketched in figure 11.7. For definitivness fix the parameters to $a = 6$, $b = -1$.

a) *Draw a rectangle in the (x, y) plane such that its nth iterate by the Hénon map intersects the rectangle 2^n times.*

b) *Construct the inverse of the (12.10).*

c) *Iterate the rectangle back in the time; how many intersections are there between the n forward and m backward iterates of the rectangle?*

d) *Use the above information about the intersections to guess the (x, y) coordinates for the two fixed points, a 2-cycle point, and points on the two distinct 3-cycles from table 11.1. The exact cycle points are computed in exercise 17.10.*