Physics 136b

Homework associated with **Chapter 17,** Convection
Available Feb 26, 2001
Due, **Monday** March 5

As usual, if any problem is trivial for you, do not do it – simply state that it is trivial and pick some other problem or make up your own. Contact me (djs@gps.caltech.edu) if you have a question or concern about the problems, all three of which are new.

1. The China Syndrome

This refers to a 1979 movie of the same name, and concerns the possible consequences of a nuclear reactor melt down. The proposed scenario is silly, but the underlying fluid dynamical problem is interesting and has relevance to the formation of metal cores in solid planets.

Consider a spherical body, radius R, that is melting its way down into the solid Earth. It does so by melting the column of rock in its path. This melt then flows as thin layer around the surface of the sphere. It is assumed that the sphere remains intact and has a density $\rho_m > \rho_r$, the density of the neighboring rock. The energy required to melt the rock is dominated by the latent heat L and the source of heat is (possibly) an intrinsic heat flux provided at the surface of the sphere (F) plus gravitational energy release as the sphere sinks. This is a complex problem and we will seek only order of magnitude understanding. The other relevant parameters are: ΔT_1 = the difference between the ambient rock temperature and its melting point, ΔT_2 = a typical temperature difference within the melt film, i.e. $k\Delta T_2/\delta$ is the heat flow outwards across the film of thickness δ and thermal conductivity k; C_p = specific heat of the material (liquid or solid), η = dynamical viscosity of the melt layer, g = gravitational acceleration.

(a) Explain why it is plausible that

$$[\rho C_p (\Delta T_1+\Delta T_2) + \rho L]u_0 \sim k\Delta T_2/\delta \sim \eta u^2/\delta + F$$

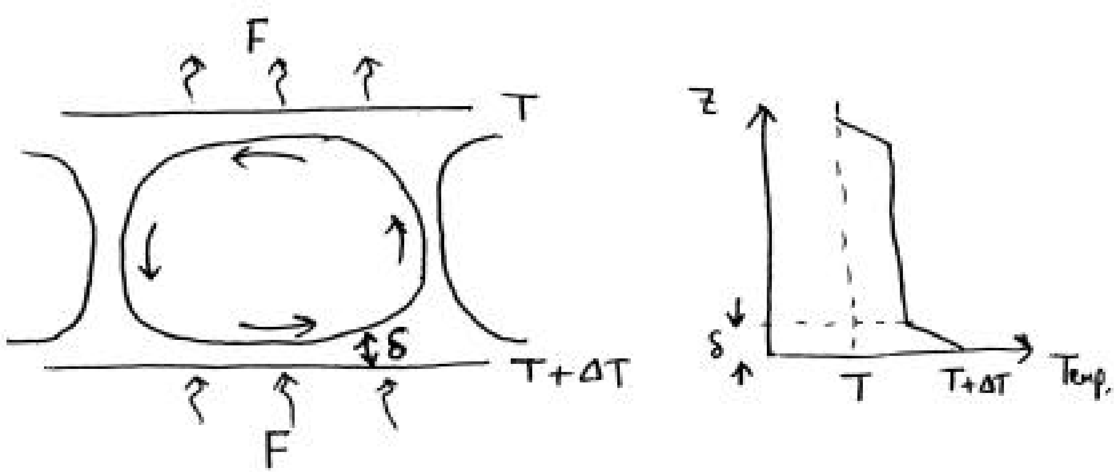
where u_0 is the velocity of the sphere (positive downwards) and u is the mean velocity of the melt within the film (upwards). Because I'm basically giving you the answer, take particular care to understand and explain the (often hidden) assumptions that lie behind these estimates.

(b)Write down the other equations that complete specification for the problem (one that relates u to the known pressure gradient in the melt layer and one that relates u to u_0). Show that there is a critical sphere radius above which there exist solutions even when F =0 . These are "runaway" solutions in which gravitational energy release is sufficient to cause lubricated descent of the sphere. What is the critical size? (Use L ~ 4 x 10^9 erg/g, ρ_m ~8 g/cm^3, ρ_r~4 g/cm^3, g ~ 10^3cm/sec^2 and ignore the ΔT terms.)

