$$\begin{cases}(\partial_t - \nu\Delta)\Delta_h \psi = \hat{z}\cdot(\nabla\times\vec{b}) \\ (\partial_t - \nu\Delta)\Delta\Delta_h \varphi = -\hat{z}\cdot(\nabla\times\nabla\times\vec{b})\end{cases}$$

	$r=R$	$z=\pm h$
	$\Delta^2\partial_\theta\varphi - r\Delta\partial_{rz}\psi = 0$	$\Delta\Delta_h\psi = 0$
u_r	$\frac{1}{r}\psi_\theta + \varphi_{rz} = 0$	$\frac{1}{r}\psi_\theta + \varphi_{rz} = 0$
u_z	$\Delta_h\varphi = 0$	$\Delta_h\varphi = 0$
u_θ	$-\psi_r + \frac{1}{r}\varphi_{\theta z} = 0$	$-\psi_r + \frac{1}{r}\varphi_{\theta z} = r\Omega_\pm$
gauge	$\varphi = 0$	$\varphi = 0$

Particular solution		Homogeneous solutions		
$E f_\psi^P = b_\psi$	$f_\psi^P\vert = 0$	$E f_\psi^{ki} = 0$	$f_\psi^{1i}(j) = \delta_{ij}$	$f_\psi^{2i}(j) = 0$
$\Delta_h\psi^P = f_\psi^P$	$\psi^P\vert = 0$	$\Delta_h\psi^{ki} = f_\psi^{ki}$	$\psi^{1i}(j) = 0$	$\psi^{2i}(j) = \delta_{ij}$
$E g_\varphi^P = b_\varphi$	$g_\varphi^P\vert = 0$	$E g_\varphi^i = 0$	$g_\varphi^i(j) = \delta_{ij}$	
$\Delta f_\varphi^P = g_\varphi$	$f_\varphi^P\vert = 0$	$\Delta f_\varphi^i = g_\varphi^i$	$f_\varphi^i(j) = 0$	
$\Delta_h\varphi^P = f_\varphi$	$\varphi^P\vert = 0$	$\Delta_h\varphi^i = f_\varphi^i$	$\varphi^i(j) = 0$	

$$\psi = \psi^P + \sum_{i\in B}\sum_{k=1}^{2} c_\psi^{ki}\psi^{ki}$$

$$\varphi = \varphi^P + \sum_{i\in B} c_\varphi^i \varphi^i$$

$$0 = \left(\frac{1}{r}\psi_\theta + \varphi_{rz}\right)(j) = \underbrace{\frac{1}{r}\partial_\theta\psi^P(j)} + \sum_{i\in B}\sum_{k=1}^{2} c_\psi^{ki}\underline{\frac{1}{r}\partial_\theta\psi^{ki}(j)} + \underbrace{\partial_{rz}\varphi^P(j)} + \sum_{i\in B} c_\varphi^i\underline{\partial_{rz}\varphi^i(j)}$$

$$\begin{pmatrix} 0 \\ r\Omega_\pm \end{pmatrix} = \left(-\psi_r + \frac{1}{r}\varphi_{\theta z}\right)(j) = \underbrace{-\partial_r\psi^P(j)} + \sum_{i\in B}\sum_{k=1}^{2} c_\psi^{ki}\underline{(-)\partial_r\psi^{ki}(j)} + \underbrace{\frac{1}{r}\partial_{\theta z}\varphi^P(j)} + \sum_{i\in B} c_\varphi^i\underline{\frac{1}{r}\partial_{\theta z}\varphi^i(j)}$$

$$\begin{pmatrix} 0 \\ \\ 0 \end{pmatrix} \begin{pmatrix} r\Delta\partial_{rz}\psi - \Delta^2\partial_\theta\varphi \\ \\ \Delta\Delta_h\psi \end{pmatrix} = \begin{pmatrix} \underbrace{\frac{1}{r}\Delta\partial_{rz}\psi^P(j)} + \sum_{i\in B}\sum_{k=1}^{2} c_\psi^{ki}\underline{\frac{1}{r}\Delta\partial_{rz}\psi^{ki}(j)} - \underbrace{\Delta^2\partial_\theta\varphi^P(j)} \\ -\sum_{i\in B} c_\varphi^i\underline{\Delta^2\partial_\theta\varphi^i(j)} \\ \underbrace{\Delta\Delta_h\psi^P(j)} + \sum_{i\in B}\sum_{k=1}^{2} c_\psi^{ki}\underline{\Delta\Delta_h\psi^{ki}(j)} \end{pmatrix}$$

$\underbrace{\qquad}$ = elements of rhs vector

$\underline{\qquad}$ = elements of influence matrix

Questions:

