Chapter 15 Waves

1.Exercise 15.3 Ship Waves

(a) Consider a plane wave with frequency ω_0 and wave vector $\vec{k_0}$ as measured in the water's frame, and ω and $\vec{k}$ as measured in the boat's frame. For an observer with position $\vec{x_0}$ in the water's frame and $\vec{x}$ in the boat's frame, the phase he measures is:

$\vec{k_0} \cdot \vec{x_0} - \omega_0 t$ in terms of water's frame variables

and $\vec{k} \cdot \vec{x} - \omega t$ in terms of boat's frame variables

Since the phase is invariant under the change of reference frame, we can equate the above two expressions and then differentiate both sides with respect to t, noting that for an observer moving together with the boat, $d\vec{x_0}/dt = \vec{u}, d\vec{x}/dt = 0$, we get:

$\omega = \omega_0 - \vec{k_0} \cdot \vec{u}$

By looking at Fig 15.3 and use $\vec{k}$ to denote the wave vector as measured in the water's frame(as the text does), we get

$\omega = \omega_0 + uk\cos\phi$

(b) θ is the angle between $\vec{V_{g0}} - \vec{u}$ and $\vec{u}$, elementary trigonometry then gives(see the sketch on the right),

$\tan\theta = V_{g0}\sin\phi/(u + V_{g0}\cos\phi)$

For stationary wave pattern $\omega = 0$, using the $\omega(k)$ we got in part (a), we see that

$\omega_0(k) = -uk\cos\phi$

(c) For capillary waves, $\omega_0 \approx \sqrt{(\gamma/\rho)k^3}, V_{g0} = \partial\omega_0/\partial k = (3/2)\sqrt{(\gamma/\rho)k}$

Plugging these and $u = -\omega_0/(k\cos\phi)$ into the expression for $\tan\theta$, we get

$\tan\theta = (3\tan\phi)/(1 - 2\tan^2\phi)$

Capillary wave pattern for a given θ exists only when we can find some $\phi \in (\pi/2, \pi)$ (i.e. only forward waves can contribute to the pattern) satisfying the above equation. And it's easy to show that indeed for any θ we can find such a ϕ given by:

$\tan\phi = (-3 - \sqrt{9 + 8\tan^2\theta})/(4\tan\theta)$ when $\theta < \pi/2$

and $\tan\phi = (-3 + \sqrt{9 + 8\tan^2\theta})/(4\tan\theta)$ when $\theta > \pi/2$

For gravity waves, $\omega_0 \approx \sqrt{g_e k}$, and $V_{g0} = (1/2)\sqrt{g_e/k}$, and we get

$\tan\theta = (-\tan\phi)/(1 + 2\tan^2\phi)$

Only when $\theta < \arcsin(1/3)$ can we find some $\phi \in (\pi/2, \pi)$ satisfying this equation, which is:

$\tan\phi = (-1 \pm \sqrt{1 - 8\tan^2\theta})/(4\tan\theta)$ (both solutions are valid)

This means that the gravity-wave pattern is confined to a trailing wedge with an opening angle $\theta_{gw} = 2\arcsin(1/3)$.

Q.E.D.

2.Tsunamis from Japan

We can treat this as a 2-dimensional problem, i.e., only consider the horizontal components of the velocity, which are almost independent of z. In what follows, ∇ is the 2-dimensional derivative operator.

(a) The mass per unit area is $\rho(D + \xi)$, and the mass flux per unit length is $\rho(D + \xi)\vec{v} \approx \rho D\vec{v}$, to the first order in perturbation. Then by mass conservation,

$\partial[\rho(D + \xi)]/\partial t + \nabla \cdot (\rho D\vec{v}) = 0$

$\Rightarrow \partial\xi/\partial t + \nabla \cdot (D\vec{v}) = 0$ (i) (assuming constant ρ)

The Navier-Stokes equation in this case is: $\partial\vec{v}/\partial t = -\nabla P/\rho + \vec{g}$, whose vertical component tells us $P = \rho g(\xi - z)$, and whose horizontal components then tell us $\partial\vec{v}/\partial t = -g\nabla\xi$ (ii)

Applying ∂t to both sides of eqn. (i) and then plugging in eqn (ii), we get

$\partial^2\xi/\partial^2 t = g\nabla\cdot(D\nabla\xi)$

(b) By plugging in a plane wave solution to the wave equation, we find the dispersion relation:

