Chapter 16 Supersonic Flow

Problem 1. The "Evaporating" of a Planet

(a) It's straightforward to derive these equations(you just have to write out the relevant conservations laws assuming spherical symmetry) so I omit the derivation here.

(b) The mass conservation equation can be rewritten as $\rho = |dM/dt|/(4\pi r^2 u)$. Also, we know that $P = \rho c^2/\gamma$

Plugging the above expressions into energy conservation equation and integrating once, we get

$$\frac{1}{2}u^2 + \frac{1}{\gamma-1}c^2 - \frac{GM}{r} = F(r) \qquad \text{(I)}$$

where we define $F(r) \equiv [\int_{r_m}^{r} \Gamma(r) 4\pi r^2 dr + \text{integration constant}]/|dM/dt|$

Using equation (I) to express c in terms of u, we can turn the momentum conservation equation into a first-order differential equation for $u(r)$

$$\frac{1}{u}\frac{du}{dr}\{\frac{\gamma+1}{2}u^2 - (\gamma-1)[F(r) + \frac{GM}{r}]\}$$

$$= (2\gamma-3)\frac{GM}{r^2} + (\gamma-1)[\frac{2}{r}F(r) - 4\pi r^2\Gamma(r)/|dM/dt|] - (\gamma-1)\frac{u^2}{r} \qquad \text{(II)}$$

Now let's use equation (I) to eliminate $F(r)$ from equation (II), after some algebra, the resulting equation turns out to be:

$$(u^2 - c^2)\frac{1}{u}\frac{du}{dr} = \frac{2}{r}[c^2 - \frac{GM}{2r} - \frac{\gamma-1}{2}\frac{\Gamma r}{\rho u}] \qquad \text{(III)}$$

At the critical radius r_s, the LHS of equation (III) vanishes and we get

$$u_s^2 = c_s^2 = \frac{GM}{2r_s} + \frac{\gamma-1}{2}(\frac{\Gamma r}{\rho u})_s$$

(c) Evaluate equation (I) at r_m and r_s respectively, and then take the difference. Neglecting u_m^2 and P_m/ρ_m(i.e. c_m^2) relative to GM/r_m, we get the claimed expression for $|dM/dt|$

(d) When the UV flux isn't very high, we should expect the sonic radius r_s to be much larger than r_m. Also the expression for u_s^2 we got in part (b) suggests that $(\frac{\Gamma r}{\rho u})_s$ is of the same order of magnitude as $\frac{GM}{2r_s}$. Thus we can retain only the $\frac{GM}{r_m}$ term in the denominator of the RHS of the equation we got in part (c). The result is then:

$$|dM/dt| \approx \int_{r_m}^{r_s} 4\pi r^2 \rho\kappa F_{UV} dr/(GM/r_m) \approx F_{UV}\frac{r_m}{GM}4\pi\int_{r_m}^{\infty} r^2\rho\kappa dr \approx 4\pi r_m^3 F_{UV}/(GM)$$

One way to understand this result is: the gravitational binding energy of the planet is $E_{GB} \sim 3GM^2/5r_m$, while the UV energy flux is $4\pi r_m^2 F_{UV}$

Then $dE_{GB}/dt \sim 4\pi r_m^2 F_{UV}$ gives the same result up to some overall constant of order unity.

(e) Plugging the given numbers into the formula got in part (d), we find the survival time of the planet to be 4.4×10^8 year.

Q.E.D.

Problem 2 Projectile Entering the Earth Atmosphere

(a) The drag stress on the projectile is the downstream pressure P_2 of the shock front (see B&T 16.5) and is given by

$P_2 \approx P_1[2\gamma/(\gamma+1)](v_1^2/c_1^2)$ (see B&T eqn. (16.43))

Using $c_1^2 = \gamma P_1/\rho_1$, we get $P_2 \approx [2/(\gamma+1)]\rho_1 v_1^2$. For air, $\gamma \approx 1.4$, we get $P_2 = \rho_1 v_1^2$ up to an overall proportionality constant of order unity. In what follows we omit the subscripts of P, ρ, and v.

The equation of motion is therefore $mdv/dt = -\rho v^2 A$, where $m = (4\pi/3)r^3\rho_0$, $A = 4\pi r^2$, and $\rho = \rho_g \exp(-z/H)$

Writing d/dt as $v(d/dz)$, we get

$(1/v)(dv/dz) = [3\rho_g/(\rho_0 r)]\exp(-z/H)$ with initial condition $v(z=\infty) = v_0$

(b) Solving the above equation of motion is straightforward and we get

$v = v_0 \exp[(-3\rho_g H/\rho_0 r)e^{-z/H}]$

(c) Using the solution we got in (b) and differentiating $P = \rho v^2$ with respect to z, we find that at $z = H\ln(6\rho_g H/\rho_0 r)$ the drag stress reaches a maximum value of
$P_{\max} = (r/6eH)\rho_0 v_0^2 = (r/10km) \times 4 \times 10^5 bar$ after plugging all the numbers.

We see that $P_{\max} = 100bar$ when $r = 2.5m$

This explains why small projectiles can reach the ground intact while large ones will ”explode” in the air.

Q.E.D.

Problem 3 Stellar Winds [by A. Dvoretskii]

(i) The energy in the stellar wind grows linearly with time, so

$$E = \epsilon t = K\rho_0 R^5 t^{-2}$$

following the Sedov-Taylor analysis. Hence,

$R \propto t^{\frac{3}{5}}$ and

$$v = \dot{R} = \frac{3}{5}\frac{R(t)}{t}$$

The speed of the post-shock gas is related to the speed of the shock front via strong shock jump conditions

$$u = \frac{2}{\gamma+1}\dot{R}$$

(ii) The star has been losing its mass at a constant rate and the speed of the wind is constant. Conservation of flux therefore gives $\rho \propto \frac{1}{R^2}$.

Hence,

$$E \approx \rho V\dot{R}^2 \approx \frac{1}{R^2}R^3\left(\frac{R}{t}\right)^2 \approx \frac{R^3}{t^2}$$

Therefore, $R \propto t^{\frac{2}{3}}$ and

$$v = \dot{R} = \frac{2}{3}\frac{R(t)}{t}$$

Q.E.D.