1) Subtract function satisfying BCs $u_\theta = r\Omega_\pm$ at $z = \pm h$

2) Special operator E assuming that $\Delta\Delta_h \psi = 0$ at $z = \pm h$

$$(\partial_t - \nu\Delta)\Delta_h \psi = h_\psi$$

$$u_\theta^{BC} = r\,\ell(z) \quad \text{where} \quad \ell(z) = \frac{\Omega_+ - \Omega_-}{2h} z + \frac{\Omega_+ + \Omega_-}{2}$$

$$\vec{u} = \nabla \times \psi \hat{e}_z + \nabla \times \nabla \times \varphi \hat{e}_z$$

$$\nabla \times \psi \hat{e}_z = \hat{e}_r \frac{1}{r}\partial_\theta \psi - \hat{e}_\theta \partial_r \psi$$

$$\nabla \times \nabla \times \varphi \hat{e}_z = \underbrace{\hat{e}_r \partial_{zr}\varphi + \hat{e}_\theta \frac{1}{r}\partial_{z\theta}\varphi}_{\nabla_h \partial_z \varphi} - \hat{e}_z \underbrace{\left(\frac{1}{r}\partial_r r \partial_r \varphi + \frac{1}{r^2}\partial_\theta^2 \varphi\right)}_{\Delta_h \varphi}$$

$$u_r = \frac{1}{r}\partial_\theta \psi + \partial_{zr}\varphi$$

$$u_\theta = -\partial_r \psi + \frac{1}{r}\partial_{z\theta}\varphi$$

$$u_z = -\Delta_h \varphi$$

$$\Delta \vec{u} = \hat{e}_r\left(\Delta u_r - \frac{2}{r^2}\partial_\theta u_\theta\right) + \hat{e}_\theta\left(\Delta u_\theta + \frac{2}{r^2}\partial_\theta u_r\right) + \hat{e}_z \Delta u_z$$

$$\nabla \times \vec{u} = \hat{e}_r\left(\frac{1}{r}\partial_\theta u_z - \partial_z u_\theta\right) + \hat{e}_\theta\left(\partial_z u_r - \partial_r u_z\right) + \hat{e}_z\left(\frac{1}{r}\partial_r r u_\theta - \frac{1}{r}\partial_\theta u_r\right)$$

$$= \hat{e}_r\left(-\frac{1}{r}\partial_\theta \Delta_h \varphi + \partial_{zr}\psi - \frac{1}{r}\partial_{zz\theta}\varphi\right)$$

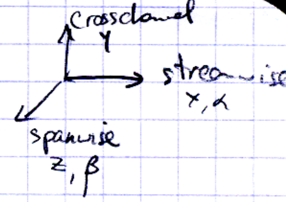
$$+ \hat{e}_\theta\left(\frac{1}{r}\partial_{z\theta}\psi + \partial_{zzr}\varphi + \Delta_h \partial_r \varphi\right)$$

$$+ \hat{e}_z\left(-\frac{1}{r}\partial_r r \partial_r \psi + \frac{1}{r}\partial_r \cancel{\partial_{z\theta}\varphi} - \frac{1}{r^2}\partial_{\theta\theta}\psi - \frac{1}{r}\cancel{\partial_{z\theta r}\varphi}\right)$$

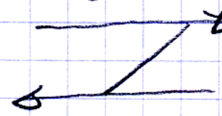
$$= \hat{e}_r\left(\partial_{zr}\psi - \frac{1}{r}\partial_\theta \Delta\varphi\right) + \hat{e}_\theta\left(\frac{1}{r}\partial_{z\theta}\psi + \partial_r \Delta\varphi\right) + \hat{e}_z\left(-\Delta_h \psi\right)$$

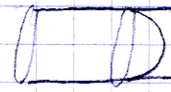
$$= \nabla_h \partial_z \psi - \nabla \times \Delta\varphi \hat{e}_z - \hat{e}_z \Delta_h \psi$$

Plane Channel Flows $\vec{U} = U(y)\hat{e}_x$



Plane Poiseuille	Plane Couette	Pipe Poiseuille
$U(y) = 1-y^2$	$U(y) = y$	$U(r) = 1-r^2$





Plane Poiseuille	Plane Couette	Pipe Poiseuille
linear instability for $Re = 5772$ (first numerical verif of spectral methods)	linearly stable for all Re (proved, Romanov)	thought linearly stable for all Re
turb for $Re \gtrsim 1000$	lab & comp: turb for $Re \gtrsim 325$	turb for $Re > 2000$

What is linear stability?