(c)Estimate the time it takes for the sphere to sink to the core (~3 x 10^3 km away) and the thickness of the melt layer, in two cases: (i) The nuclear reactor case:R=10^4 cm, F = 10^8 erg/cm^2.sec ; (ii) the core formation case: F=0 and R = twice the critical size found in (b).

A paper on this subject is Turcotte and Emerman, *J. Geophys. Res.* **88 Suppl.** B91-B96 (1983).

2. Boundary Layer theory for High Viscosity Convection



For simplicity, consider free slip top and bottom boundary conditions with heat flow F. At large Prandtl number and high Rayleigh number (but not so high that the convection becomes chaotic), a simple "conveyor belt" picture describes the fluid motions. Consider, for example, the bottom of the convecting layer. Material flows along the bottom boundary for a horizontal distance of order the depth of the fluid layer = D , and heat diffuses up into a thermal boundary layer of thickness δ. This hot sheet of fluid then rises to the top boundary, where it releases its heat and becomes colder than the interior fluid. The fluid, now cooled, then sinks under negative buoyancy and the entire cycle is repeated. The only significant vertical temperature gradients are in the top and bottom boundary layers; nearly all the rest of the fluid in between is neutrally buoyant.

(a) By considering the diffusion of heat into the bottom or out of the top, derive a relationship between δ, the typical fluid velocity v, the size of the system L and thermal diffusivity κ.

(b) By considering the gravitational forces and drag forces for upwelling (hot) or downwelling (cold) sheets of thickness $\sim\delta$, derive an approximate relationship involving v, L, δ, ν (the kinematic viscosity of the fluid), the temperature difference driving the convection, gravitational acceleration g and coefficient of thermal expansion α.

(c) At the top or bottom, the heat flux F must be entirely conductive. Hence show that the heat flux for the system as a whole is larger than $k\Delta T/D$ (the value if conduction alone operated through the entire layer) by a factor of $\sim Ra^{1/3}$ where $Ra = g\alpha\Delta T D^3/\nu\kappa$.

(d) Estimate the convective velocities for Earth, assuming $\nu \sim 10^{21}$ cm^2/sec, $\alpha \sim 2 \times 10^{-5} K^{-1}$, $\kappa \sim 10^{-2}$ cm^2/sec, $D \sim 3 \times 10^8$ cm, $\Delta T \sim 10^3$ K and compare with plate motions (few cm/year).

This is a standard scaling analysis and can be found, for example in the textbook *Geodynamics* by Turcotte and Schubert.

3. Inviscid Convection with Rotation

Consider a fluid in which the vertical temperature gradient deviates from adiabaticity by an amount β, defined so that a positive value means superadiabatic (i.e., potentially unstable to convection). In other words, the linearized energy equation takes the form

$$\partial\theta/\partial t = \beta u_z$$

where thermal diffusion has been neglected, u_z is the vertical component of fluid motion and θ is the temperature difference relative to the adiabat. The fluid is rotating , so the linearized, inviscid equation of motion takes the form

$$\partial \mathbf{u}/\partial t + 2\,\Omega\ \mathbf{x}\ \mathbf{u} = -\nabla p/\rho - \mathbf{g}\alpha\theta$$

where Ω is the angular velocity, p is hydrodynamic pressure, ρ is the constant density of the incompressible fluid (excluding the effect of thermal expansion), **g** is the gravity vector (vertically downwards), and α is the coefficient of thermal expansion. Assume incompressibility. For disturbances in the variables of the form exp[i(σt + **k.r**)], show that

$$\sigma^2 = g\alpha\beta(k_x^2+k_y^2)/k^2 - (2\Omega.\mathbf{k})^2/k^2$$

In the case of rapid rotation defined as $\Omega^2 >> g\alpha\beta$ (a common situation for large scale convection in giant planets and stars), show that the unstable modes must take the form of fluid motions in which the scale length of variation perpendicular to the rotation vector is much smaller than the scale parallel to the rotation vector. In other words, Taylor columns.