Physics 136b

Homework associated with **Chapter 18,** MHD

Available March 5, 2001

Due, **Monday** March 12 in Xinkai Wu's mailbox, East Bridge.

As usual, if any problem is trivial for you, do not do it – simply state that it is trivial and pick some other problem or make up your own. Contact me (djs@gps.caltech.edu) if you have a question or concern about the problems, two of which are new.

1. **Flux Ropes**

Consider a velocity field of the form

$$\mathbf{u} = (-\alpha x/2, -\alpha y/2, \alpha z)$$

in the usual Cartesian coordinates, where α is a positive quantity. This is an incompressible, irrotational flow field that has the tendency of aligning fluid elements parallel to the z-axis. It is somewhat like the local form of flow at the base of a convective upwelling or the top of a convective downwelling along the z-axis. Given the nature of this flow and the way it advects field lines, it is not unreasonable to seek a steady solution for the magnetic field in the form:

$$\mathbf{B} = (0,0, B(x,y))$$

(a) Find the partial differential equation that B satisfies.

(b) Show that this equation is satisfied by

$$B(x, y) = B_0 \exp[-\alpha (x^2+y^2) /4D_M]$$

where D_M is the magnetic diffusivity ($1/\mu_0\kappa_e$, where μ_0 is the permeability of free space and κ_e is the electrical conductivity.) You may need to resort to a demonstration by substitution here, though you should test your ability on PDEs to see if you can deduce this without brute force substitution.

(c) Determine the Lorentz force acting on the fluid. How quickly would this Gaussian flux rope disperse (by the accelerated flow arising from the Lorentz force) if the pressure gradient required to sustain it were suddenly turned off? (Order of magnitude is sufficient here.) Estimate this for plausible Earth core parameters ($B_0 \sim 10^{-3}$ Tesla, $\alpha \sim 10^{-10}$ sec^{-1}, $D_M \sim 1\text{m}^2/\text{sec}$, fluid density $\rho \sim 10^4$ kg/m^3). You'll find it is fast compared to the natural convective timescale α^{-1}.

This question is based in part on p.49 of H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*.

2. Exercise 18.3 in the book (The Earth's Bow Shock).

3. **Parker Dynamo waves and the Sunspot Cycle**

We consider a local Cartesian coordinate system in the Northern Hemisphere of the Sun, in which x represents the Southward direction, y is the azimuthal coordinate and z is the vertical coordinate. We assume that the magnetic field has the form

$$\mathbf{B} = B\,(x,z)\,\mathbf{e_y} + \nabla\ \mathbf{x}\ (A(x,z)\mathbf{e_y})$$

representing the toroidal and poloidal fields respectively, where $\mathbf{e_y}$ is a unit vector in the y-direction. We also assume that there is a differential rotation, i.e. a velocity field

$$\mathbf{u} = (\ 0,\ U(z),\ 0\ )$$

and we assume that small scale (cyclonic, helical motions) provide a source term that creates poloidal field from toroidal field at a rate α x $(B(x,z)\ \mathbf{e_y}\ )$. This is called the alpha effect (where alpha has the dimensions of a velocity).

(a) Confirm that the two differential equations describing this system take the form

$$\partial A/\partial t = \alpha B + D_M\ \ {}^2 A$$

$$\partial B/\partial t = (\partial A/\partial x).(dU/dz) + D_M \quad {}^2B$$

(b) Consider a solution in which A, B behave as $\exp(\sigma t + i(k_x x + k_z z))$, and assume dU/dz and α are slowly varying relative to the spatial variation of the solution. Find the dispersion relationship (the dependence of σ on α, D_M, k_x, k_z, dU/dz). Show that one of these solutions can be in the form of a growing wave that propagates towards the equator, and provide the circumstances (relative signs of α, k_x, dU/dz) that must be satisfied.

This is based on the analysis given on p.212 of H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*. It is also in E. N. Parker's monograph. This simple model can explain the "butterfly diagram" (incidence of sunspots as a function of latitude and time.)