Reynolds Orr equation: Take $U+u$, $P+p$ for U, P satisfying N-S

$$\frac{\partial}{\partial t}(U+u) + (U+u)\cdot\nabla(U+u) = -\nabla(P+p) + \nu\nabla^2(U+u) \qquad \nabla\cdot(U+u) = 0$$

$$\frac{\partial u}{\partial t} + U\cdot\nabla u + u\cdot\nabla U + u\cdot\nabla u = -\nabla p + \nu\nabla^2 u \qquad \nabla\cdot u = 0$$

Scalar product with u to derive eq for disturbance energy

$$u_i\frac{\partial u_i}{\partial t} + u_i U_j \partial_j u_i + u_i u_j \partial_j U_i + u_i u_j \partial_j u_i = -u_i \partial_i p + \nu u_i \partial_j^2 u_i$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u_i^2\right) + U_j \partial_j\left(\tfrac{1}{2}u_i^2\right) = -u_i u_j S_{ij} - 2\nu s_{ij}^2 \quad -\partial_j\left(\tfrac{1}{2}u_i u_i u_j + u_j p - 2\nu u_i s_{ij}\right)$$

Reynolds Orr equation:

$$\frac{\partial E_V}{\partial t} \equiv \frac{\partial}{\partial t}\tfrac{1}{2}u_i^2 = -\underbrace{\int u_i u_j S_{ij}}_{\text{exchange energy with mean flow}} - \underbrace{\frac{2}{R}\int s_{ij}^2}_{\text{dissipation}}$$

where $S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$

$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

No nonlinear terms $u_i u_j u_k$

So energy of perturbation grows via linear mechanisms below Re_E, no growth at all

Instantaneous growth rate $\frac{1}{E_V}\frac{\partial E_V}{\partial t}$ is independent of disturbance amplitude

	Re_E	Re_L
plane Poiseuille	49.6	5772
plane Couette	20.7	∞
pipe Poiseuille	81.5	∞
Blasius	—	315

linear stability equs. for $U(y)\hat{e}_x$

$$\frac{\partial \vec{u}}{\partial t} + U\frac{\partial \vec{u}}{\partial x} + vU'\hat{e}_x = -\nabla p + \frac{1}{R}\nabla^2\vec{u} \quad \text{and} \quad \nabla\cdot\vec{u} = 0$$

$$\nabla\cdot \Rightarrow \underbrace{\frac{\partial \nabla\cdot u}{\partial t}}_{0} + \underbrace{\nabla\cdot\left(U\frac{\partial \vec{u}}{\partial x}\right) + \frac{\partial}{\partial x}(vU')}_{} = -\nabla^2 p + \underbrace{\frac{1}{R}\nabla^2\nabla\cdot\vec{u}}_{0}$$

$$U\frac{\partial \nabla\cdot\vec{u}}{\partial x} + \frac{\partial \vec{u}}{\partial x}\cdot\nabla U + U'\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x}U' + U'\frac{\partial v}{\partial x} \Rightarrow \nabla^2 p = -2U'\frac{\partial v}{\partial x}$$

$$\frac{\partial \nabla^2 v}{\partial t} + \underbrace{\nabla^2\left(U\frac{\partial v}{\partial x}\right)} = \underbrace{-\frac{\partial}{\partial y}\nabla^2 p} + \frac{1}{R}\nabla^4 v$$

$$-\frac{\partial}{\partial y}\left(-2U'\frac{\partial v}{\partial x}\right) = 2U'\frac{\partial^2 v}{\partial x\partial y} + 2U''\frac{\partial v}{\partial x}$$

$$(\nabla^2 U)\frac{\partial v}{\partial x} + 2\nabla U\cdot\nabla\left(\frac{\partial v}{\partial x}\right) + U\nabla^2\frac{\partial v}{\partial x}$$

$$U''\frac{\partial v}{\partial x} + 2U'\frac{\partial^2 v}{\partial x\partial y} + U\frac{\partial}{\partial x}\nabla^2 v$$

$$\frac{\partial}{\partial t}\nabla^2 v + U\frac{\partial}{\partial x}\nabla^2 v - U''\frac{\partial v}{\partial x} - \frac{1}{R}\nabla^4 v = 0$$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} - \frac{1}{R}\nabla^4\right]v = 0$$

Orr-Sommerfeld equation for velocity normal to plates

Normal vorticity: $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right]\eta = -U'\frac{\partial v}{\partial z}$$

BCs $\quad u, v, w = 0$ at walls at $y = \pm 1$

$\Updownarrow$

$v, v', \eta = 0$ at walls

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial w}{\partial z} = 0$ at walls $\Rightarrow \frac{\partial v}{\partial y} = 0$ at walls