$\omega = k\sqrt{gD}\sqrt{1 - i(\nabla D/D)\cdot(\vec{k}/k^2)} \approx k\sqrt{gD}\,[1 - (i/2)(\nabla D/D)\cdot(\vec{k}/k^2)]$

The imaginary part of ω only affects the decaying(or growing) of the the wave amplitude but not it's propagation direction, and furthermore it's smaller than the real part by a factor of $\lambda/l \ll 1$ (where λ is the wave length and l is the scale over which D varies). Thus we take $\omega \approx k\sqrt{gD}$.

Using the Hamilton's equations of motion introduced in Chapter 6 Geometrical Optics, we get

$d\vec{x}/dt = \nabla_{\vec{k}}\omega = \sqrt{gD}\,(\vec{k}/k)$

and $d\vec{k}/dt = -\nabla_{\vec{x}}\omega = -(k/2)\sqrt{g/D}\,\nabla_{\vec{x}}D$

Thus we see the direction of wave propagation is always deflected towards the shollower part of the ocean.

(c) Create in the bottom of the Pacific Ocean a ridge going from Japan to LA(with equi-depth contours being elliptical curves and LA being at the focus)! This ridge will act as a lens, focusing those Tsunamis towards LA. Note that only very slight deflection(ver slight difference in ocean depth) is sufficient: Assume Japan extends 500km long, and the distance between Japan and LA is about 10,000km. Then only about 500km/10000km ~ 5% change in the ocean depth is needed. .

Q.E.D.

3.Solitary Waves in a Deformable Conduit

(a) Assuming that a vary slowly with height, the solution for this part can be found in Section 12.4.5 Blood Flow B&T. The only difference between there and here is that here we no longer neglect gravity. By adding a $\rho_1 g$ term to the driving force, we get [see eqn.(12.72) Poiseuille's Law B&T],

$Q = -\pi a^4(\partial p_1/\partial z + \rho_1 g)/(8\eta_1)$

(b) The force per unit area on the conduit wall applied by fluid-2 consists of two parts: the pressure contribution p_2 and the viscous contribution $f_{viscous}$. Since fluid-2 has no vertical motion, we have p_2 =constant$-\rho_2 gz$. Now let's calculate $f_{viscous}$ in cylindrical coordinates.

The velocity field due to the change in a is: $\vec{v} = v_r\vec{e_r}$, with $v_r = (a/r)(\partial a/\partial t)$

$f_{viscous} = T_{rr} = -2\eta_2\sigma_{rr} = -2\eta_2[(2/3)\partial v_r/\partial r - (1/3)v_r/r] =)\ 2\eta_2(a/r^2)(\partial a/\partial t)$ where we have used formulas analogous to those in Box 10.2 B&T

$f_{viscous}(r = a) = 2\eta_2(1/a)(\partial a/\partial t)$

Thus the force per unit area applied by fluid-2 on the wall is constant$-\rho_2 gz + 2\eta_2(1/a)(\partial a/\partial t)$, and similarly the force per unit area applied by fluid-1 on the wall is $p_1 + 2\eta_1(1/a)(\partial a/\partial t)$. Equating these two forces and noting that $\eta_2 \gg \eta_1$, we get

$p_1 = -\rho_2 gz + 2\eta_2(1/a)(\partial a/\partial t)$ +constant

Thus the PDE relating Q and a is:

$Q = (\pi a^4/8\eta_1)\{\rho_2 g - \rho_1 g - \partial[2(\eta_2/a)(\partial a/\partial t)]/\partial z\}$

The other PDE relating Q and a is given by mass conservation(i.e. volume conservation, assuming constant ρ) for fluid-1 as follows,

$\frac{d}{dt}\int_{z_1}^{z_2}\pi a^2 dz = Q(z_1) - Q(z_2) \Rightarrow 2\pi a(\partial a/\partial t) = -\partial Q/\partial z$

(c) $2\pi a(\partial a/\partial t) = \partial(\pi a^2)/\partial t = -\partial Q/\partial z$

let $Q = Q_0 + Q_1 f(z - ct)$, then $\pi a^2 = (Q_1/c)f(z - ct)$, where we've set the additive constant to zero without loss of generality.

