4.1.1 Instantaneous rate of shear

The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \,.$$

The infinitesimal deviation vector δx is thus transported along the trajectory $x(x_0, t)$, with time variation given by

$$\frac{d}{dt} \delta x_i(x_0, t) = \sum_j \frac{\partial v_i}{\partial x_j}(x) \bigg|_{x=x(x_0,t)} \delta x_j(x_0, t) \,. \tag{4.1}$$

As both the displacement and the trajectory depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0, t) \to x(t) \to x$, $\delta x_i(x_0, t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

$$\dot{x}_i = v_i(x) \,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x) \delta x_j \tag{4.2}$$

governs the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in T\mathcal{M}_x$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix* or *velocity gradients matrix*

$$A_{ij}(x) = \frac{\partial}{\partial x_j} v_i(x) \tag{4.3}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow. A swarm of neighboring points of $x(t)$ is instantaneously sheared by the action of the stability matrix, $\delta x(t + \delta t) = \delta x(t) + \delta t \, A(x_n) \, \delta x(t)$. A is a tensorial rate of deformation, so it is a bit hard (if not impossible) to draw.

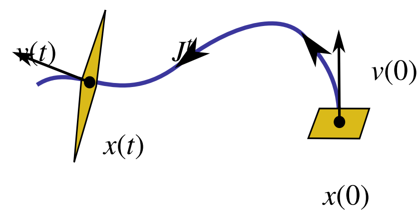
4.1.2 Finite time linearized flow

By Taylor expanding a *finite time* flow to linear order,

$$f_i^t(x_0 + \delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}} \delta x_j + \cdots \,, \tag{4.4}$$

one finds that the linearized neighborhood is transported by the Jacobian matrix

Figure 4.1: For finite times a local frame is transported along the orbit and deformed by Jacobian matrix J^t. As J^t is not self-adjoint, an initial orthogonal frame is mapped into a non-orthogonal one.



$$\delta x(t) = J^t(x_0)\,\delta x_0\,, \qquad J^t_{ij}(x_0) = \frac{\partial x(t)_i}{\partial x(0)_j}\,, \qquad J^0(x_0) = \mathbf{1}\,. \tag{4.5}$$

For example, in 2 dimensions the Jacobian matrix for change from initial to final coordinates is

$$J^t = \frac{\partial(x\,,y\,)}{\partial(x_0, y_0)} = \left[\begin{array}{cc} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{array}\right].$$

The Jacobian matrix is evaluated on a trajectory segment that starts at point $x_0 = x(t_0)$ and ends at point $x_1 = x(t_1)$, $t_1 \geq t_0$. As the trajectory $x(t)$ is deterministic, the initial point x_0 and the elapsed time t in (4.5) suffice to determine J, but occasionally we find it helpful to be explicit about the initial and final times and state space positions, and write

$$J^{t_1-t_0}_{ij} = J_{ij}(t_1;t_0) = J_{ij}(x_1,t_1;x_0,t_0) = \frac{\partial x(t_1)_i}{\partial x(t_0)_j}\,. \tag{4.6}$$

The map f^t is assumed invertible and differentiable so that J^t exists.

4.1.3 Co-moving frames

J describes the deformation of an infinitesimal neighborhood at a finite time t in the co-moving frame of $x(t)$. This deformation of an initial frame at x_0 into a non-orthogonal frame at $x(t)$ is described by the eigenvectors and eigenvalues of the Jacobian matrix of the linearized flow (see figure 4.1),

$$J^t\,\mathbf{e}^{(j)} = \Lambda_j\,\mathbf{e}^{(j)}\,, \qquad j = 1,2,\cdots,d\,. \tag{4.7}$$

Λ_k will always denote the kth *eigenvalue* (the *stability multiplier*) of the finite time Jacobian matrix J^t.
$\lambda^{(k)}$ is the kth *stability exponent*, with real part $\mu^{(k)}$ and phase $\omega^{(k)}$:

$$\Lambda_k = e^{t\lambda^{(k)}} \qquad \lambda^{(k)} = \mu^{(k)} + i\omega^{(k)}\,. \tag{4.8}$$

As J^t is a real matrix, its eigenvalues are either real or come in complex conjugate pairs,

$$\{\Lambda_k, \Lambda_{k+1}\} = \{e^{t(\mu^{(k)}+i\omega^{(k)})}, e^{t(\mu^{(k)}-i\omega^{(k)})}\}\,,$$

with magnitude $|\Lambda_k| = |\Lambda_{k+1}| = \exp(t\mu^{(k)})$. The phase $\omega^{(k)}$ describes the rotation velocity in the plane spanned by the pair of real eigenvectors, $\{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$, with one period of rotation given by $T = 2\pi/\omega^{(k)}$.

$J^t(x_0)$ depends on the initial point x_0 and the elapsed time t. For notational brevity we omitted this dependence, but in general both the eigenvalues and the eigenvectors, $\Lambda_j = \Lambda_j(x_0, t)$, $\cdots$, $\mathbf{e}^{(j)} = \mathbf{e}^{(j)}(x_0, t)$, also depend on the trajectory traversed.

Nearby trajectories diverge exponentially with time along *unstable directions* and approach each other along *stable directions*; however, the distance between trajectories both increases and decreases with time along *marginal directions* at rates slower than exponential. The relative path of nearby trajectories (i.e., diverging, approaching or changing) corresponds to the eigenvalues of the Jacobian matrix with magnitude larger than, smaller than, or equal to 1. In the literature, the adjectives *neutral*, *indifferent* and *center* are often used instead of 'marginal'. Attracting, or stable directions are sometimes called 'asymptotically stable', and so on.

4.2 Computing the Jacobian matrix

know the equations of motion, also know stability matrix A, the instantaneous rate of shear of an infinitesimal neighborhood $\delta x_i(t)$ of the trajectory $x(t)$.
do not know is the finite time deformation
In terms of differential equations, the relationship between these two matrices is found by taking the time derivative of (4.5) and replacing $\dot{\delta x}$ by (4.2)

$$\frac{d}{dt}\,\delta x(t) = \frac{dJ^t}{dt}\,\delta x_0 = A\,\delta x(t) = AJ^t\,\delta x_0\,.$$

Hence the matrix elements of the $[d\times d]$ Jacobian matrix satisfy the 'tangent linear equations'

$$\frac{d}{dt}J^t(x_0) = A(x)\,J^t(x_0)\,, \quad x = f^t(x_0)\,, \quad \text{initial condition } J^0(x_0) = \mathbf{1}\,. \tag{4.10}$$

For autonomous flows, the matrix of velocity gradients $A(x)$ depends only on x, not time, while J^t depends on both the state space position and time. Given a numerical routine for integrating the equations of motion, evaluation of the Jacobian matrix requires minimal additional programming effort; one simply extends the d-dimensional integration routine and integrates the d^2 elements of $J^t(x_0)$ concurrently with $f^t(x_0)$.

So now we know how to compute Jacobian matrix J^t given the stability matrix A, at least when the d^2 extra equations are not too expensive to compute. Mission accomplished.