$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} = 0$ at walls $\Rightarrow \frac{\partial \eta}{\partial y} = 0$ at walls

v, v', η bounded in far field

Equations are <u>homogeneous</u> in x, z, t $\Rightarrow \begin{Bmatrix} v \\ \eta \end{Bmatrix} = \begin{Bmatrix} v \\ \eta \end{Bmatrix}(y)\, e^{i\alpha x} e^{i\beta z} e^{-i\alpha c}$

c may be complex

$c_i > 0 \Rightarrow$ instability

$$\left[(-i\alpha c + Ui\alpha)(D^2 - k^2) - U''i\alpha - \frac{1}{R}(D^2 - k^2)^2\right]v = 0$$

$$\left[-i\alpha c + Ui\alpha - \frac{1}{R}(\nabla^2 - k^2)\right]\eta = -U'i\beta v$$

$k^2 = \alpha^2 + b^2$

$D \equiv \frac{\partial}{\partial y}$

Note v eqn stands alone but η eqn forced by v

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rayleigh equation: OS without viscosity (1880)

$$\left[(-i\alpha c + U i\alpha)(D^2-k^2) - U'' i\alpha\right] v = 0$$

$$(D^2-k^2)v = \frac{U'' i\alpha v}{-i\alpha c + U i\alpha} = \frac{U'' v}{U-c}$$

$$\int_{-1}^{1} dy\, v^* (D^2-k^2) v = \int_{-1}^{1} dy\, \frac{v^* U'' v}{U-c}$$

$$\int_{-1}^{1} dy \frac{|v|^2 U''}{U-c} = \int dy \frac{|v|^2 U''}{|U-c|^2}\left(U-(c_r - i c_I)\right) = \int_{-1}^{1} dy \frac{|v|^2 U''(U-c_r)}{|U-c|^2} + i c_i \int_{-1}^{1} dy \frac{|v|^2 U''}{|U-c|^2}$$

Imag part of eqn: $c_I \int dy \frac{|v|^2 U''}{|U-c|^2} = 0$

Instability: $c_I > 0$ $\Rightarrow$ integral must be zero $\Rightarrow$ U'' must change sign somewhere

$\Rightarrow$ U must have $\boxed{\text{inflection point } U''=0}$

Fjortoft used real part of eqn to conclude that U' must be <u>max</u> (not min) for instability

Return to OS: Squires Theorem (1933)

$$\left[(U-c)(D^2-k^2) - U'' - \frac{1}{i\alpha R}(D^2-k^2)^2\right] v = 0$$

$$\left[(U-c)(D^2-\alpha^2) - U'' - \frac{1}{i\tilde{\alpha}\tilde{R}}(D^2-\tilde{\alpha}^2)^2\right] v = 0$$

$\tilde{\alpha} = k$

$\tilde{\alpha}\tilde{R} = \alpha R$

$\tilde{\beta} = 0$

$$\Rightarrow (\alpha, \beta, R) \Rightarrow \left(\sqrt{\alpha^2+\beta^2},\ 0,\ \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} R\right)$$

higher x wavenum

longer x wavelength

2D only

lower Reynolds #

$\Rightarrow$ Solve only 2D problem

But 2D results don't explain transition to turbulence

Laurette Tuckerman
Georgia Tech Lecture Nov 7 '02
"All that is known about pipe
and channel flows"

Transient growth

$u = \sum c_k \psi_k$

$A u_k = \lambda_k u_k$ with all $|\lambda_k| < 1$

If ψ_k orthonormal

$1 = \|u\|^2 = \sum c_k^2 \|\psi_k\|^2 = \sum c_k^2$

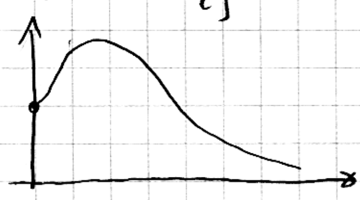
$1 = \|Au\|^2 = \sum c_k^2 \lambda_k^2 \|\psi_k\|^2 = \sum c_k^2 \lambda_k^2 < \sum c_k^2 = 1$

But if ψ_k not orthogonal

$1 = \|u\|^2 = \sum_{ij} c_i c_j \langle \psi_i, \psi_j \rangle$

$\|Au\|^2 = \sum_{ij} \lambda_i c_i \lambda_j c_j \langle \psi_i \psi_j \rangle$ may not be smaller than $\|u\|^2$

G(t)



where $\frac{\partial u}{\partial t} = Lu$ $\quad G(t) = \max_{u_0} \frac{\|u(t)\|}{\|u_0\|^2}$ $\quad G = \max_t G(t)$

Flows	max $G(t) \times 10^3$	t_{max}	α	β
plane Poiseuille	$0.2 R^2$	$.08 R$	0	2
plane Couette	$1.2 R^2$	$.12 R$	$35/R$	1.6
pipe Poiseuille	$0.1 R^2$	$.05 R$	0	1
Blasius bdy layer	$1.5 R^2$	$.78 R$	0	0.65