Now $Q = \pi a_0^2 v_0$ far away from the solitary wave
Define $f \equiv 1$ far away from the solitary wave.
Then $\pi a_0^2 = Q_1/c, \pi a_0^2 v_0 = Q_0 + Q_1$, so $Q_{01} = \pi a_0^2(v_0 - c)$
Now $Q = (\pi a^4/8\eta_1)\Delta\rho g - (\pi a^4/8\eta_1)\eta_2\partial[(\partial a^2/\partial t)/a^2]/\partial z$
$= (\pi a^4/\pi a_0^4)v_0\pi a_0^2 - (\pi a^4/8\eta_1)\eta_2\partial[(\partial a^2/\partial t)/a^2]/\partial z$
So
$\pi a_0^2(v_0 - c) + \pi a_0^2 cf = \pi a_0^2 v_0 f^2 + (\pi\eta_2/8\eta_1)cf^2(f'/f)'$
where f' means $df/d\xi$, $\xi \equiv z - ct$
Divide by $\pi a_0^2 v_0$
$(1 - c/v_0) + (c/v_0)f = f^2 + kf^2(f'/f)'$
where we have absorbed a bunch of constants into k.
but $f^2(f'/f)' \equiv (f^3/2)d[(f')^2/f^2]/df$
thus $(1 - c/v_0)/f^3 + c/(v_0 f^2) - 1/f = (k/2)d[(f')^2/f^2]/df$
Now $f' = 0$ at $f = A$(peak of the wave) and $f = 1$(far away from the wave)
Therefore
$(1 - c/v_0)\int_1^A df/f^3 + (c/v_0)\int_1^A df/f^2 - \int_1^A df/f = 0$
$(1 - c/v_0)(1 - 1/A^2)/2 + (c/v_0)(1 - 1/A) - \ln A = 0$
So $c/v_0 = 2(\ln A - 1/2 + 1/2A^2)/(1 - 2/A + 1/A^2)$

Q.E.D.

4. Kursk Submarine Disaster

(a) Consider the total energy E_{total} of the system consisting of the bubble and the surrounding water.

$E_{total} = U + E_{water}$, where U is the internal energy of the gas in the bubble, and E_{water} is the kinetic energy of the surrounding water(the kinetic engery of the gas is negligible).

Assuming a spherically sysmetric velocity field for the water: $\vec{v} = v_r\vec{e_r} = (a^2/r^2)(da/dt)$, and integrating the velocity field from $r = a$ to $r = \infty$, we find $E_{water} = 2\pi\rho_0 a^3(da/dt)^2$

Now this bubble-water system does work "on the infinity" at a rate $4\pi a^2(da/dt)P_0$, where P_0 is the ambient pressure.

Then by engergy conservation,
$-4\pi a^2(da/dt)P_0 = dE_{total}/dt = (\partial U/\partial V)_s 4\pi a^2(da/dt) + 2\pi\rho_0[3a^2(da/dt)^3 + 2a^3(da/dt)(d^2a/dt^2)]$

Using $(\partial U/\partial V)_s = -P$ with P being the pressure of the gas, and simplifying the above eqn., we get

$[P(a) - P_0] = (1/2)\rho_0[3(da/dt)^2 + 2a(d^2a/dt^2)]$

Linearizing the above equation around the equilibrium, $a = a_0 + \delta a$, where a_0 satisfies $P(a_0) = P_0$, we get the equation for small oscillation of the radius,

$d^2\delta a/dt^2 + (c^2/a_0^2)(3\rho_g/\rho_0)\delta a = 0,$ where $c = [(\partial P/\partial\rho_g)_s]^{1/2}$

Thus the frequency of small oscillation is given by

$\omega = (c/a_0)(3\rho_g/\rho_0)^{1/2}$

(b) At a depth of $\sim 100m$, $P_0 \sim 11 atmosphere$, take $\rho_g \sim 1kg/m^3, c \sim 340m/s$, we solve for a_0 from the above expression with $\omega = 1.45Hz$ and get $\alpha_0 \sim 13m$. And the work that must be done to create it is $\sim (4\pi a_0^3/3)P_0 \sim 8 \times 10^{16} ergs \sim 2$ metric tons of TNT equivalent, which agrees with the "expert" estimates.

Q.E.D.